Nuclear structure & decays

Contents

1 Structure of matter and energy scales **2**

2 Binding Energy **2**
2.1 Semi-empirical mass formula 3

3 Decays and reactions **7**
3.1 Alpha Decays 9
3.1.1 Rate calculation for α decays 10
3.2 Beta decays 11
3.2.1 Fermi theory 13

4 Nuclear Scattering **16**
4.1 Cross sections 16
4.2 Resonances and the Breit-Wigner formula 18
4.3 Nuclear scattering and form factors 20

5 Key points **23**

Appendices **24**

.A Natural units **24**

.B Tools **25**
.B.1 Decays and the Fermi Golden Rule 25
.B.2 Density of states 25
.B.3 Fermi G.R. example 26
.B.4 Lifetimes and decays 26
.B.5 The flux factor 27
.B.6 Luminosity 27

.C Shell Model § **28**

.D Gamma decays § **30**

Nuclear structure & decays

1 Structure of matter and energy scales

Subatomic physics deals with objects of the size of the atomic nucleus and smaller. We cannot see subatomic particles directly, but we may obtain knowledge of their structures by observing the effect of projectiles that are scattered from them. The resolution any such probe is limited to of order the de Broglie wavelength,

$$\lambda = \frac{h}{p} \tag{1}$$

where h is Planck's constant, and p is the momentum of the projectile. If we wish to resolve small distances, smaller than the size of the atom, we will need to do so with probes with high momenta. Smaller objects also tend to have larger binding energies holding them together, so require larger energies to excite their internal components. Some typical sizes of objects are given below, together with the momementum of the projectile required to resolve their size, and typical binding energies in electron-volt (eV) units.

keV	10^3 eV
MeV	10^6 eV
GeV	10^9 eV
TeV	10^{12} eV

Object	Size	$p = \frac{h}{\lambda}$	Binding energy
Atom	10^{-10} m	$10\,\mathrm{keV}/c$	$\sim$ eV
Nucleus	$\sim 10^{-15}$ m	$1\,\mathrm{GeV}/c$	$\sim$ MeV
Quark	$< 10^{-19}$ m	$>$ TeV$/c$	$>$ TeV

We can see that small objects also tend to have high binding energies, and hence probes of large energy will be required in order to excite them or break them up. The momenta are indicated in units of eV$/c$ where c is the speed of light. These units make it easy to compare the momentum of the projetile to its corresponding energy $E = pc$ for the case of a massless probe such as a photon. The most convenient unit for describing the size of nuclei is the femtometer 10^{-15} m.[1] No sub-structure has yet been found for quarks even when using very high energy (TeV) probes.

2 The Nuclear Periodic Table and Binding Energy

Nuclei are found to be made out of two constituents: protons and neutrons. We label nuclei by their **atomic number** Z which is the number of protons they contain, by their neutron number N, and by their **mass number** $A = Z + N$.

[1] The unit of 10^{-15} m or femtometer is sometimes called the 'fermi' reflecting the many seminal contributions of the Italian physicist Enrico Fermi to the field of nuclear physics.

The symbol used to identify a nucleus is

$$ {}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X}_{\mathrm{N}} $$

where X is the name of the chemical element. For example the Carbon-14 nucleus, which contains 8 neutrons and 6 protons is denoted ${}^{14}_{6}\mathrm{C}_{8}$. Since the element's name specifies the number of electrons, and hence the atomic number Z, and since $A = N + Z$, we can fully specify the nucleus by just the symbol for the chemical and the mass number,

$$ {}^{\mathrm{A}}\mathrm{X} \quad \text{e.g.} \quad {}^{14}\mathrm{C}. $$

Isotopes	Same Z
Isotones	Same N
Isobars	Same A

Notation for related nuclei

Most nuclei are spherical in shape. The nuclear radius r can be measured in scattering experiments, and follows the general rule

$$ \boxed{r = r_0 A^{1/3}} \tag{2} $$

where the constant r_0 is the characteristic nuclear size and is about 1.2×10^{-15} m. The fact that r is proportional to $A^{1/3}$ indicates that the volume of the nucleus $V \propto r^3$ is proportional to the mass number A. Each proton or neutron is therefore making an equal contribution to the overall nuclear volume.

2.1 Binding energy and the semi-empirical mass formula

The mass $m(A, Z)$ of the nucleus containing Z protons and $A - Z$ neutrons should be given by the mass of its consituents, less the mass associated with the binding energy. The mass-energy is therefore

$$ m(A, Z)c^2 = Zm_p c^2 + (A - Z)m_n c^2 - B(A, Z), \tag{3} $$

where $m_p \approx 938.3\,\mathrm{MeV}/c^2$ and $m_n \approx 939.6\,\mathrm{MeV}$ are the masses of the proton and neutron respectively. In nuclear physics it is convenient to measure energies in units of MeV and masses in units of MeV/c^2. Using these units makes it easy for us to convert from mass to mass-energy and vice versa. By assuming such units, we can omit the factors of c^2 in what follows. [2]

We can build up a functional form for the binding energy $B(A, Z)$ by considering the forces between the nuclear constituents. To find the full quantum mechanical ground state for all of the protons and neutrons would be a very difficult problem. However we can understand a great deal about nuclear behaviour by building up a model of the mass which encapsulates its key features. This we will do over the rest of the section, building up towards the **semi-empirical mass formula** of equation (5). The 'semi-empirical' means that the model is built partly partly by demanding agreement with data, and partly from our understanding of the underlying theory.

Firstly, we will need an attractive force in order to hold the nucleus together against the mutual electrostatic replusion of its constituent protons. That force must be

[2] For more on 'natural units' see appendix .A.

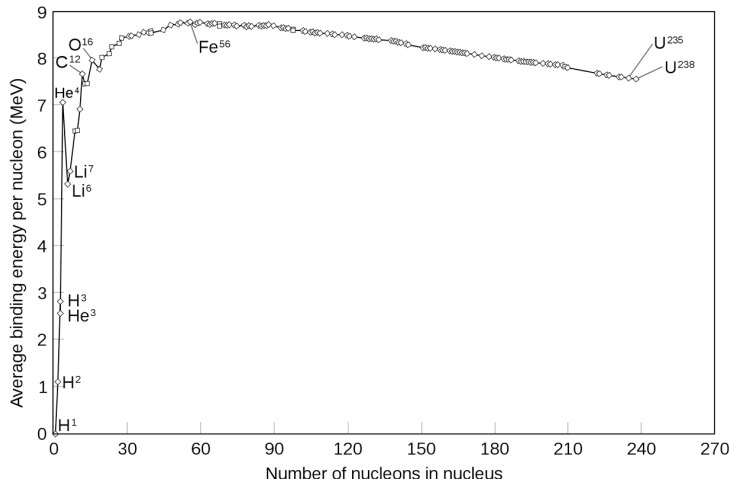


Figure 1: Binding energy per nucleon (B/A) as a function of A for some common nuclei. Data taken from [1]. Plot from [source].

very strong, since the Coulomb electrostatic repulsion between a pair of protons, each of charge e and separated by distance $d \approx 1\,\text{fm}$, is

$$\begin{aligned} F &= \frac{e^2}{4\pi\epsilon_0\, d^2} \\ &\approx 230\,\text{N} \end{aligned}$$

which is macroscopic – comparable to the weight of a child.

What form should that nucleon-nucleon attractive force take? We can get clues about the force by looking at the binding energy per nucleon B/A is shown for some common nuclei, shown in Figure 1. For nuclei this binding energy is typically of order 8 MeV per nucleon. It can be seen that the most stable nuclei are found around ^{56}Fe. Different behaviours can be seen in different regions. There is a broad flattish plateau for the central region $30 < A < 200$ for which $B/A \approx 8\,\text{MeV}$. For A below about 30 the binding energy per nucleon is smaller than the plateau value and is spiky. There is a systematic drop in B/A for large A, particularly for $A > 200$.

To obtain a value of B/A that is rather flat, we cannot permit strong attractions between each of the constituent nucleons and every one of the others. If every nucleon felt an attraction to each of the others, then the binding energy would be expected to grow as approximately $B \propto A(A-1) \sim A^2$, and hence B/A would be approximately proportional to A. Such a linear growth in B/A is ruled out by the data (Figure 1).

To obtain the flat B/A found in nature we must assume that the strongly attractive force acts only between **nearest neighbour** nucleons. In this way, each nucleon binds to the same number of nearest neighbours, independently of the size of the nucleus, and hence the binding energy per nucleon is the same regardless of the

nuclear size,

$$B \approx \alpha A$$

where α is a constant with units of energy. The use of nearest-neighbour interactions indicates that the force must either be short-range, or screened from long-range interactions by the effects of the nucleons in between.

In modelling a nearest-neighbour force we ought to make a correction for the fact that those nucleons on the surface have fewer neighbours. To correct for the reduced number of binding opportunities on the surface we reduce the binding energy by an amount proportional to the surface area, leading to the corrected formula

$$B \approx \alpha A - \beta A^{\frac{2}{3}}. \tag{4}$$

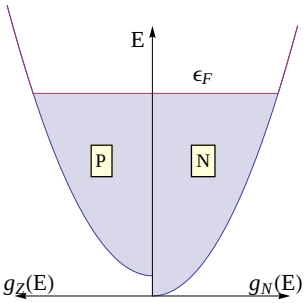
The new contribution is negative since it reduces the binding energy. We have made use of the observation (2) that since the volume of the nucleus scales as $r^3 \propto A$, the surface area scales as $r^2 \propto A^{2/3}$.

