<Propagators Green's functions & Scattering>
Batch IX

A typical High Energy physics experiment consists of a beam of particles incident on either a target or another beam, the beam & target or the beam + beam interact in the interaction region, and scatter.

Far from the interaction region we have detection apparatuses that measure the 4-momenta etc of these particles.

The angular distribution + numbers of the scattered particles, and their identities help us understand the nature of fundamental interactions.

To develop a formalism to calculate the angular distributions, number and kind of particles produced + scattered we make a few assumptions.

(i) That the potential responsible for the interactions is very weak compared to the energies of these particles.

(ii) That the time during which the interaction takes place and the region over which the interactions take place are tiny compared to the time to and distance to the point where the scattered particles are detected.

Lets assume that the wave function for a particular particle is known at point x and time t:

$\psi(x,t)$

According to Huygens principle, each point x acts as a source for the wave function at another point x' at a time t'.

This "constant" of proportionality is actually a function. We express the relation between ψ at (x, t) and ψ at (x', t') by the equation:

$$\psi(x', t') = i\int d^3x \, G(x', t'; x\, t)\, \psi(x, t) \quad -(1)$$
$$t' > t$$

Thus:

(i) The wave function of a particle is known at (x, t) i.e $\psi(x, t)$.

(ii) Every point on $\psi(x, t)$ contributes to $\psi(x', t')$. The amplitudes are proportional and are related by $G(x', t'; x, t)$, but since every point on ψ contributes (at x, t) we must integrate over all x.

(iii) $G(x', t'; x, t)$ depends on the nature of the wave equation, there are many such wave equations; the Schrödinger the Klein-Gordon, and the Dirac equations. This is known as the "Green's function" or propagator....

Now lets introduce an interaction potential:

We first write down the free (in the absence of potential) wave function:

$\phi(x, t)$

Now assume that a very weak potential V is switched on very briefly at time t_1 at location x_1 for a very brief interval Δt_1. Assume now that the change in the wave function is $\Delta\psi(x_1, t_1)$

NB: I have set $\hbar = c = 1$ for this and all further lectures.

Thus we have:

$$\psi(x_1, t_1) = \phi(x_1, t_1) + \Delta\psi(x_1, t_1) \quad \text{—— (2)}$$

"complete" wave function solution to $H = H_0 + V$

free particle wave function evaluated at x_1, t_1, solution to H_0

change in ψ due to presence of $V \rightarrow$ Remember V only switches on at t_1, very small

We can write:

$$i\frac{\partial}{\partial t_1}\psi(x_1, t_1) = \hat{H}_0\psi(x_1,t_1) + V\psi(x_1, t_1)$$

H_0 is the free particle Hamiltonian

We rewrite this and remind ourselves that $V = V(x, t_1)$

$$i\frac{\partial}{\partial t_1}\psi(x_1, t_1) - \hat{H}_0\psi(x_1, t_1) = V(x, t_1)\psi(x_1, t_1)$$

Using equation (ii) above:

$$\left(i\frac{\partial}{\partial t_1} - \hat{H}_0\right)\left[\phi(x_1,t_1) + \Delta\psi(x_1, t_1)\right] = V(x_1, t_1)\left[\phi(x_1,t_1) + \Delta\psi(x_1,$$

Since $\phi(x_1, t_1)$ is the free particle solution

$$\left(i\frac{\partial}{\partial t_1} - \hat{H}_0\right)\phi(x_1, t_1) = 0$$

and we have:

$$i\frac{\partial \Delta\psi(x_1,t_1)}{\partial t_1} - \hat{H}_0\,\Delta\psi(x_1, t_1) = V(x_1, t_1)\,\phi(x_1, t_1) + V(x_1, t_1) \times \Delta\psi(x,$$

Since both V and $\Delta\psi$ are small quantities we can write:

$$i\frac{\partial\Delta\psi(x, t_1)}{\partial t_1} - \hat{H}_0\,\Delta\psi(x_1, t_1) = V(x_1, t_1)\phi(x_1, t_1)$$

Integrating to first order in t and recalling that $\Delta\psi$ & Δt_1 are tiny:

$$i\,\Delta\psi(x_1, t_1) = V(x_1, t_1)\phi(x_1, t_1)\,\Delta t_1$$

$$\Delta\psi(x_1, t_1) = -iV(x_1, t_1)\phi(x_1, t_1)\Delta t_1 \quad \text{—— (3)}$$

So the interaction was switched at t_1 in the location x_1 for Δt_1, now what does this newly perturbed wave look like at point x', t'?

