Non-relativistic scattering

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1 Scattering theory

We are interested in a theory that can describe the scattering of a particle from a potential $V(\mathbf{x})$. Our Hamiltonian is

$$H = H_0 + V.$$

where H_0 is the free-particle kinetic energy operator

$$H_0 = \frac{p^2}{2m}.$$

In the absence of the potential V the solutions of the Hamiltonian could be written as the free-particle states satisfying

$$H_0|\phi\rangle = E|\phi\rangle.$$

These free-particle eigenstates could be written as momentum eigenstates $|\mathbf{p}\rangle$, but since that isn't the only possibility we hold off writing an explicit form for $|\phi\rangle$ for now. The full Schrödinger equation is

$$(H_0 + V)|\psi\rangle = E|\psi\rangle.$$

We define the eigenstates of H such that in the limit where the potential disappears ($V \to 0$), we have $|\psi\rangle \to |\phi\rangle$, where $|\phi\rangle$ and $|\psi\rangle$ are states with the same energy eigenvalue. (We are able to do this since the spectra of both H and $H + V$ are continuous.)

A possible solution is[1]

$$|\psi\rangle = \frac{1}{E - H_0} V|\psi\rangle + |\phi\rangle. \quad (1)$$

By multiplying by $(E - H_0)$ we can show that this agrees with the definitions above. There is, however the problem of the operator $1/(E - H_0)$ being singular. The singular behaviour in (1) can be fixed by making E slightly complex and defining

$$\boxed{|\psi^{(\pm)}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V|\psi^{(\pm)}\rangle}. \quad (2)$$

This is the **Lippmann-Schwinger** equation. We will find the physical meeting of the $(\pm)$ in the $|\psi^{(\pm)}\rangle$ shortly.

[1] Functions of operators are defined by $f(\hat{A}) = \sum_i f(a_i)|a_i\rangle\langle a_i|$. The reciprocal of an operator is well defined provided that its eigenvalues are non-zero.

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1.1 Scattering amplitudes

To calculate scattering amplitudes we are going to have to use both the position and the momentum basis, the incoming beam is (almost) a momentum eigenstate, and V is a function of position $\mathbf{x}$. If $|\phi\rangle$ stands for a plane wave with momentum $\hbar\mathbf{k}$ then the wavefunction can be written

$$\langle \mathbf{x}|\phi\rangle = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}.$$

We can express (2) in the position basis by bra-ing through with $\langle\mathbf{x}|$ and inserting the identity operator $\int d^3x'\, |\mathbf{x}'\rangle\langle\mathbf{x}'|$

$$\langle \mathbf{x}|\psi^{(\pm)}\rangle = \langle \mathbf{x}|\phi\rangle + \int d^3x' \left\langle \mathbf{x}\middle|\frac{1}{E - H_0 \pm i\epsilon}\middle|\mathbf{x}'\right\rangle \langle \mathbf{x}'|V|\psi^{(\pm)}\rangle. \qquad (3)$$

The solution to the Greens function defined by

$$G_{\pm}(\mathbf{x},\mathbf{x}') \equiv \frac{\hbar^2}{2m}\left\langle \mathbf{x}\middle|\frac{1}{E - H_0 \pm i\epsilon}\middle|\mathbf{x}'\right\rangle$$

is

$$G_{\pm}(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}.$$

Using this result we can see that the amplitude of interest simplifies to

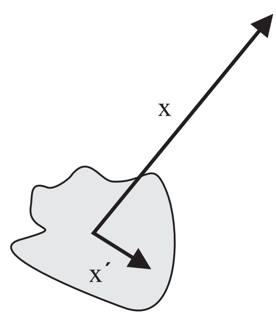
$$\langle \mathbf{x}|\psi^{(\pm)}\rangle = \langle \mathbf{x}|\phi\rangle - \frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x' \frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}')\langle \mathbf{x}'|\psi^{(\pm)}\rangle \qquad (4)$$

where we have also assumed that the potential is local in the sense that it can be written as

$$\langle \mathbf{x}'|V|\mathbf{x}''\rangle = V(\mathbf{x}')\delta^3(\mathbf{x}' - \mathbf{x}'').$$

The wave function (4) is a sum of two terms. The first is the incoming plane wave. For large $r = |\mathbf{x}|$ the spatial dependence of the second term is $e^{\pm ikr}/r$. We can now understand the physical meaning of the $|\psi^{(\pm)}\rangle$ states; they represent outgoing $(+)$ and incoming $(-)$ spherical waves respectively. We are interested in the outgoing $(+)$ spherical waves – the ones which have been scattered from the potential.