These two terms (4) in this first approximation to the binding energy are known as the **volume term** and the **surface term** respectively. Together they form what is known as the **liquid drop model**, since a similar result would be found for a drop of fluid with nearest neighbour interactions and a surface tension parameterised by β. The liquid drop model is consistent with the observation that each nucleon requires the same volume of space, in agreement with equation (2).

So far, so good. However there is nothing in this liquid drop model to prevent the growth of arbitrarily large nuclei. Such large nuclei are not observed in nature, so we must be missing something. The obvious candidate is the **Coulomb repulsion**, which interacts over long distances, and so will tend to push larger nuclei apart. This electrostatic repulsion between protons will reduce the binding energy by an amount proportional to $Z(Z-1) \approx Z^2$ because every proton feels the repulsion from all of the other protons (not just nearest neighbours). The binding energy will be reduced by the electrostatic binding energy which can be parameterised by

$$\epsilon \frac{Z^2}{A^{\frac{1}{3}}}.$$

Here ϵ is a another constant with dimensions of energy, which we will calculate a value for in the examples. The Coulomb repulsion energy is inversely proportional to the radius of the nucleus, and hence to $A^{\frac{1}{3}}$, since the potential energy of a uniform sphere of charge Q is proportional to Q^2/r.



The density of states $g(E)$ for protons and neutrons as a function of energy E.

Two further terms are required to give a good match between our model and the data. Both of them are quantum mechanical in origin.

Firstly there is an **asymmetry term**. The origin of this term is as follows. Since protons are identical fermions, the Pauli exclusion principal states that no two of them may exist in the same state. Nor may any neutron occupy the same state as any other neutron. However it is possible for a proton and a neutron to exist in the same state since the two particles are not identical. The allowed states are therefore distinct, and are separately filled for the protons compared to the neutrons.

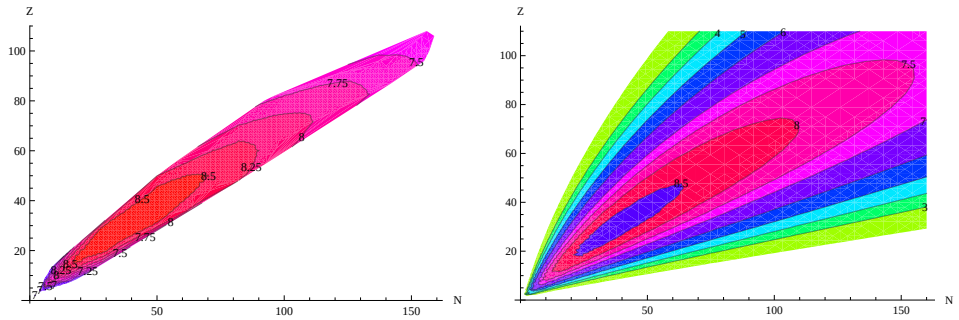


Figure 2: Diagram showing binding energies as a function of proton and neutron number for **(a)** data [1] and **(b)** the Semi-Empirical Mass Formula.

We can work out the size of the asymmetry effect by calculating the number of states available. Neutrons and protons are both fermions, and so obey Fermi-Dirac statistics. The temperatures we are interested in are small compared to the chemical potential ($k_B T \ll \mu$). Under these circumstances the Fermi-Dirac distribution tends towards a step function — all levels are filled up to some energy level, known as the **Fermi Energy** ϵ_F, with all states with energy above ϵ_F left vacant.

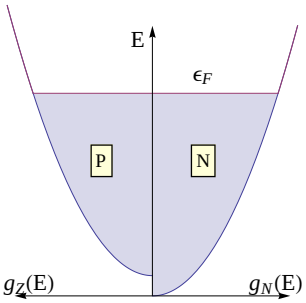
$$p(E_i) = \frac{1}{e^{(E_i-\mu)/k_B T}+1}$$

The Fermi-Dirac function gives the probability $p(E_i)$ of filling a state with energy E_i for a system at temperature T and with chemical potential μ. k_B is the Boltzmann constant.

At large mass number A the Coloumb repulsion term would tend to favour larger N and smaller Z, since neutrons do not suffer from the Coulomb repulsion as protons do. However this energic advantage of neutrons over protons will be partially cancelled out by the fact that the additional neutrons must (on average) be placed in higher energy levels than additional protons, since all of the lower-energy neutron states will already be filled.

The density of available states is found to be proportional to $E^{\frac{1}{2}}$. In the examples we show that this leads to an energy equation of the form

$$\gamma \frac{(N-Z)^2}{A}.$$



The density of states $g(E)$ for protons and neutrons as a function of energy E.

This **asymmetry term** reduces the binding energy, doing so most when the difference between the number of protons and of neutrons is largest.

Finally there is a **pairing term** which accounts for the observation that nuclei with either even numbers of protons (Z even) or with even numbers of neutrons (N even) tend to be more stable than those with odd nuclei. The pairing term is zero for odd-A nuclei. Even A nuclei have two possibilities. If both Z and N are even then the nucleus is more tightly bound and have an extra binding contribution, so B is increased by δ. If both Z and N are odd then the nucleus is less tightly bound and so B is decreased by δ.

N	Z	pairing term
even	even	δ
even	odd	0
odd	even	0
odd	odd	$-\delta$

Putting all five terms together we obtain a formula for the binding energy,

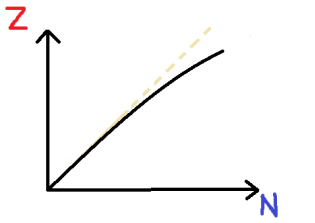
$$B(A,Z) = \alpha A - \beta A^{\frac{2}{3}} - \gamma \frac{(A-2Z)^2}{A} - \epsilon \frac{Z^2}{A^{\frac{1}{3}}} + \delta(A,Z),$$

Volume	Surface	Asymmetry	Coulomb	Pairing
α	β	γ	ϵ	δ
15.835	18.33	23.2	0.71	$11.2/\sqrt{A}$

Figure 3: Typical values of the SEMF parameters (in MeV). From Bowler.

having eliminated N in favour of A. Substituting this into the formula defining the binding energy (3) we obtain the **semi-empirical mass formula** (SEMF)

$$M(A,Z) = Zm_p + (A-Z)m_n - \alpha A + \beta A^{\frac{2}{3}} + \gamma\frac{(N-Z)^2}{A} + \epsilon\frac{Z^2}{A^{\frac{1}{3}}} - \delta(N,Z). \quad (5)$$



Sketch of the shape of the valley of stability.

Other than for $A < 30$, where our approximations are less valid, the SEMF gives a rather good description of the binding energies of the observed nuclei (Figure 2). In particular the SEMF correctly predicts the shape of the curved **valley of stability** in the Z, N plane within which the stable nuclei are found. The relative numbers of protons and neutrons along this valley reflects a trade-off between the Coulomb and asymmetry terms. At low A the asymmetry term favours $N = Z$. At larger A the Coulomb term starts to compete with the asymmetry term, reducing the ratio of protons to neutrons.

It is energetically favourable for nuclei far from that valley to migrate towards it by nuclear decay, in the ways we describe in the following section.

3 Decays and reactions

A table of the nuclides can be found in Figure 4. The stable long-lived nuclides lie along the valley of stability where the binding energy per nucleon is largest. The valley lies along $N \approx Z$ for light nuclei but has $N > Z$ for heavier nuclei. Nuclei far from that valley, and very heavy nuclei, tend to be unstable against nuclear decay.

While unstable nuclei will decay spontaneously, other reactions can be initiated by firing projectiles at a nucleus. Reactions are said to be **elastic** if the final state contains the same set of particles e.g. the elastic scattering of a photon from a nucleus, via an excited intermediate state:

$$X + \gamma \longrightarrow X^* \longrightarrow X + \gamma.$$

Reactions are **inelastic** if there is a change in particle content during the reaction e.g. radiative capture of a neutron

$$^{\mathrm{A}}\mathrm{X} + n \longrightarrow {}^{\mathrm{A+1}}\mathrm{X}^* \longrightarrow {}^{\mathrm{A+1}}\mathrm{X} + \gamma.$$

For all nuclear decays and reactions we define the Q value to be amount of energy