Lets look at equation (1) page 2

$$\psi(x', t') = i\int d^3x\, G(x', t'; x, t)\, \psi(x, t) \quad — (1).$$

$\psi(x, t)$ is in fact $\psi(x_1, t_1) = \phi(x_1, t_1) + \Delta\psi(x_1, t_1)$

$$\psi(x', t') = i\int d^3x_1\, G(x', t'; x_1, t_1)\left[\phi(x_1, t_1) + \Delta\psi(x_1, t_1)\right]$$

$$\psi(x', t') = i\int d^3x_1\, G(x', t'; x_1, t_1)\,\phi(x_1, t_1)$$
$$+ i\int d^3x_1\, G(x', t'; x_1, t_1)\,(-i)\,V(x_1, t_1)\,\phi(x_1, t_1) \times \Delta t_1$$

Now by definition of G

$$i\int d^3x_1\, G(x', t'; x_1, t_1)\,\phi(x_1, t_1) = \phi(x', t')$$

it's simply the piece of the wave that would have propagated there if no interaction had taken place...

And finally:

$$\boxed{\psi(x', t') = \phi(x', t') + \underbrace{\int d^3x_1\, \Delta t_1\, G_0(x', t'; x_1, t_1)\, V(x_1, t_1) \times \phi(x_1, t_1)}_{(4)}.}$$

Thus $\psi(x', t')$ i.e the wave function at x', t' when an interaction V is switched on at x_1, t_1 for time Δt_1 is:

(i) The free particle wave function ϕ evaluated at x', t'

PLUS

(ii) $V \times \phi(x_1, t_1) \times G(x', t'; x_1, t_1)\, \Delta t_1$ integrated over d^3x_1

We can also identify $\Delta\psi(x', t')$ by writing $\psi(x', t')$ as $\phi(x', t') + \Delta\psi(x', t')$

$\therefore$ using (4)

$$\Delta\psi(x', t') = \int d^3x_1\, g_0(x', t'; x_1 t_1)\, \phi(x_1, t_1) V(x_1, t_1) \Delta t_1$$

A brief stop to review concepts + notation

(a) $g_0(x', t'; x_1, t_1)$ is a <u>free particle</u> propagator i.e its simply a function relating free wave solutions at various points.
We will determine its <u>form</u> later.

(b) Note the semi-colon ";" between the initial and final points in g_0..

Now let's examine (4) again

$$\psi(x', t') = \phi(x', t') + \int d^3x_1\, \Delta t_1\, g_0(x', t'; x_1, t_1) V(x, t_1) \times \phi(x_1, t_1)$$

Consider now the wave function <u>before</u> its interaction at x_1, t_1 ; ie at x, t

$\psi(x', t') =$

$$i\int d^3x\, g_0(x', t'; x, t)\, \phi(x, t)$$

$$+ i\int d^3x \int d^3x_1\, \Delta t_1\, g_0(x', t'; x_1, t_1) V(x_1, t_1)\, g_0(x_1 t_1; x, t) \times \phi(x, t) \times (-i)$$

$$\therefore \psi(x', t') = i\int d^3x \Big\{ g_0(x', t'; x, t) + \int d^3x_1\, \Delta t_1\, g_0(x', t'; x_1 t_1) \cdot V(x, t_1)\, g_0(x_1 t_1; x, t) \Big\} \phi(x, t)$$