We want to know the amplitude of the outgoing wave at a point $\mathbf{x}$. For practical experiments the detector must be far from the scattering centre, so we may assume $|\mathbf{x}| \gg |\mathbf{x}'|$.



We define a unit vector $\hat{\mathbf{r}}$ in the direction of the observation point

$$\hat{\mathbf{r}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

and also a wave-vector $\mathbf{k}'$ for particles travelling in the direction $\hat{\mathbf{x}}$ of the observer,

$$\mathbf{k}' = k\hat{\mathbf{r}}.$$

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Far from the scattering centre we can write

$$\begin{aligned} |\mathbf{x}-\mathbf{x}'| &= \sqrt{r^2 - 2rr'\cos\alpha + r'^2} \\ &= r\sqrt{1 - 2\frac{r'}{r}\cos\alpha + \frac{r'^2}{r^2}} \\ &\approx r - \hat{\mathbf{r}}\cdot\mathbf{x}' \end{aligned}$$

where α is the angle between the $\mathbf{x}$ and the $\mathbf{x}'$ directions.

It's safe to replace the $|\mathbf{x}-\mathbf{x}'|$ in the denominator in the integrand of (4) with just r, but the phase term will need to be replaced by $r - \hat{\mathbf{r}}\cdot\mathbf{x}'$. So we can simplify the wave function to

$$\langle\mathbf{x}|\psi^{(+)}\rangle \xrightarrow{r \text{ large}} \langle\mathbf{x}|\mathbf{k}\rangle - \frac{1}{4\pi}\frac{2m}{\hbar^2}\frac{e^{ikr}}{r}\int d^3x'\, e^{-i\mathbf{k}'\cdot\mathbf{x}'}V(\mathbf{x}')\langle\mathbf{x}'|\psi^{(+)}\rangle$$

which we can write as

$$\langle\mathbf{x}|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}\left[e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{r}f(\mathbf{k}',\mathbf{k})\right].$$

This makes it clear that we have a sum of an incoming plane wave and an outgoing spherical wave with amplitude $f(\mathbf{k}',\mathbf{k})$ given by

$$f(\mathbf{k}',\mathbf{k}) = -\frac{1}{4\pi}(2\pi)^3\frac{2m}{\hbar^2}\langle\mathbf{k}'|V|\psi^{(\pm)}\rangle. \tag{5}$$

We will ignore the interference between the first term which represents the original 'plane' wave and the second term which represents the outgoing 'scattered' wave, which is equivalent to assuming that the incoming beam of particles is only approximately a plane wave over a region of dimension much smaller than r.

We then find that the partial cross-section $d\sigma$ — the number of particles scattered into a particular region of solid angle per unit time divided by the incident flux[2] — is given by

$$d\sigma = \frac{r^2|j_{\text{scatt}}|}{|j_{\text{incid}}|}d\Omega = |f(\mathbf{k}',\mathbf{k})|^2\, d\Omega.$$

This means that the differential cross section is given by the simple result

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\mathbf{k}',\mathbf{k})|^2.}$$

The differential cross section is simply the mod-squared value of the scattering amplitude.

[2] Remember that the flux density is given by $\mathbf{j} = \frac{\hbar}{2im}[\psi^*\nabla\psi - \psi\nabla\psi^*]$.

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1.2 The Born approximation

If the potential is weak we can assume that the eigenstates are only slightly modified by V, and so we can replace $|\psi^{(\pm)}\rangle$ in (5) by $|\mathbf{k}\rangle$.

$$f^{(1)}(\mathbf{k}',\mathbf{k}) = -\frac{1}{4\pi}(2\pi)^3\frac{2m}{\hbar^2}\langle\mathbf{k}'|V|\mathbf{k}\rangle. \quad (6)$$

This is known as the **Born approximation**. Within this approximation we have the simple result that

$$\boxed{f^{(1)}(\mathbf{k}',\mathbf{k}) \propto \langle\mathbf{k}'|V|\mathbf{k}\rangle.}$$

Up to some constant factors, the scattering amplitude is found by squeezing the perturbing potential V between incoming and the outgoing momentum eigenstates of the free-particle Hamiltonian.