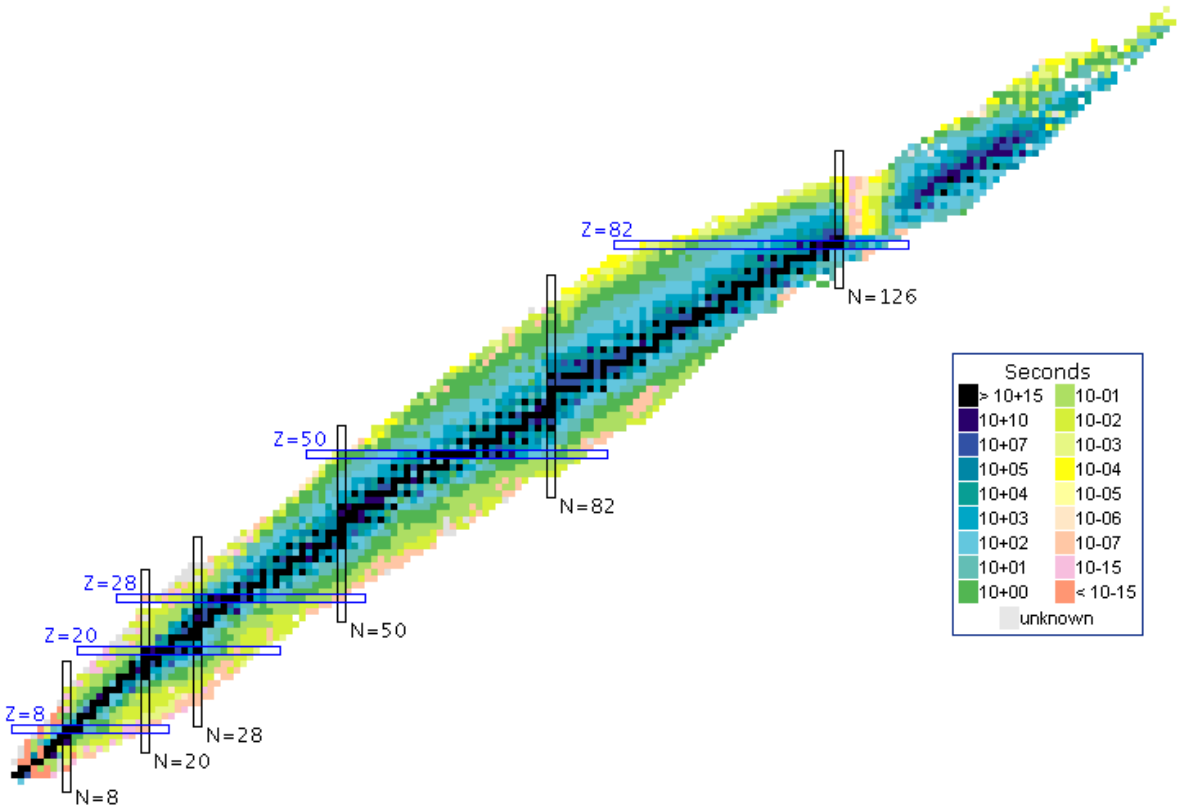


Figure 4: Table of the nuclides as a function of the number of neutrons (N, on the x-axis) and the number of protons (Z on the y-axis). Darker colours represent longer-lived nuclides, which can be found in the 'valley of stability'. The 'Magic Numbers' indicate particularly stable nuclei and are described in the shell model (see appendix .C). From http://www.nndc.bnl.gov/nudat2/.

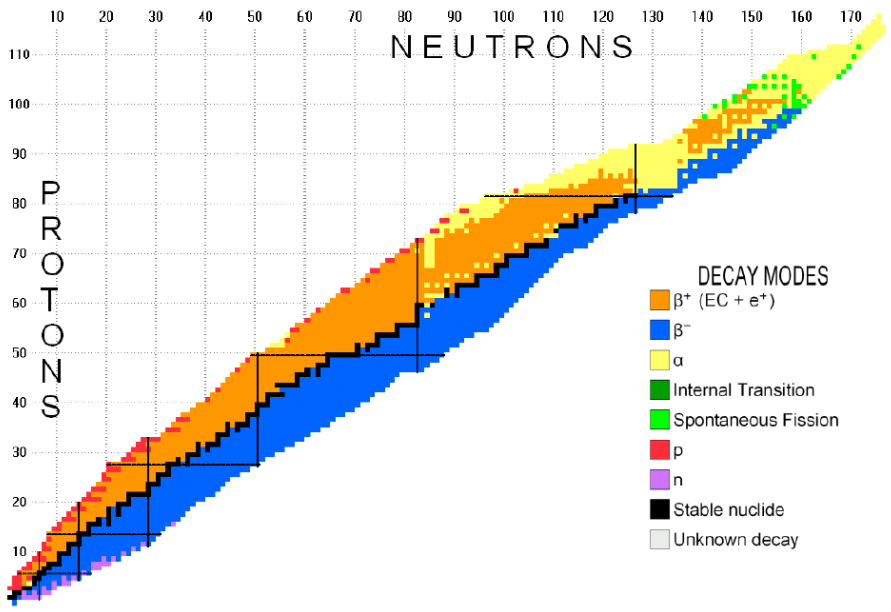
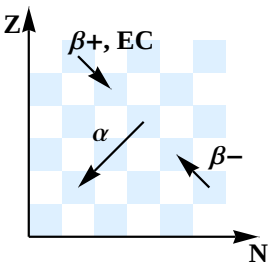


Figure 5: Decay modes of the nuclei. From [1].

'released' by the decay,

$$\boxed{Q = \sum M_i - \sum M_f}$$

The first sum is over the masses of the initial particles in the decay (including their binding energies), while the second sum is over the masses of the final-state particles (including their binding energies). A positive Q value shows that a reaction is energetically favourable.



The changes in (Z, A) induced by various decays.

3.1 Alpha Decays

Alpha decays occur when (usually heavy) nuclei eject an 'α particle', that is a helium nucleus containing two protons and two neutrons. The decay is

$$\boxed{{}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X} \longrightarrow {}^{\mathrm{A-4}}_{\mathrm{Z-2}}\mathrm{Y} + {}^{4}_{2}\mathrm{He}.}$$

The change of mass number of the heavy nucleus is $\Delta A = -4$ and the change in its atomic number is -2.

The process of decay of heavy nuclei is often via sequential chains involving both alpha and beta decays. Since $\Delta A = -4$ for α decays and $\Delta A = 0$ for β and γ decays we see that for any nucleus starting with mass number A (and if it only decays via alpha, beta or gamma decay processes) all other nuclei in that chain must have some other set of mass numbers $A' = A - 4m$, where m is an integer indicating the number of alpha decays that have occured.

There are therefore *four* non-overlapping decay chains for the heavy elements, corresponding to $A = n$, $A = n + 1$, $A = n + 2$, and $A = n + 3$ respectively, where n is an integer. So for example a chain initiated by $^{238}\mathrm{U}$ will decay via nuclides with mass numbers separated by four units $A = \{238,\ 234,\ 230,\ \ldots,\ 4n+2, \ldots\}$. This occurs via the series of decays:

$$^{238}\mathrm{U} \xrightarrow{\alpha} {}^{234}\mathrm{Th} \xrightarrow{\beta} {}^{234}\mathrm{Pa} \xrightarrow{\beta} {}^{234}\mathrm{U} \xrightarrow{\alpha} {}^{230}\mathrm{Th} \xrightarrow{\alpha} {}^{226}\mathrm{Th} \quad \text{etc.}$$

The other three types of chain can, for example, be initiated from decays of the following isotopes: $^{232}\mathrm{Th}$ (leading to nuclides in the $A = 4n$ series); $^{237}\mathrm{Np}$ (leading to nuclides in the $A = 4n + 1$ series); and $^{239}\mathrm{Pu}$ (leading to nuclides in the $A = 4n + 3$ series).

3.1.1 Rate calculation for α decays

We can model the α decay as a process in which 'proto α particles' are pre-formed inside the nucleus. Each is assumed to have a large number of collisions with the edge of the nucleus, but a small probability on each collision of **tunnelling** through the Coulomb barrier and escaping.

If the Q value of the decay is positive, then the decay is energetically favourable, but it may still be suppressed by a large tunnelling factor. Let us try to model the probability of tunnelling through the barrier. We will assume that the large exponential in the quantum tunnelling factor will dominate the calculation of the rate of decay, so we will neglect differences in the probability of formation of the proto-alpha particle, and its rates of hitting the barrier.

The time independent Schrödinger equation defines the energy eigenstate $|\Psi\rangle$,

$$E|\Psi\rangle = \left(\frac{p^2}{2m} + V\right)|\Psi\rangle.$$

where E is the energy of the alpha particle, p is the momentum operator, m is its mass, and V is the potential in which it moves.

For simplicity, we will ignore the spherical geometry and treat the problem as one-dimensional in the radial direction r so that for a state with energy Q,

$$Q\langle r|\Psi\rangle = \left(-\frac{1}{2m}\frac{\partial^2}{\partial r^2} + V(r)\right)\langle r|\Psi\rangle, \tag{6}$$

where we use natural units such that $\hbar = \frac{h}{2\pi} = c = 1$ (see appendix .A). In the Dirac notation $\langle r|\Psi\rangle$ represents the wave function — that is the amplitude to find the alpha particle located between r and $r + dr$. Without losing any generality we can write the wave function as the exponential of some other function $\eta(r)$,

$$\langle r|\Psi\rangle = \exp[\eta(r)]. \tag{7}$$

After inserting (7) into (6) and dividing by $\exp(\eta)$ we find

$$Q = -\frac{1}{2m}\left[\eta'' + (\eta')^2\right] + V(r),$$

where the primes indicate derivatives by r. We can model the potential $V(r)$ felt by any α particle by the function

$$V(r) = \begin{cases} \text{const} & r < R_a \\ \frac{zZ\alpha_{\rm EM}}{r} & r > R_a \end{cases}$$

where inside the nucleus V is large and negative, and outside the nucleus it is given by the Coulomb potential and hence characterised by the charges z and Z of the α-particle and the daughter nucleus respectively. The constant $\alpha_{\rm EM}$ in the Coulomb potential is the dimensionless electromagnetic fine structure constant

$$\alpha_{\rm EM} = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

Within the barrier the potential is smoothly varying, so η should be a smoothly varying function of (r). We then expect $\eta'' \ll (\eta')^2$, and we can safely neglect the η'' term compared to the $(\eta')^2$.[3]

The tunnelling probability can be found from the ratio of the mod-squared amplitudes:

$$P = \frac{|\langle R_b|\Psi\rangle|^2}{|\langle R_a|\Psi\rangle|^2} = e^{-2G}.$$

where G (> 0) is given by

$$-G = \eta(R_b) - \eta(R_a) = -\sqrt{2m}\int_{R_a}^{R_b} dr\,\sqrt{V(r) - Q}.$$

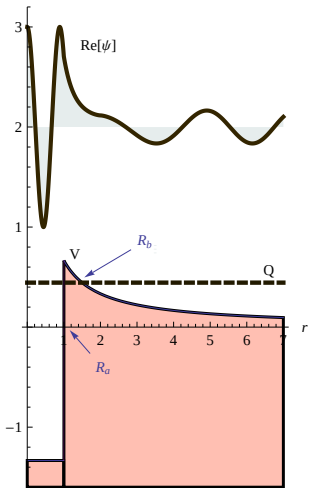
The minus sign before the radical ensures that we select the exponentially falling solution. The inner limit of the integration is the radius of the nucleus

$$R_a \approx r_0 A^{\frac{1}{3}},$$

and the outer limit

$$R_b = \frac{Z_1 Z_2 \alpha_{\rm EM}}{Q},$$

the radius for which $Q > V$, i.e. where the α particle enters the classically allowed region.