By the definition of the Green's function or propagator we can write down the Green function for the case where the particle is not "free" or is subject to the presence of a potential at some time in the past therefore:

$$\mathcal{G}(x',t';x,t) = g_0(x',t';x,t) + \int d^3x_1\,\Delta t_1\, G_0(x',t';x_1,t_1) \times V(x_1,t_1)\, G_0(x_1,t_1;x,$$

$\mathcal{G}$ in the presence of a V in the past (not free!!)

free particle propagator

two free particle propagators × V

Now consider the switching on of a potential V (same) at t_2 and x_2 for Δt_2, with $t_2 > t_1$ and $t_2 < t'$.

$$\Delta\psi(x',t') = i\int d^3x_2\, g_0(x',t';x_2,t_2)\, V(x_2 t_2)(-i)\,\psi(x_2 t_2)\Delta$$

$$\Delta\psi(x',t') = \int d^3x_2\, g_0(x',t';x_2,t_2) V(x_2,t_2)\psi(x_2,t_2)\Delta t_2$$

But $\psi(x_2,t_2)$ is a combination of a free wave arriving at x_2 + a wave scattered at x_1; since this is an "already scattered" wave we didn't use ϕ to denote it.

$\therefore$

$$\psi(x_2 t_2) = \phi(x_2 t_2) + \int d^3x_1\, \mathcal{G}(x_2,t_2;x_1,t_1)\, V(x_1,t_1)\phi(x_1,t_1)\Delta t_1$$

$$\therefore\ \Delta\psi(x',t') = \int d^3x_2\, g_0(x',t';x_2,t_2) V(x_2,t_2)\Delta t_2 \Big\{\phi(x_2,t_2) + \int d^3x_1\, \mathcal{G}(x_2,t_2;x_1,t_1)\phi(x_1,t_1)\Delta t_1\Big\}$$

and so the "complete" wave at x',t' is:

$$\psi(x',t') = \phi(x',t') + \int d^3x_2\, g_0(x',t';x_2 t_2) V(x_2 t_2)\Delta t_2 + \int d^3x_2\int d^3x_1\, g_0(x',t';x_2,t_2)V(x_2,t_2) \times \mathcal{G}(x_2,t_2;x_1,t_1)V(x_1,t_1)\phi(x_1,t_1) \times\Delta t_1 \times \Delta t_2$$

We can now write down the $\psi(x',t')$ for a series of scatterings each for a small time interval Δt_N

$$\psi(x',t') = \phi(x',t') + \sum_{\substack{i=1\\(t_i<t')}}^{N}\int d^3x_i\,\Delta t_i\, g_0(x',t';x_i,t_i)V(x_i,t_i) \times\phi(x_i,t_i)$$

$$+\sum_{\substack{i,j\\t_j<t_i\\t_i<t'}}\int d^3x_i\int d^3x_j\,\Delta t_i\Delta t_N\, g_0(x',t';x_j,t_j)V(x_j,t_j)\,\mathcal{G}(x_j,t_j;x_i,t_i) \times\phi(x_i,t_i)V(x_i,t_i) + \sum_i\sum_j\sum_k(\text{etc})\ldots$$

We now switch to a continuos integral over each time, rather than a sum over discrete points, so $(x', t') \to (x')$ and $(x_N, t_N) \to (x_N)$

$$\psi(x') = \phi(x') + \int d^4x_1\, G_0(x'; x_1) V(x_1) \phi(x_1)$$
$$+ \int d^4x_1 \int d^4x_2\, G_0(x'; x_2) V(x_2) G(x_2; x_1) V(x_1) \phi(x_1)$$
$$+ \int d^4x_1 \int d^4x_2 \int d^4x_3\, G(x'; x_3) V(x_3) G(x_3; x_2) V(x_2)$$
$$\times G(x_2; x_1) \phi(x_1) + \text{etc} \ldots$$

We can now rewrite $\psi(x')$ as:

$$\psi(x') = i \int d^4x \Big\{ G_0(x'; x) + \int d^4x_1\, G_0(x'; x_1) V(x_1) G(x_1; x)$$
$$+ \int d^4x_1 \int d^4x_2\, G_0(x'; x_2) V(x_2) G_0(x_2'; x_1) V(x_1)$$
$$+ \cdots \Big\} \phi(x)$$

We expect that this series converges due to the smallness of V.