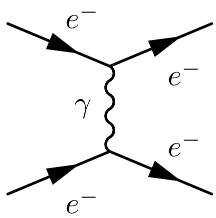
Expanding out (6) in the position representation (by insertion of a couple of completeness relations $\int d^3x'\,|\mathbf{x}'\rangle\langle\mathbf{x}'|$) we can write

$$f^{(1)}(\mathbf{k}',\mathbf{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x' e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}V(\mathbf{x}').$$

This result is telling us that scattering amplitude is proportional to the **3d Fourier transform** of the potential. By scattering particles from targets we can measure $\frac{d\sigma}{d\Omega}$, and hence infer the functional form of $V(r)$. This result is used, for example, in the nuclear form factor (Section **??**).

2 Virtual Particles

One of the insights of subatomic physics is that at the microscopic level forces are caused by the exchange of force-carrying **particles**. For example the Coulomb force between two electrons is mediated by excitations of the electromagnetic field – i.e. photons. There is no real 'action at a distance'. Instead the force is transmitted between the two scattering particles by the exchange of some unobserved photon or photons. The mediating photons are emitted by one electron and absorbed by the other. It's generally not possible to tell which electron emitted and which absorbed the mediating photons – all one can observe is the net effect on the electrons.



Other forces are mediated by other force-carrying particles. In each case the messenger particles are known as **virtual particles**. Virtual particles are not directly observed, and have properties different from 'real particles' which are free to propagate.

To illustrate why virtual particles have unusual properties, consider the elastic scattering of an electron from a nucleus, mediated by a single virtual photon. We can assume the nucleus to be much more massive than the electron so that it is approximately stationary. Let the incoming electron have momentum $\mathbf{p}$ and the outgoing, scattered electron have momentum $\mathbf{p}'$. For elastic scattering, the energy of the

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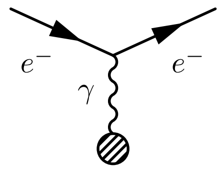
electron is unchanged $E' = E$. The electron has picked up a change of momentum $\Delta\mathbf{p} = \mathbf{p}' - \mathbf{p}$ from absorbing the virtual photon, but absorbed no energy. So the photon must have energy and momentum



$$\begin{aligned} E_\gamma &= 0 \\ \mathbf{p}_\gamma &= \Delta\mathbf{p} = \mathbf{p}' - \mathbf{p}. \end{aligned}$$

The exchanged photon carries momentum, but no energy. This sounds odd, but is nevertheless correct. What we have found is that for this *virtual* photon, $E_\gamma^2 \neq p_\gamma^2$. The particular value $E_\gamma = 0$ is special to the case we have chosen, but the general result is that for *any virtual particle* there is an energy-momentum invariant[3] which is *not* equal to the square of its mass

$$\mathsf{P} \cdot \mathsf{P} = E^2 - \mathbf{p} \cdot \mathbf{p} \neq m^2.$$

Such virtual particles do *not* satisfy the usual energy-momentum invariant and are said to be '**off mass shell**'.

Note that we would not have been able to escape this conclusion if we had taken the alternative viewpoint that the electron had emitted the photon and the nucleus had absorbed it. In that case the photon's momentum would have been $\mathbf{p}_\gamma = -\Delta\mathbf{p}$. The square of the momentum would be the same, and the photon's energy would still have been zero.

These exchanged, virtual, photons are an equally valid solution to the (quantum) field equations as are the more familiar travelling-wave solutions of 'real' on-mass-shell photons. It is interesting to realise that all of classical electromagnetism is actually the result of very many photons being exchanged.

3 The Yukawa Potential

There is a type of potential that is of particular importance in subatomic scattering, which has the form (in natural units)

$$\boxed{V(r) = \frac{g^2}{4\pi} \frac{e^{-\mu r}}{r}.} \tag{7}$$

This is known as the **Yukawa potential**. The constant g^2 tells us about the depth of the potential, or the size of the force. When $\mu = 0$ (7) has the familiar $1/r$ dependence of the electrostatic and gravitational potentials. When μ is non-zero, the potential also falls off exponentially with r, with a characteristic length of $1/\mu$.

To understand the meaning of the μ term it is useful to consider the relativistic wave equation known as the Klein-Gordan equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2 \right) \varphi(\mathbf{r}, t) = 0. \tag{8}$$

[3] We used sans serif capitals P to indicate Lorentz four-vectors $\mathsf{P} = (E, p_x, p_y, p_z)$. The dot product of two Lorentz vectors $\mathsf{A} \cdot \mathsf{B} = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3$ is a Lorentz invariant scalar.