Real part of $\langle r|\Psi\rangle$ (as calculated in the WKBJ approximation) and $V(r)$ for a thin Coulomb potential barrier.

3.2 Beta decays, electron capture

There are three related nuclear decay processes which are all mediated by the **weak nuclear interaction**. Neutron-rich isotopes can decay through the emission of an electron e and an anti-neutrino $\bar{\nu}_e$ in the **beta decay** a process:

$$^{\rm A}_{\rm Z}{\rm X} \longrightarrow {}^{\rm A}_{\rm Z+1}{\rm Y} + e^- + \bar{\nu}_e. \tag{8}$$

[3] This is known as the WKBJ approximation. It is a good approximation if many wave-lengths (or in the classically forbidden regions, as here, many factors of $1/e$) of the wave-function occur before the potential changes significantly.

The effect is to increase the atomic number by one, but to leave the mass number unchanged. At the level of the individual nucleons the reaction is

$$n \longrightarrow p + e^- + \bar{\nu}_e. \tag{9}$$

The emitted electron can be observed and its energy measured. The associated anti-neutrino has a very small interaction probability, and so is expected to escape unobserved. Long before neutrinos were observed, Wolfgang Pauli realised that an additional, invisible, massless particle was required in order to conserve energy and momentum in the decay (8). His arguments ran as follows. The emitted electrons are observed to have a variety of different kinetic energies, up to Q. Meanwhile the mass difference between the parent and daughter nucleus is fixed to a single value. The energy given to the recoiling daughter nucleus is small and is fixed by momentum conservation, so it can't be responsible for the deficit observed when the electron has energy less than Q. Energy conservation is then only possible if the total energy Q can be shared between the electron and some other unobserved partice – the (anti-)neutrino.

Pauli also argued that without the neutrino the reaction (9) would violate angular momentum conservation. Adding the angular momenta of just the two observed final state spin-half particles – the electron and the proton – according to the rules of quantum mechanical angular momentum addition we would find

$$\frac{1}{2} \oplus \frac{1}{2} = 0 \text{ or } 1.$$

Neither of the possibilities of total angular momentum $s = 0$ or $s = 1$ match the spin of the initial neutron, which has $s = \frac{1}{2}$. However by adding a third spin-half particle to the final state – the $s = \frac{1}{2}$ anti-neutrino – we can reconstruct a state which has total angular momentum equal to that of the proton ($s = \frac{1}{2}$) since

$$\frac{1}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} = \frac{1}{2} \text{ or } \frac{3}{2}.$$

Isotopes which have a surplus of protons can proceed via one of two processes. The first is the emission of a positively charge anti-electron. This is known as **positron emission** or β^+ decay

$${}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X} \longrightarrow {}_{\mathrm{Z}-1}{}^{\mathrm{A}}\mathrm{Z} + e^+ + \nu_e \tag{10}$$

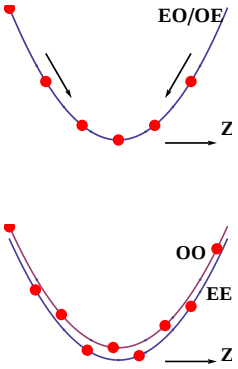
The positron is the anti-particle of the electron. It has the same mass as the electron, but positive charge.

The second method of decay of proton-rich nuclei is by the nucleus removing one of the atomic electrons, the **electron capture** process:

$${}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X} + e^- \longrightarrow {}_{\mathrm{Z}-1}{}^{\mathrm{A}}\mathrm{Z} + \nu_e \tag{11}$$

These two processes ((10) and (11)) result in the same change to the nucleus, and so compete with one another to reduce the Z number of proton-rich nuclei. When considering whether electron capture or β^+ decay will dominate we note that

- The Q value for positron emission is $2 \times m_e c^2$ smaller than that for the corresponding electron capture.
- Electron capture relies on there being a substantial overlap of an electron wave-function with the nucleus.



Mass as a function of Z for nuclides of the same A, for odd-A nuclei (above) and even-A nuclei (below). The even-A case has two curves separated by 2δ.

When viewed at the level of the nuclear consitutents, all three of the interactions above — β decay (8), β^+ decay (10) and electron capture (11) — involve the interaction of four particles: a proton, a neutron, an (anti-)electron and an (anti-) neutrino.

$$n \longrightarrow p + e^- + \bar{\nu}_e \tag{12}$$
$$p \longrightarrow n + e^+ + \nu_e, \tag{13}$$
$$p + e^- \longrightarrow n + \nu_e. \tag{14}$$

We note that all of the reactions (12)–(14) are assumed to be occurring inside the complex environment of the nucleus. Of these three reactions, only neutron decay (12) can occur in isolation, since it is the only one with $Q > 0$, (the neutron being about $1.3\,\mathrm{MeV}/c^2$ heavier than the proton). The other two reactions (13)–(14) occur only within a nucleus, when the energy released from the rearrangement of the nuclear constituents is sufficient to compensate for the endothermic nature of the reaction at the level of the individual nucleon.

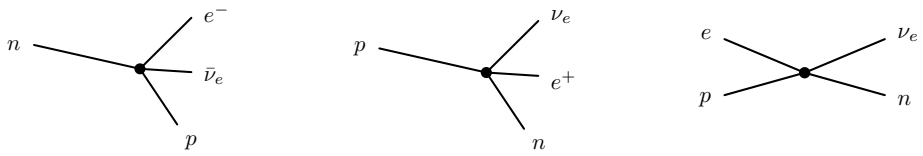
We note that all of three transitions — β^-, β^+, and e^- capture — leave the mass number A unchanged. The parent and daughter nuclei are isobars. The decay processes (12)–(14) allow transitions between isobars, and mean that for odd-A nuclei for any value of A there is usually only one stable isobar, that for which the mass of the system is minimum.

Even-A nuclei may have one stable isobar, but can also have two or very occasionally three. Multiple stable states are possible for A even because the binding curves for (even-Z, even-N) and (odd-Z odd-N) are separated by 2δ, where δ is the pairing energy in the SEMF. For an even-even nucleus, since all of the reactions (12)–(14) change both $|Z|$ and $|N|$ by one they result in an odd-odd nucleus, and so a transition to the higher of the two curves. None of the reactions permit a change in Z of two units, and the probability of two such reactions happening at once is extremely small, so an even-even nucleus with some Z can be stable provided that both the neighbouring nuclei with $Z + 1$ and $Z - 1$ have larger mass.

3.2.1 Fermi theory of beta decays

To understand the lifetimes of the nuclei, we wish to calculate the expected rates for $\beta^\pm$ decays. We follow the method and approximations of Enrico Fermi.

If we put the initial state particles on the left hand side of the diagram, and the final state particles on the right hand side, then we obtain the following three diagrams for the reactions (12)–(14).



We shall assume that each of these four-particle interactions will happen at a single point in space. The amplitude for each reaction is given by the same constant — a four-body coupling constant which tells us the amplitude for each interaction at that point in space. For each of the three diagrams that coupling is the **Fermi constant**,

$$\boxed{G_F \approx 1.17 \times 10^{-5}\,\mathsf{GeV}^{-2}}$$

which has units of inverse energy squared.

In using a single, constant factor, we implicitly make the simplifying assumption that the four-body interaction does not depend on the spins of the incoming or outgoing particles. To simplfy calculations we will also follow Fermi in assuming that the wave-functions of the electron and the anti-neutrino can be represented by plane waves. This ignores the effect of the Coulomb attraction between the electron and the nucleus, and is a good approximation provided that the electron energy is sufficiently high.

We recall that in quantum mechanics, the rate of some a transition from an initial state to a final state characterised by a continuum of energy levels is given by the **Fermi Golden Rule**[4]

$$\Gamma = \frac{2\pi}{\hbar}|A_{fi}|^2 \frac{dN}{dE_f} \tag{15}$$

Here the transition rate Γ is given in terms of the amplitude A_{fi} connecting the initial and the final states, and the degeneracy $\frac{dN}{dE_f}$ of states at the final energy. It is the A_{fi} and $\frac{dN}{dE_f}$ that we shall have to calculate.