Since $\psi(x') = i \int d^4x\, G(x'; x) \phi(x)$
we can write $G(x'; x)$, the propagator in the presence of a perturbation V as:

$$G(x'; x) = G_0(x'; x) + \int d^4x_1\, G_0(x'; x_1) V(x_1) G_0(x_1; x)$$
$$+ \int d^4x_1 \int d^4x_2\, G_0(x'; x_2) V(x_2) G_0(x_2; x_1) V(x_1) G_0(x_1; x)$$
$$+ \int d^4x_1 \int d^4x_2 \int d^4x_3\, G_0(x'; x_3) V(x_3) G_0(x_3; x_2) V(x_2) G(x_2; x_1) V(x_1)$$
$$\times G_0(x_1; x) + \text{higher order} \ldots \quad (5).$$

One more time $G(x'; x)$ is the propagator or Greens function in the presence of a potential

$G_0(x'; x)$ is the free particle propagator.

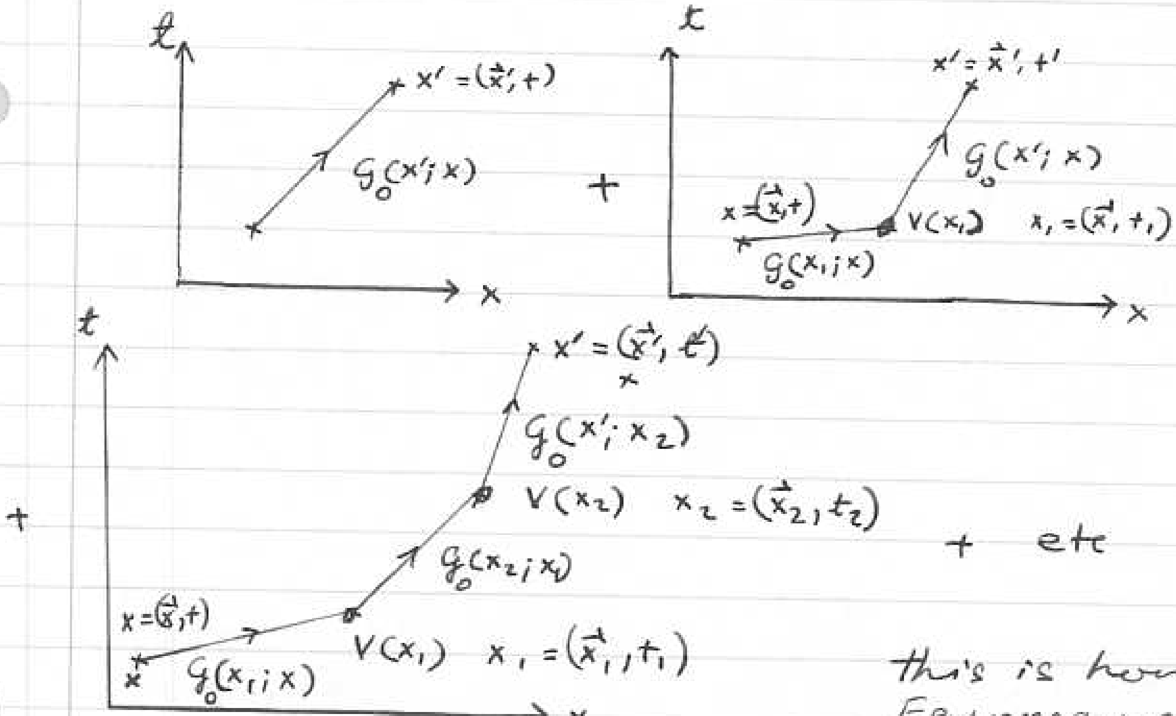
We can examine equation (5) on the previous page again; note that since the series is infinite I can insert a bracket just before the first $G_0(x_N;x)$ an recover a $G(x_N;x)$ like so:

$$G(x';x) = G_0(x';x) + \int d^4x_1\, G_0(x';x_1)V(x_1)\Big\{G_0(x_1;x)$$
$$+ \int d^4x_2\, G(x_2;x_1)V(x_2)G(x_1;x)$$
$$+ \int d^4x_2 \int d^4x_3\, G(x_3;x_2)V(x_2)G(x_2;x_1)V(x_1)G_0(x_1;x)$$
$$+ \ldots \text{ etc} \Big\}$$

within the curly brackets we simply have $G(x_1;x)$

$$\therefore G(x';x) = G_0(x';x) + \int d^4x_1\, \underset{\text{free}}{G_0(x';x_1)}\, V(x_1)\, \underset{\substack{\text{presence}\\ \text{of potential}}}{G(x_1;x)}$$

Note that from equation (5) we can represent $G(x';x)$ diagrammatically as:



this is how Feynmann graphs are born.

Enough of the formalism, lets now see how we can find out what these G_0's actually are.

Now, recall, we had assumed that the perturbation would only be switched on at time $t' > t$.

Recall the definition of the θ function

$$\theta(t'-t) = 0 \quad \text{for} \quad t > t'$$
$$= 1 \quad \text{for} \quad t' > t$$

clearly what we want is:

$$\theta(t'-t)\,\psi(x') = i\int d^3x\, G(x';x)\,\psi(x) \quad — (6)$$

A representation of the θ function is

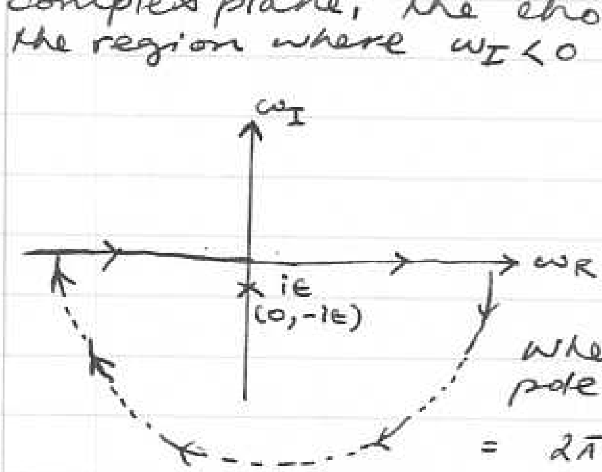
$$\underset{\tau = t'-t}{\theta(t'-t)} = \theta(\tau) = \lim_{\epsilon\to 0} -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega + i\epsilon}\, d\omega$$

This is a contour integral: lets examine it a little:

If $t'-t > 0$ and ω is complex

$$\lim_{\epsilon\to 0} -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i(\omega_R + i\omega_I)\tau}\, d\omega}{\omega + i\epsilon} = \lim_{\epsilon\to 0} -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\omega_R}\, e^{\omega_I \tau}}{\omega + i\epsilon}\, d\omega$$

Note that if $\tau > 0$, then the exponential is damped only if we are in the lower half complex plane, the choice of contour is thus to the region where $\omega_I < 0$:



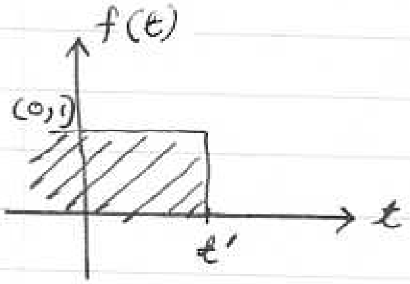
Note: there are no poles in the upper half plane so if $\tau < 0$ the integral is only defined in the upper plane $\therefore$ its simply $= 0$.

When $\tau > 0$ we pick up the pole in the lower half plane

$$= 2\pi i\,(-1)(-1)\frac{1}{2\pi i}\, e^0 = 1 \rightarrow \text{Satisfying the condition of the } \theta \text{ function.}$$

contour is traversed clockwise!