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The Yukawa potential and the range of forces

The electromagnetic force is mediated by excitations of the electromagnetic force, i.e. photons. The photon is massless so the electrostatic potential falls as $1/r$. The exponential fall-off of (7) is removed since $\mu = 0$, and so electromagnetism is effective even at large distances.

By contrast, the weak nuclear force, which is mediated by particles with μ close to 100 GeV is feeble at distances larger than about $1/(100\,\mathrm{GeV}) \approx (197\,\mathrm{MeV\,fm})/(100\,\mathrm{GeV}) \sim 10^{-18}\,\mathrm{m}$. This makes it short-range in nature, and so it appears to be weak. (In fact the coupling constant g for the so-called 'weak' force is actually larger than that for the electromagnetic force.)

This is the relativistic wave equation for spin-0 particles. The plane-wave solutions to (8) are

$$\begin{aligned}\phi(\mathsf{X}) &= A\exp\left(-i\mathsf{P}\cdot X\right)\\ &= A\exp\left(-iEt + i\mathbf{p}\cdot\mathbf{x}\right).\end{aligned}$$

These solutions require the propagating particles to be of mass $\mu = \sqrt{E^2 - p^2}$. The Klein-Gordon equation is therefore describing excitations of a field of particles each of mass μ. The Yukawa potential is another solution to the field equation (8). The difference is that the Yukawa potential describes the *static* solution due to virtual particles of mass μ created by some source at the origin.

The scattering amplitude of a particle bouncing off a Yukawa potential is found to be

$$\langle\mathbf{k}'|V_{\mathrm{Yukawa}}|\mathbf{k}\rangle = -\frac{g^2}{4\pi\,(2\pi)^3}\frac{1}{\mu^2 + |\Delta\mathbf{k}|^2}. \tag{9}$$

We can go some way towards interpreting this result as the exchange of a virtual particle as follows. We justify the two factors of g as coming from the points where a virtual photon is either created or annihilated. This **vertex factor** g is a measure of the interaction or 'coupling' of the exchanged particle with the other objects. There is one factor of g the point of creation of the virtual particle, and another one at the point where it is absorbed.

The other important factor in the scattering amplitude (9) is associated with the momentum and mass of the exchanged particle:

$$-\frac{1}{\mu^2 + |\Delta\mathbf{k}|^2}$$

In general it is found that if a virtual particle of mass μ and four-momentum P is exchanged, there is a **propagator factor**

$$\boxed{\frac{1}{\mathsf{P}\cdot\mathsf{P} - \mu^2}} \tag{10}$$

in the scattering amplitude. This relativistically invariant expression is consistent

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Vertex factors in electromagnetism

In electromagnetism we require that at a vertex where a photon interacts with a particle, the vertex factor g should to be proportional to the charge of the particle $\mathcal{Q}e$. For a particle of charge $\mathcal{Q}_1 e$ scattering from a field generated by another particle of charge $\mathcal{Q}_2 e$, we seek a ($\mu = 0$) Yukawa potential of the form

$$V_{\rm EM} = \frac{(\mathcal{Q}_1 e)(\mathcal{Q}_2 e)}{4\pi\epsilon_0 r}.$$

For scattering from a Coulomb potential we can therefore use the Yukawa result (9) by making the substitution

$$\frac{g^2}{4\pi} \Rightarrow \frac{\mathcal{Q}_1\mathcal{Q}_2 e^2}{4\pi\epsilon_0}.$$

This identification shows that the vertex factors g are just dimensionless measures of the charges of the particle. The vertex factor for a charge $\mathcal{Q}e$ is $\mathcal{Q}g_{\rm EM}$ where

$$\frac{g_{\rm EM}^2}{4\pi} = \alpha_{\rm EM} \approx \frac{1}{137}.$$

with our electron-scattering example, where the denominator was:

$$\begin{aligned} \mathsf{P}\cdot\mathsf{P} - \mu^2 &= E^2 - p^2 - \mu^2 \\ &= 0 - |\Delta\mathbf{k}|^2 - \mu^2 \\ &= -\left(\mu^2 + |\Delta\mathbf{k}|^2\right) \end{aligned}$$

Note that the propagator (13) becomes singular as the particle gets close to its mass shell. i.e. as $\mathsf{P}\cdot\mathsf{P} \to \mu^2$. It is only because the exchanged particles are *off* their mass-shells that the result is finite.

The identification of the vertex factors and propagators will turn out to be very useful when we later try to construct more complicated scattering processes. In those cases we will be able to construct the most important features of the scattering amplitude by writing down:

- an appropriate vertex factor each time a particle is either created or annihilated and
- a propagator factor for each virtual particle.