To be concrete, let us consider the beta decay reaction (12). We denote the initial nuclear wave-function by $\langle \mathbf{x}|\Psi_i\rangle$, the final nuclear wave-function by $\langle \mathbf{x}|\Psi_f\rangle$. The electron and anti-neutrino wave-functions are approximated as plane waves

$$\langle \mathbf{x}|\phi_e\rangle \equiv \phi_e = \exp\left(i\,\mathbf{p}_e \cdot \mathbf{x}\right) \tag{16}$$
$$\langle \mathbf{x}|\phi_\nu\rangle \equiv \phi_\nu = \exp\left(i\,\mathbf{p}_\nu \cdot \mathbf{x}\right). \tag{17}$$

We can now write down the initial state $|\Psi_i\rangle$, which is just that of the parent nucleus

$$|\Psi_i\rangle = |\psi_i\rangle,$$

and final state $|\Psi_f\rangle$, which is the product of the daughter nucleus state $|\psi_f\rangle$, the electron state $|\phi_e\rangle$ and the anti-neutrino state $|\phi_{\bar{\nu}_e}\rangle$

$$|\Psi_f\rangle = |\psi_f\rangle \times |\phi_e\rangle \times |\phi_{\bar{\nu}_e}\rangle.$$

[4] For a refresher, see appendix .B.1

The matrix element $\mathcal{A}_{fi}$ controls the transition from the initial to the final state

$$\mathcal{A}_{fi} = \langle \Psi_f | \mathcal{A} | \Psi_i \rangle$$

It can be obtained by working in the position representation and recognising that the amplitude G_F associated with the point-like interaction (12) should be integrated over the volume of the nucleus,

$$\mathcal{A}_{fi} = \int d^3x \; G_F \; \phi_e^* \, \phi_\nu^* \, \psi_f^* \, \psi_i .$$

The ϕ and ψ terms are the position representations (wave functions) of the four particles, and in the final state are found in complex conjugate form. The integral sums over the amplitudes for the point-like reaction to occur anywhere in the nucleus, since the reaction could have occurred anywhere within.

To perform the integral we first Taylor expand the exponentials in the plane wave functions (16)–(17) for the electron and the neutrino. The expansion is useful because the exponents $\mathbf{p} \cdot \mathbf{x}$ are small.[5] The product of ϕ_e^* and ϕ_ν^* can therefore be written

$$e^{-i(\mathbf{p}_e + \mathbf{p}_\nu)\cdot \mathbf{x}} \approx 1 - i(\mathbf{p}_e + \mathbf{p}_\nu) \cdot \mathbf{x} + \ldots \tag{18}$$

Provided that the first term in this expression does not vanish when performing the integral, it can be expected to dominate, and the whole integral can be approximated by

$$\begin{aligned} \mathcal{A}_{fi} &= G_F \int d^3x \; \psi_f^* \, \psi_i \\ &\equiv G_F \; M_{\rm nucl} \end{aligned}$$

where in the lower line $M_{\rm nucl}$ denotes the overlap integral between the neutron in the parent nucleus and the proton in the daughter nucleus. The size of the quantity $M_{\rm nucl}$ depends on the participating nuclei, and is known as the **nuclear matrix element**.

In some particularly simple cases $M_{\rm nucl}$ can be calculated analytically. In particular, if the initial-state neutron, and the final-state proton happen to inhabit the same state within a nucleus, the overlap integral is maximal, i.e. for those nuclei

$$|M_{\rm nucl}| = 1.$$

An example of a maximum overlap integral is found for the simplest case of the isolated neutron decay. [6]

To complete the job of calculating Γ we need to find the density of states factor $\frac{dN}{dE_f}$. The density of states for the outgoing electron can be calculated from the density of states inside a box[7],

$$dN = \frac{d^3\mathbf{p}}{(2\pi)^3}.$$

[5] The size of $\mathbf{x}$ is of order the typical nuclear size, i.e. $\sim 10\,$fm, which in natural units is $10\,\text{fm}/(197\,\text{MeV fm}) \sim 10^{-1}\,\text{MeV}^{-1}$. The typical momenta of the out-going particles are of order MeV, so the dot products in the exponents are of order 10^{-1}.

[6] Such decays are called 'super-allowed'.

[7] See appendix .B.2 for the source of this term.

Assuming spherical symmetry the angular integrals yield 4π so

$$dN = \frac{4\pi\ p^2 dp}{(2\pi)^3}.$$

A similar result holds for the neutrino. The states allowed by the daughter nucleus are fixed by total momentum conservation, so provide no further contribution to the density of states. The recoil energy of the heavy daughter nucleus is negligible, so conservation of energy gives

$$E_e + E_\nu = Q,$$

where E_e is the kinetic energy of the electron. Hence the rate of decays that yield electrons with momenta between p_e and $p_e + dp_e$ is

$$d\Gamma(p_e) = G_F^2 |M_{\rm nucl}|^2 \frac{(Q - E_e)^2}{2\pi^3} p_e^2 dp_e.$$

In the relativistic limit where $E_e \gg m_e$ we can perform the integral and obtain the simple result

$$\boxed{\Gamma_\beta \propto Q^5.}$$

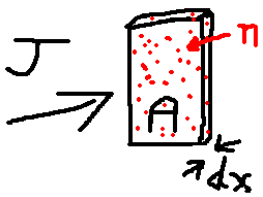
i.e. the rate depends on the fifth power of the available energy.

> 'Forbidden' decays (*Non examinable*)
>
> We have assumed that the first term in (18) will dominate, but it can vanish due to selection rules. For example, if the nuclear matrix element has odd parity the first term vanishes, since then we are integrating the product of an odd and an even function. In that case, the next term in the series is required, and the reaction rate is suppressed. Such decays are said to be 'first forbidden'. In general the larger the change in angular momentum required in the nuclear transition, the further along the series one will need to go to find a non-zero term, and the slower will be the decay.

4 Nuclear Scattering

The structure of the nucleus can be probed by scattering projectiles from it. Those projectiles might be protons, electrons, muons, or indeed other nuclei.

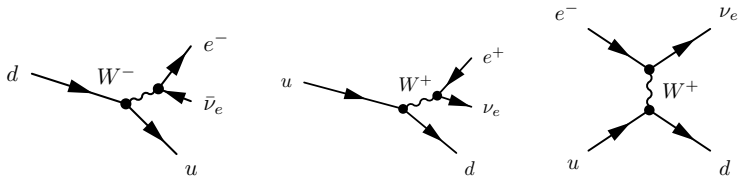
4.1 Cross sections



Many experiments take the form of scattering a beam of projectiles into a target.

Provided that the target is sufficiently thin that the flux is approximately constant within that target, the rate of any reaction W_i will be proportional to the flux of

We will later find that Beta decays are mediated at very small length scales ($\sim 10^{-18}$ m) by charged spin-1 force-carrying particles known as $W^{\pm}$ bosons.



The Feynman diagrams above show the $W^{\pm}$ bosons responsible for β^- decay, β^+ decay, and electron capture. For probes with wavelength $\lambda \gg \lambda_{c,W}$, where

$$\lambda_{c,W} = \frac{\hbar}{m_W c}$$

is the W boson Compton wavelength, or equivalently for probes with momentum $p \ll m_W$, the small-distance behaviour of the interaction is not apparent. We do not resolve the W boson and instead we get what appears to be a single four-body interaction.

incoming projectiles J (number per unit time) the number density of scattering centres n in the target (number per unit volume), and the width δx of the target

$$W_i = \sigma_i \, n \, J \, \delta x. \tag{19}$$

The constant of proportionality σ_i has dimensions of area. It is known as the **cross section** for process i and is defined by

$$\boxed{\sigma_i = \frac{W_i}{n \, J \, \delta x}} \tag{20}$$

We can get some feeling for why this is a useful quantity if we rewrite (19) as

$$W_i = \underbrace{(n \, A \, \delta x)}_{N_{\rm target}} J \; \underbrace{\frac{\sigma}{A}}_{P_{\rm scatt}}$$

where A is the area of the target. Here $N_{\rm target}$ is the total number of targets illuminated by the projectile, and the cross section can be interpreted as the **effective area** presented to the beam per target for which a particular reaction can be expected to occur.

The total rate of loss of beam is given by $W = \Sigma W_i$, and the corresponding total cross section is therefore

$$\sigma = \sum_i \sigma_i.$$

We could choose to quote cross sections in units of e.g. fm^2 or in natural units of GeV^{-2}, however the most common unit used in nuclear and particle physics is the so-called **barn** (b) where

$$\boxed{1 \, \text{barn} = 10^{-28} \, \text{m}^2}$$

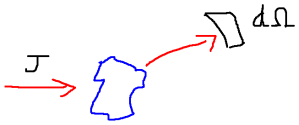
As well as scattering experiments, the size of various nuclei can be determined by other methods, including:

- Shifts in the energy-levels of atomic electrons from the change of their Coulomb potential caused by the finite size of the nucleus.
- Muonic equivalents of the above. Muons are about 200 times heavier than electrons, so their "Bohr raduis" is about 200 times smaller. One observes the series of x-rays from the atomic (muonic) transitions

We can convert the barn to natural units of MeV^{-2} using the $\hbar c$ conversion constant as follows

$$\begin{aligned} 1\,\text{barn} &= 10^{-28}\,\text{m}^2 \\ &= 100\,\text{fm}^2/(197\,\text{MeV fm})^2 \\ &= 0.00257\,\text{MeV}^{-2}. \end{aligned}$$

The 'differential cross section' $\frac{d\sigma}{d\Omega}$ is the cross section per unit solid angle of scattered particle. It is defined to be the rate of scattering per target per unit incoming flux density per unit solid angle ($d\Omega$) of deflected particle.