Note that the θ function (by its very definition) looks like:



its derivative with respect to t is infinite and peaked at $t'=t$ its actually simply a dirac delta function.

$$\therefore \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega+i\epsilon}\, d\omega = \theta(\tau)$$

$$\frac{d\theta(\tau)}{dt'} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\tau} = \delta(\tau)$$

Now lets try and find a real G:
recall $\psi(x')$ satisfies the equation:

$$\left\{\frac{i\partial}{\partial t'} - \hat{H}(x')\right\}\psi(x') = 0$$

now put in $\theta(t'-t)\psi(x')$ in place of

$$\left\{\frac{i\partial}{\partial t'} - \hat{H}(x')\right\}\theta(t'-t)\psi(x') = \underbrace{\left\{\frac{i\partial}{\partial t'} - \hat{H}(x')\right\}\psi(x')}_{=0}\times\theta(t'-t) + \frac{i\partial}{\partial t'}\theta(t'-t)\psi(x')$$

$$\left\{\frac{i\partial}{\partial t'} - \hat{H}(x')\right\}\times\theta(t'-t)\psi(x') = i\delta(t'-t)\psi(x')$$

Now simply write $\psi(x') = \int d^3x\, \delta(\vec{x}'-\vec{x})\,\psi(x)$ on the right
and $\psi(x') = i\int d^3x\, G_0(x';x)\,\psi(x)$ on the left

$$\Rightarrow \cancel{i}\int d^3x \left[\frac{i\partial}{\partial t'} - \hat{H}(x')\right] G_0(x';x)\psi(x) = \cancel{i}\delta(t'-t)\int d^3x\, \psi(x)\,\delta^3(\vec{x}'-\vec{x})$$

you can simply cancel the $\psi(x)$ since all the operators acting on it are functions of x'

$$\Rightarrow \frac{i\partial G_0(x';x)}{\partial t'} - \hat{H}(x')\, G_0(x';x) = i\,\delta^3(\vec{x}-\vec{x}')\,\delta(t'-t)$$
$$= i\delta^4(x-x') \qquad \text{— (6)}$$

Thus the propogator satisfies the equation

$$\hat{O}\, g(x';x) = \delta^4(x'-x) \quad \text{— (7)}$$

where $\hat{O}$ can be any wave equation, Dirac, Klein-Gordon, Schrödinger etc, etc (even classical).

In order to solve equation (7) or (6) (previous page) consider

(i) $g_0(x';x)$ can only be a function of the difference of x' and x $\therefore$ $g_0(x';x) = g_0(x'-x)$

(ii) Just like any other mathematical function in the Universe we can write the Greens function/propagator as a Fourier transform, thus:

$$g_0(x'-x) = \int \frac{d^3p\, d\omega}{(2\pi)^4}\, e^{i\vec{p}\cdot(\vec{x}'-\vec{x})} \times e^{-i\omega(t'-t)}\, g_0(P,\omega)$$

(this must be determined)

(iii) We can also re-write the Dirac-Delta function on the right hand side of (6) + (7) simpley by using the same variables in the Fourier transform) so equation (6) becomes:

$$\left(\frac{i\partial}{\partial t'} + \frac{1}{2m}\nabla'^2\right) \int \frac{d^3p\, d\omega}{(2\pi)^4}\, e^{i\vec{p}\cdot(\vec{x}'-\vec{x})-i\omega(t'-t)}\, g_0(P,\omega)$$

$$= \int \frac{d^3p\, d\omega}{(2\pi)^4}\, e^{i\vec{p}\cdot(\vec{x}'-\vec{x})-i\omega(t'-t)}$$

Simply by the definition of the Dirac Delta function.