By multiplying together these factors we get the scattering amplitude.

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Key concepts

- The amplitude for scattering from a potential can be solved iteratively, using the **Lippman-Schwinger** equation:

$$\boxed{|\psi^{(\pm)}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |\psi^{(\pm)}\rangle}$$

- The leading **Born approximation** to the scattering amplitude is

$$\boxed{f^{(1)}(\mathbf{k}', \mathbf{k}) \propto \langle \mathbf{k}' | V | \mathbf{k} \rangle}$$

- The differential cross-section is given in terms of the scattering amplitude by

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\mathbf{k}', \mathbf{k})|^2}$$

- Forces are transmitted by **virtual** mediating particles which are off-mass-shell:

$$\mathsf{P} \cdot \mathsf{P} = E^2 - \mathbf{p} \cdot \mathbf{p} \neq m^2$$

- The **Yukawa potential** for an exchanged particle of mass μ and coupling g is

$$\boxed{V(r) = \frac{g^2}{4\pi} \frac{e^{-\mu r}}{r}} \qquad (11)$$

- The scattering amplitude contains a **vertex factors** g for any point where particles are created or annihilated

- The relativistic **propagator** factor is

$$\boxed{\frac{1}{\mathsf{P} \cdot \mathsf{P} - \mu^2}}$$

for each virtual particle.

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.A Beyond Born: non-relativistic propagators §

Non examinable

To see how things develop if we don't want to rashly assume that $|\psi^{\pm}\rangle \approx |\phi\rangle$ it is useful to define a **transition operator** T such that

$$V|\psi^{(+)}\rangle = T|\phi\rangle$$

Multiplying the Lippmann-Schwinger equation (2) by V we get an expression for T

$$T|\phi\rangle = V|\phi\rangle + V\frac{1}{E - H_0 + i\epsilon}T|\phi\rangle.$$

Since this is to be true for any $|\phi\rangle$, the corresponding operator equation must also be true:

$$T = V + V\frac{1}{E - H_0 + i\epsilon}T.$$

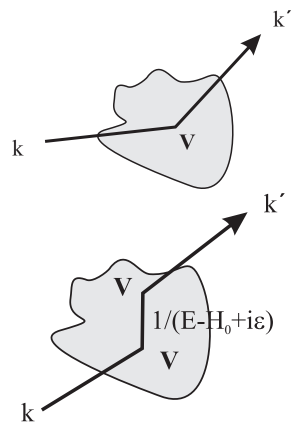
This operator is defined recursively. It is exactly what we need to find the scattering amplitude, since from (5), the amplitude is given by

$$f(\mathbf{k}', \mathbf{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\langle\mathbf{k}'|T|\mathbf{k}\rangle.$$

We can now find an iterative solution for T:

$$T = V + V\frac{1}{E - H_0 + i\epsilon}V + V\frac{1}{E - H_0 + i\epsilon}V\frac{1}{E - H_0 + i\epsilon}V + \dots \tag{12}$$

We can interpret this series of terms as a sequence of the operators corresponding to the particle interacting with the potential (operated on by V) and propagating along for some distance (evolving as it goes according to $\frac{1}{E-H_0+i\epsilon}$).



The operator

$$\boxed{\frac{1}{E - H_0 + i\epsilon}} \tag{13}$$

is the non-relativistic **propagator**. Propagators are central to much of what we will do later on, so it is a good idea to try to work out what they mean. Physically the propagator can be thought of as a term in the expansion (12) which is giving a contribution the amplitude for a particle moving from an interaction at point A to another at point B. Mathematically it is a Greens function solution to the Lippmann-Schwinger equation in the position representation (3).

We are now in a position to quantify what we meant by a 'weak' potential earlier on. From the expansion (12) we can see that the first Born approximation (6) will be useful if the matrix elements of T can be well approximated by its first term V.

When is this condition likely to hold? Remember that the Yukawa potential was proportional to the square of a dimensionless coupling constant $\propto g^2$. If $g^2 \ll 1$ then successive applications of V introducing higher and higher powers of g and can

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usually be neglected. This will be true for electromagnetism, since the dimensionless coupling relevant for electromagnetism is related to the fine structure constant

$$\frac{g^2}{4\pi} = \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

Since $\alpha \ll 1$, we can usually get away with just the first term of (12) for electric interactions (i.e. we can use the Born approximation).

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