4.2 Resonances and the Breit-Wigner formula

Sometimes the projectile can be absorbed by the nucleus to form a compound state, then later reabsorbed. The time-energy uncertainty relationship of quantum mechanics tells us that if a state has only a finite lifetime (of order Δt), then it has an uncertaintity on its energy ΔE given by

$$\Delta E \Delta t \sim \hbar.$$

Using the fact that the decay rate $\Gamma = 1/\tau$, and using natural units to set $\hbar = 1$, we find that

$$\Delta E \sim \Gamma. \tag{21}$$

In these units, the uncertainty in the rest-energy of a particle is equal to the rate of its decay. This means that if we take a set of identical unstable particles, and measure the mass of each, we will expect to get a range of values with width of order Γ.

Long-lived intermediate states have small Γ and hence well-defined energies. We tend to think of these reasonably long-lived intermediate states as 'particles'. The neutron is an example of an unstable state that lives long enough for the word 'particle' to be meaningfully applied to it.

Short-lived intermediate states have large widths and less well defined energies. When the intermediate state is so short-lived that its width Γ is similar to its mass, then the decay is so rapid that it is no longer useful to think of it as a particle — it's really some transition through which the state happens to be momentarily passing.

The decay width can be generalised to a particle which has many different decay modes. The rate of decay into mode i is given Γ_i. The total rate of decay is given by the sum over all possible decay modes

$$\Gamma = \sum_{i=1\ldots n} \Gamma_i.$$

The fraction of particles that decay into final state i, is known as the **branching ratio**

$$\mathcal{B} = \frac{\Gamma_i}{\Gamma}.$$

The quantity Γ_i is known as the **partial width** to final state i, whereas the sum of all partial widths is known as the **total width**.

We can develop these ideas more quantitatively by considering the general process

$$A + B \longrightarrow O \longrightarrow C + D. \tag{22}$$

The initial particles A and B collide to form an unstable intermediate O, which then decays to the final state C and D (which may or may not have the same particle content as the initial state). An example of a familiar process is the absorption and then emission of a photon by an atom, with an intermediate excited atomic state,

$$A + \gamma \longrightarrow A^* \longrightarrow A + \gamma.$$

Alternatively the reaction could represent an inelastic nuclear interaction, for example the nuclear absorption of a neutron to form a heavier isotope followed by its de-excitation

$$^{25}\mathrm{Mg} + n \longrightarrow {}^{26}\mathrm{Mg}^* \longrightarrow {}^{26}\mathrm{Mg} + \gamma.$$

Other reactions can create and annihilate other types of particle.

The reaction will proceed most rapidly when the energies of the incoming particles are correctly tuned to the mass of the intermediate. The reaction rate will be fastest when the energy of $A + B$ is equal to the rest-mass energy E_O of the intermediate state. The energy need only match E_O to within the uncertainty Γ in the energy of the intermediate.

Under the condition that the transition from $A + B$ to $C + D$ proceeds **exclusively** via the intermediate state '0', of mass m_0 and that the width of the intermediate is not too large ($\Gamma \ll m_0$), the probability for the scattering process, as a function of total energy E takes the familiar Lorentzian shape

$$p(E) \propto \frac{1}{(E - E_0)^2 + \Gamma^2/4}. \tag{23}$$

This is the same peaked shape as seen in resonances in situations involving oscillators, and so the excited intermediate state is often called a **resonance**, and the process is known as resonant scattering.

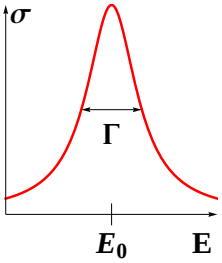
Taking into account density of states and flux factors, and the possibilities of decay into multiple different final states, the overall cross-section for the process (22) is

given by the famous **Breit-Wigner** formula

$$\sigma_{i\to 0\to f} = \frac{\pi}{k^2}\frac{\Gamma_i\Gamma_f}{(E-E_0)^2+\Gamma^2/4}. \qquad (24)$$

Since excited states are very common, this is an important result not just in nuclear and particle physics, but also in any process where excitations are found. The terms in this equation are as follows:

- Γ_i is the *partial width* of the resonance to decay to the initial state $A+B$
- Γ_f is the *partial width* of the resonance to decay to the final state $C+D$
- Γ is the *full width* of the resonance at half-maximum (and equal to the sum $\sum_j \Gamma_j$ over all possible decay modes)
- E is the centre-of-mass energy of the system
- E_0 is the characteristic rest mass energy of the resonance
- k is the wave-number of the incoming projectile in the centre-of-mass frame which is equal to its momentum in natural units.



The Breit-Wigner line shape.

The cross-section is non-zero at any energy, but has a sharp peak at energies E close to the rest-mass-energy E_0 of the intermediate particle. Longer lived intermediate particles have smaller Γ and hence sharper peaks.

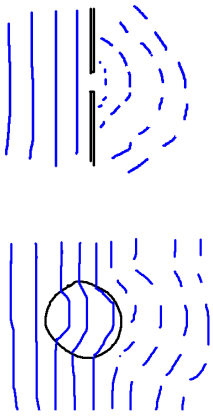
Resonant scattering experiments can tell us about the excited states of nuclei, and hence provide further information about nuclear structure and interactions. All sorts of particles which are too short-lived to travel macroscopic distances can nevertheless be created as intermediate states and studied from the properties of their Breit-Wigner peaks.

4.3 Nuclear scattering and form factors

When a target only slightly perturbs the wave-function of the projectile, the resulting scattering behaives rather like optical diffraction.

In optics, the properties of a microscopic aperture can be understood from the pattern obtained when light, of wavelength similar to the size of the aperture, is diffracted by that apperture. Far from the aperture, the optical pattern observed is the two dimensional Fourier transformation of the aperture function. This is true even if the optical aperture is too small to observe directly.

Now consider scattering a wave from a three-dimensional projectile. Again, the observed diffraction pattern comes from a Fourier transform of the object, but now the aperture function is replaced with the potential $V(\mathbf{x})$.


Diffractive scattering from a slit and from an object.

In the Born approximation, which is valid for weak potentials, the amplitude $f(\Delta\mathbf{k})$ for scattering a projectile such that its change in momentum is $\Delta\mathbf{k}$, is proportional to the 3D Fourier transform of the scattering potential V,

$$f(\Delta\mathbf{k}) = A\int d^3x\; V(\mathbf{x})\; e^{-i\Delta\mathbf{k}\cdot\mathbf{x}} \tag{25}$$

where A is a normalising constant. The probability to scatter into some small angle $d\Omega$ is then proportional to $|f(\Delta\mathbf{k})|^2$.

Let us consider the scattering of a projectile of charge z from a nucleus with charge Z and spherically symmetric local charge density $\rho(r)$, centred at the origin. The potential at some point $\mathbf{x}'$ is given by summing over the Coulomb potentials from distributed charges at all other locations $\mathbf{x}''$,

$$\begin{aligned} V(\mathbf{x}') &= \frac{ze^2}{4\pi\epsilon_0}\int d^3x''\;\frac{\rho(\mathbf{x}'')}{|\mathbf{x}'-\mathbf{x}''|} \\ &= z\alpha\int d^3x''\;\frac{\rho(\mathbf{x}'')}{|\mathbf{x}'-\mathbf{x}''|} \end{aligned}$$

where in the second step we again use the relation (valid in natural units) that the electromagnetic fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0}$.

Substituting this form of the potential into the Born relation (25) we find

$$f(\Delta\mathbf{k}) = z\alpha A\int d^3x'\int d^3x'' e^{-i(\Delta\mathbf{k})\cdot\mathbf{x}'}\frac{\rho(\mathbf{x}'')}{|\mathbf{x}'-\mathbf{x}''|}.$$

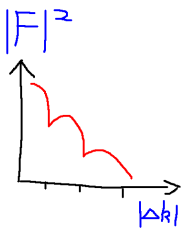
We may simplify this expression by defining a new variable $\mathbf{X} = \mathbf{x} - \mathbf{x}'$. This change of variables allows us to factorize the two integrals, giving the result

$$f(\Delta\mathbf{k}) = \underbrace{\left[\frac{1}{Z}\int d^3x'\rho(x')e^{-i\boldsymbol{\Delta k}\cdot\mathbf{x}'}\right]}_{\text{Form Factor}} \times \underbrace{Zz\alpha A\left[\int d^3X\frac{e^{-i\boldsymbol{\Delta k}\cdot\mathbf{X}}}{|\mathbf{X}|}\right]}_{\text{Rutherford}}. \tag{26}$$

The scattering amplitude from a distributed charge is therefore equal to the product of two terms. The second term can be recognised as the Rutherford scattering amplitude – the amplitude that would be obtained from scattering from a point charge density $\rho(\mathbf{x}) = Z\delta(\mathbf{x})$. The second term therefore tells us nothing about the internal structure of the nucleus. All of the interesting information about the nuclear structure is encapsulated in the first term,

$$\boxed{F_{\rm nucl}(\Delta\mathbf{k}) = \int d^3x\; N(\mathbf{x})\; e^{-i\boldsymbol{\Delta k}\cdot\mathbf{x}}}$$

which is known as the **nuclear form factor**. The form factor is the three-dimensional Fourier transform of the normalised charge density $N(\mathbf{x}) = \rho(\mathbf{x})/Z$. All of the interesting information about the size and structure of the nucleus is found in $F_{\rm nucl}(\Delta\mathbf{k})$. We will find interesting scattering — that is interesting 'diffraction patterns' — if the exponent is of order unity. For this to be true the de Broglie wave-length of



Sketch of a nuclear form factor diffraction pattern.

the projectile must be of the same order as the nuclear size, as was noted in the introduction to this chapter.