<u>Next step:</u> operate $\frac{i\partial}{\partial t'} + \frac{1}{2m}\nabla'^2$ on the right

and remove the integral over $\int \frac{d^3p\,d\omega}{(2\pi)^4}$ and the exponential on both sides:

$$\left(\omega - p^2/2m\right) g_0(p,\omega) = 1$$

$$\therefore \quad g_0(p,\omega) = \frac{1}{\omega - p^2/2m}$$

this is known as the momentum space Greens function or propagator in momentum space of the lowly (as opposed to Dirac!) Schrödinger equation.

with $g_0(p,\omega)$ at hand we can write
$$G_0(x'-x) : \int \frac{d^3p}{(2\pi)^4}\, d\omega \, \frac{e^{i\vec{p}\cdot(\vec{x}'-\vec{x})+i\omega(t-t')}}{\omega - p^2/2m}$$

I can split this up:
$$G_0(x'-x) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}'-\vec{x})} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t')}}{\omega - p^2/2m}$$

and rewrite this again:

$$\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}'-\vec{x})} \times \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t')}}{\omega - p^2/2m + i\epsilon}$$

Lets examine the second integral in the product. Note $\omega = p^2/2m - i\epsilon$ is a pole in the lower half complex plane thus by the same argument that we made for $\lim_{\epsilon\to 0} \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\omega + i\epsilon}\, d\omega = \theta(\tau)$.

the second integral in the product is

$$\frac{(-i2\pi)}{(2\pi)}\, e^{\left(\frac{ip^2}{2m}+\epsilon\right)(t-t')} \times \theta(t'-t)$$

the contour of integration is closed in the lower half plane.

Thus from the previous page we have:

$$g_0(x'-x) = -i\theta(t'-t)\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}') - \frac{ip^2}{2m}(t'-t)}$$

Note that this is simply:

$$g_0(x',-x) = -i\theta(t'-t)\int d^3p\, \phi_p(x',t')\phi_p^*(x,t)$$

where the ϕ's are simply the free particle wave solutions to the Schrödinger equation.

with $\phi_p(x) = \frac{e^{-i\vec{p}\cdot\vec{x}' - \frac{ip^2t'}{2m}}}{(2\pi)^{3/2}}$ $\phi_p^*(x) = \frac{e^{+i\vec{p}\cdot\vec{x} + \frac{ip^2}{2m}t}}{(2\pi)^{3/2}}$

Note that for a <u>discrete</u> set of solutions to the Schrödinger equation

$$g_0(x'-x) = -i\theta(t'-t)\sum_n \phi_n(x',t')\phi_n^*(x,t)$$

and that for a $g(x'-x)$; that is a valid propagator in the presence of the potential we may simply write:

$$g(x'-x) = -i\theta(t'-t)\int d^3p\, \psi(x',t')\psi^*(x,t)$$

$$\text{or } -i\theta(t'-t)\sum_n \psi_n(x',t')\psi^*(x,t).$$

(if we have discrete solutions).

Now lets take

$$g_0(x'-x) = -i\theta(t'-t)\sum_n \psi_n(x',t')\psi_n^*(x,t)$$

multiply it on the right first by $\psi_m(x,t)$ and integrate over d^3x, then by $\psi_m^*(x',t')$ and integrate over d^3x'.

So we have now the following:

"immune" to integration over x

$$\int g_0(x'-x)\,\psi_m(x,t)\,d^3x = -i\theta(t'-t)\sum_n \int \psi_n(x',t')\,\psi_n^*(x,t)\,\psi_m(x,t)\,d^3x$$

$$\Rightarrow \quad i\int g_0(x'-x)\,\psi_m(x,t)\,d^3x = -i\theta(t'-t)\sum_n \psi_n(x',t')\,\delta_{mn}$$

$$i\int g_0(x'-x)\,\psi_m(x,t)\,d^3x = \theta(t'-t)\,\psi_m(x',t') \quad — 8(a)$$

↳ this is simply the definition of the propagator.

and for the case of multiplying with $\psi_m^*(x',t')$ and integrating over d^3x':

$$\int g_0(x'-x)\,\psi_m^*(x',t')\,d^3x' = -i\theta(t'-t)\sum_n \int \psi_m^*(x',t')\,\psi_n(x',t') \times \psi_n^*(x,t)\,d^3x'$$

$$= i\int g_0(x'-x)\,\psi_m^*(x',t')\,d^3x' = \theta(t'-t)\,\psi_m^*(x,t) \quad - 8(b)$$

this time this is "immune" to the integration.