The mod-squared of the optical amplitude gives the intensity. Similarly it is $|F|^2$ which is important when considering the flux of scattered projectile particles. The rate at which particles are scattered into unit solid angle is given by

$$\frac{dN}{d\Omega} = |F_{\rm nucl}(|\mathbf{\Delta k}|)|^2 \left(\frac{dN}{d\Omega}\right)_{\rm Rutherford}. \tag{27}$$

This equation is more often written in terms of the differential cross section for scattering

$$\boxed{\frac{d\sigma}{d\Omega} = |F_{\rm nucl}(|\mathbf{\Delta k}|)|^2 \left(\frac{d\sigma}{d\Omega}\right)_{\rm Rutherford}}.$$

By examining the form factor for particles scattered with various changes in momentum $|\Delta\mathbf{k}|$ we can infer information about $N(\mathbf{x})$ and hence about the size and shape of the nuclear potential $V(\mathbf{x})$.

5 Key points

- In **natural units** (appendix .A), $\hbar = c = 1$ and

$$\boxed{[\text{Mass}] = [\text{Energy}] = [\text{Momentum}] = [\text{Time}]^{-1} = [\text{Distance}]^{-1}}$$

  A useful conversion constant is

$$\boxed{\hbar c \approx 197 \text{ MeV fm}}$$

- The nuclear mass is well described by the **semi-empirical mass formula**

$$\boxed{M(A,Z) = Zm_p + (A-Z)m_n - \alpha A + \beta A^{\frac{2}{3}} + \gamma \frac{(A-2Z)^2}{A} + \epsilon \frac{Z^2}{A^{\frac{1}{3}}} - \delta(A,Z).}$$

- The binding energy leads to a **valley of stability** in the (A, Z) plane where the stable nuclei lie
- In a reaction or decay, the Q-value is the energy released in a decay

$$\boxed{Q = \sum M_i - \sum M_f}$$

  If $Q > 0$ the reaction is **exothermic** – it gives out energy, whereas if $Q < 0$ the reaction is **endothermic**, and energy must be supplied for it to proceed.
- Alpha decay rates are dominated by quantum **tunnelling** through the Coulomb barrier.
- Beta decay rates and electron capture are governed by the **Fermi coupling constant**

$$\boxed{G_F \approx 1.17 \times 10^{-5} \text{ GeV}^{-2}}$$

- The **cross section** is defined by:

$$\boxed{\sigma_i = \frac{W_i}{n\, J\, \delta x}} \tag{28}$$

  The **differential cross section** is the cross section per unit solid angle

$$\boxed{\frac{d\sigma_i}{d\Omega}}$$

- Cross sections for sub-atomic physics are often expressed in the unit of **barns**.

$$\boxed{1 \text{ barn} = 10^{-28} \text{ m}^2}$$

- The **Breit Wigner formula** for resonant scattering is

$$\boxed{\sigma_{i\to 0\to f} \;=\; \frac{\pi}{k^2} \frac{\Gamma_i \Gamma_f}{(E-E_0)^2 + \Gamma^2/4}.}$$

- In elastic nuclear scattering the **form factor**

$$\boxed{F(|\Delta k|) = \int d^3x \ \left(\frac{\rho(x)}{Z}\right) \ e^{-i\mathbf{\Delta k}\cdot\mathbf{x}},}$$

is the 3D Fourier transform of the normalised charge density, and is related to the Rutherford scattering differential cross section by

$$\boxed{\frac{d\sigma}{d\Omega} = |F_{\rm nucl}(|\mathbf{\Delta k}|)|^2 \left(\frac{d\sigma}{d\Omega}\right)_{\rm Rutherford}.}$$

.A Natural units

In the S.I. system of units, times are measured in seconds and distances in meters. In those units the speed of light takes the value close to $3 \times 10^8\,\mathrm{ms}^{-1}$.

We could instead have chosen to use unit of time such that $c = 1$. For example we could have used units in which time is measured in seconds and distance in light-seconds. In those units the speed of light is one (one light-second per second). Using units in which $c = 1$ allows us to leave c out of our equations (provided we are careful to remember the units we are working in). Such units are useful in relativistic systems, since now the relativistic energy-momentum-mass relations are simplified to

$$E = \gamma m$$
$$p = \gamma m v$$
$$E^2 - p^2 = m^2.$$

So for a relativistic system setting $c = 1$ means that energy, mass and momentum all have the same dimensions.

Since we are interested in quantum systems, we can go further and look for units in which $\hbar$ is also equal to one. In such units the energy E of a photon will be equal to its angular frequency ω

$$E = \hbar\omega = \omega.$$

Setting $\hbar = 1$ therefore means that the units of energy are the same as the units of inverse time. Units with $\hbar = 1$ imply that time (and via $c = 1$ distance too) must have the same dimensions as inverse energy, E^{-1}.

So in our system **natural units** with $\hbar = c = 1$ we have have that all of the following dimensions are the same:

$$\boxed{[\mathrm{Mass}] = [\mathrm{Energy}] = [\mathrm{Momentum}] = [\mathrm{Time}]^{-1} = [\mathrm{Distance}]^{-1}}$$

We are still free to choose a convenient unit for all of these quantities. In subatomic physics it is common to use units of energy (or inverse energy). The nuclear energy

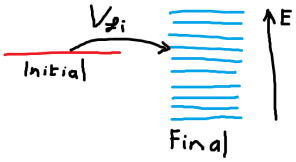
levels have typical energies of the order of 10^6 electron-volts, so we shall measure energies, momenta and masses in MeV, and lengths and times in MeV^{-1}. At the end of a calculation we might wish to recover, for example, a "real" length from one measured in our MeV^{-1} units. To do so we can make use of the conversion factor

$$\boxed{\hbar c \approx 197 \text{ MeV fm}}$$

which tells us that one of our MeV^{-1} length units corresponds to 197 fm where $1\,\text{fm} = 10^{-15}\,\text{m}$.

.B Tools for cross-section calculations

.B.1 Decays and the Fermi Golden Rule



In subatomic physics we are interested in the decays of unstable particles, such as radioactive nuclei, muons from the atmosphere, or Higgs bosons. Using time-dependent perturbation theory in quantum mechanics it is possible to show that that the transition rate of an unstable state into a continuum of other states is given by the **Fermi Golden Rule**:

$$\boxed{\Gamma = \frac{2\pi}{\hbar} |V_{fi}|^2 \frac{dN}{dE_f},} \tag{29}$$

where

- Γ is the rate of the decay
- V_{fi} is the matrix element of the Hamiltonian coupling the initial and the final states
- $\frac{dN}{dE_f}$ is the density of final states.

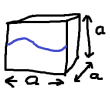
.B.2 Density of states

The density of states for a single particle within a cubic box with sides length a can be calculated as follows. The plane wave solution is of form

$$\langle \mathbf{x} | \Psi \rangle \propto \exp\left(i\mathbf{k} \cdot \mathbf{x}\right).$$

If we require periodic boundary conditions, with period a equal to the side of the box, then the values of the wavenumber k_x are constrained to $k_x = 2\pi n/a$ for integer n. Similar conditions hold for k_y and k_z. The number of momentum states within some range of momentum $d^3\mathbf{p} = d^3\mathbf{k}$ (for $\hbar = 1$) is therefore given by

$$dN = \frac{d^3\mathbf{p}}{(2\pi)^3 \hbar} \mathcal{V}$$



where $\mathcal{V} = a^3$ is the volume of the box.

.B.3 Fermi G.R. example

Consider the isotropic decay of a neutral spin-0 particle A into two massless daughters, B and C

$$A \longrightarrow B + C.$$

The Fermi G.R. gives the decay rate (in natural units) of A as

$$\Gamma = 2\pi \left|V_{fi}\right|^2 \frac{dN}{dE_f}.$$

The density of final states can be found from the allowed momenta $\mathbf{p}_B$ of particle B.

$$dN = \frac{d^3\mathbf{p}_B}{(2\pi)^3}\mathcal{V}$$

When $\mathbf{p}_B$ is fixed there is no further freedom for $\mathbf{p}_C$ since the sum of the momenta of the two final state particles is fixed by total momentum conservation. This constraint means that for the two body final state there is no additional term in the density of states for $\mathbf{p}_C$.[8] Since all decay angles are equally probable, the integrals over the angles contribute 4π, leading to

$$\Gamma = 2\pi \left|V_{fi}\right|^2 \frac{4\pi\, p_B^2}{(2\pi)^3}\frac{dp_B}{dE_f}\mathcal{V}.$$

The relativistic decay products each have momentum $|\mathbf{p}_B| = E_f/2$ so $\frac{dp_B}{dE_f} = \frac{1}{2}$. Normalising to one unstable particle in our unit volume gives $\mathcal{V} = 1$, and results in a decay rate

$$\begin{aligned}\Gamma &= \frac{1}{2\pi}\left|V_{fi}\right|^2 p_B^2 \\ &= \frac{1}{8\pi}\left|V_{fi}\right|^2 m_A^2.\end{aligned}$$

.B.4 Lifetimes and decays

The number of particles remaining at time t is governed by the decay law[9]

$$\frac{dN}{dt} = -\Gamma N,$$

where the constant Γ is the decay rate per nucleus. The equation is easily integrated to give

$$N(t) = N_0 \exp\left(-\Gamma t\right).$$

[8] For a three-body final state there would be terms in dN of the form $\frac{d^3p}{(2\pi)^3}$ for two of the three particles, the third again being fixed by momentum conservation.

[9] The decay law was discovered experimentally by Frederick Soddy (1877-1956). Soddy, who had been a scholar at Merton, was also first person to understand that radioactivity led to the transmutation of the elements — in effect making him the first true alchemist.

We can calculate the particles' average proper lifetime τ, using the probability that they decay between time t and $t + \delta t$

$$p(t)\,\delta t = -\frac{1}{N_0}\frac{dN}{dt}\,\delta t = \Gamma \exp\left(-\Gamma t\right)\,\delta t.$$

The mean lifetime is then

$$\begin{aligned}\tau &= \langle t \rangle \\ &= \frac{\int_0^\infty t\,p(t)\,dt}{\int_0^\infty p(t)\,dt} \\ &= \frac{1}{\Gamma}\end{aligned}$$

The decay law can be justified from precise experimental verification. In essence it represents a statement that the decay rate is independant of the history of the nucleus, its method of preparation and its environment. These are often excellent approximations, provided that the nucleus lives long enough that has mass $m \gg \Gamma$ where Γ is its decay width, and provided it it not bombarded with disruptive probes, such as high-energy strongly interacting particles.

.B.5 The flux factor

When calculating a cross section σ from a rate Γ, we need to take into account that for scattering from a single fixed target

$$\sigma = \frac{W}{\mathcal{J}}$$

where $\mathcal{J}$ is the flux density of incoming particles. The flux density is itself given by

$$\mathcal{J} = n_p v$$

where n_p is the number density of projectiles and v is their speed. If we normalise to one incoming particle per unit volume, then $n_p = 1$ and the cross section is simply related to the rate by

$$\sigma = \frac{W}{v}$$

.B.6 Luminosity

In a **collider** — a machine which collides opposing beams of particles — the rate of any particular reaction will be proportional to the cross section for that reaction and on various other parameters which depend on the machine set-up. Those parameters will include the number of particles in each collding bunch, their spatial distributions, and their frequency of bunch crossings.

We can define a parameter called the **luminosity** $\mathcal{L}$ which encapsulates all the relevant machine parameters. It is related to the rate W and the cross section σ by

$$\mathcal{L} = \frac{W}{\sigma}.$$

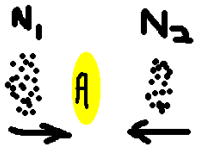
For any collider the luminosity tells us the instantaneous rate of reaction for any cross section. The product of the time-integrated luminosity and the cross section tell us the expected count of the events of that type

$$N_{\text{events, i}} = \sigma_i \int \mathcal{L}\, dt \ .$$

For a machine colliding trains of counter-rotating bunches containing N_1 and N_2 particles respectively at a bunch-crossing rate f, we can show that the luminosity is

$$\boxed{\mathcal{L} = \frac{N_1 N_2 f}{A},}$$

where A is the cross-sectional area of each bunch (perpendicular to the beam direction).



We have assumed above that the distributions of particles within each bunch is uniform. If that is not the case (e.g. in most real experiments the beams have approximately Gaussian profiles) then we will have to calculate the effective overlap area A of the bunches by performing an appropriate integral.

.C Shell Model §

Non examinable

The SEMF provides a reasonable description of the binding energies of the nuclei for $A > 30$ but only the overall structure, not the finer details.

Differences at small A (e.g. the tightly bound isotopes ${}^4_2\mathrm{He}$ and ${}^{16}_8\mathrm{O}$) are already obvious in Figure 1. Figure 6 shows in more detail the difference between the measured binding energy (per nucleon) and the SEMF prediction. Islands of particularly high stability — that is with anomalously large B/A — are clearly visible near some special values of N or Z:

$$\{2, 8, 20, 28, 50, 82, 126\}.$$

These are known as the **magic numbers**. They correspond to configurations of nuclear shells that are precisely filled with either protons or neutrons. Evidence for this shell structure can be found in the binding energies, excitation energies, abundances, spins, and magnetic moments. Some nuclei, such the Helium nucleus ${}^4_2\mathrm{He}$ have magic numbers both for N and for Z. This observation goes some way to explaining why it is that Helium nuclei are emitted by heavy particles in the process of alpha decay. The shell model gives further insight into a variety of nuclear properties, but is beyond the scope of this course.

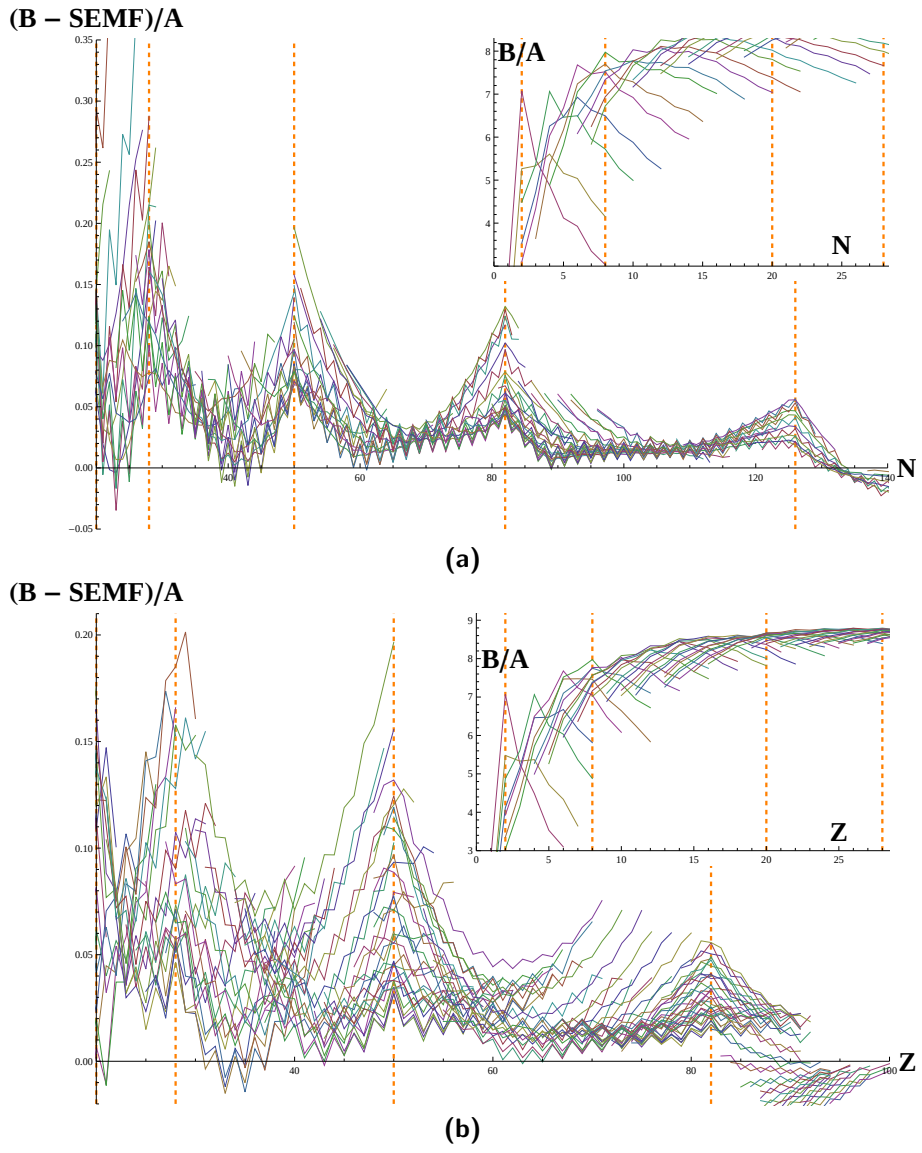


Figure 6: Difference between the measured binding energy (per nucleon) and the SEMF prediction. **(a)** The x-axis shows the number of neutrons in the nucleus; curves show isoto**p**es (same Z). **(b)** The x-axis shows the number of protons in the nucleus; curves show isoto**n**es (same N). In both cases the inset shows the binding energy per nucleon for the low-A nuclei. The magic numbers $\{2, 8, 20, 28, 50, 82, 126\}$ are marked with dashed lines.

.D Gamma decays §

Non examinable

Gamma decays are electromagnetic transitions, and are found when excited nuclear states relax to their ground states.

Similarly to the beta decay case, one can work out the rate using the Fermi golden rule. If one represents the initial nuclear wave-function by Ψ_a and the final nuclear wave-function by Ψ_b, then the appropriate matrix element is found to be

$$\langle \Psi_f | M | \Psi_i \rangle = \int d^3x \, \Psi_b^* (\mathsf{A} \cdot \hat{\mathsf{J}}) \, e^{-i\mathbf{k}\cdot\mathbf{x}} \Psi_a$$

where A represents the electromagnetic 4-potential and $\hat{\mathsf{J}} = q\hat{\mathsf{P}}/m$ is the electric 4-current operator. The electromagnetic selection rules and transitions are analogous to those of atomic physics.

Further Reading

- *"An Introduction to Nuclear Physics"*, W. N. Cottingham and D. A. Greenwood, 2001 for the basics
- *"Nuclear Physics"*, M.G. Bowler, Pergamon press, 1973
- *"Introductory Nuclear Physics"*, P.E. Hodgeson, E. Gadioli and E. Gadioli Erba, OUP, 2003
- The BNL table of the nuclides provides good reference data http://www.nndc.bnl.gov/nudat2/.

Bolwer and Hodgeson et. al. are good books which go well beyond this course.

References

[1] G. Audi, A. H. Wapstra, and C. Thibault. The 2003 atomic mass evaluation: (ii). tables, graphs and references. *Nuclear Physics A*, 729(1):337 – 676, 2003. The 2003 NUBASE and Atomic Mass Evaluations.