Note that 8(a) + 8(b) are definitions of the Greens function (or propagator),

8(a) you've seen earlier, but 8(b) requires a bit of attention:

Note that x' is a later point of time than x, thus 8(b) is representing the propagation of a signal, backward in time.

So far we have taken the first steps in developing a formalism and methodology for calculating scattering amplitudes: we have:

(i) Postulated the existence of a function $G(x';x)$ such that

$$\theta(t'-t)\psi(x') = i\int G(x';x)\psi(x)d^3x$$

Note you can replace $G(x';x)$ with $G(x'-x)$

(ii) Shown that the free particle G, i.e G_0 follows the differential equation

$$\hat{O}G(x'-x) = \delta^4(x'-x)$$

where $\hat{O}$ is the appropriate wave equation i.e Schrödinger, Dirac, Klein-Gordon → whichever problem you wish to solve.

(iii) We've come up with an expression and an integral equation which summarizes $G(x'-x)$; the "exact" propagator in terms of a series i.e (page 8)

$$G(x';x) = G(x'-x) = G_0(x'-x) + \int d^4x_1\, G_0(x'-x_1)V(x_1)G_0(x_1-x)$$
$$+ \int d^4x_1 \int d^4x_2\, G_0(x'-x_2)V(x_2)G_0(x_2-x_1)V(x_1)$$
$$\times G_0(x_1-x) + \text{etc} \ldots$$

which in term can be written as:

$$G(x'-x) = G_0(x'-x) + \int d^4x_1\, G_0(x'-x_1)V(x_1)$$
$$\times G_0(x_1-x)$$

G can be calculated to the desired level of accuracy depending on where the above series is truncated.

(iv) The <u>crux</u> of solving a perturbation theory problem is therefore knowing G_0, the initial ϕ_0 and $G_0(x'-x)$ as we found has a simple representation in terms of the un-perturbed wavefunctions ϕ: $G_0(x'-x) = i\theta(t'-t)\int d^3p\, \phi_p(x')\phi_p^*(x)$

$\int$ or Σ

We now have some building blocks at hand and we can now begin to use them in calculating scattering <u>amplitudes</u>.

Assume that a particle starts out in an initial state i, in the remote past at $t = -\infty$ undergoes some sort of scattering and in the far future we observe it in a final state f. ϕ_f is the final state wave function and ϕ_i the initial. We denote the probability amplitude for the transition by S_{fi}

$$S_{fi} = \lim_{t' \to \infty} \int d^3x' \, \phi_f^*(\vec{x}', t') \, \phi_i(\vec{x}', t')$$

by definition of a transition...

This is in turn in terms of the "exact" Green's function.

$$S_{fi} = \lim_{t' \to \infty} \lim_{t \to -\infty} i \int d^3x \int d^3x' \, \phi_f^*(x') \, \mathcal{G}(x' - x) \, \phi_i(x)$$

where $\mathcal{G}(x, x')$ is the Greens function in the presence of a potential, or the "exact" Greens function. Of course the most exact it can get depends on where the series is truncated...

Now lets expand this out and see what we get, we know that $\mathcal{G}(x', x)$ is a series..

$$i \int d^3x \int d^3x' \, \phi_f^*(x') \, G_0(x' - x) \phi_i(x) + i \int d^3x' \int d^3x \int d^4x_1 \, \phi_f^*(x') G_0(x' - x_1) V(x_1) \times G_0(x_1 - x)$$

$$+ i \int d^3x \, d^3x' \, d^4x_1 \, d^4x_2 \, \phi_f^*(x') \, G_0(x' - x_2) V(x_2) \, G_0(x_2 - x_1) \, V(x_1) \, G_0(x_1 - x) \phi_i(x)$$

+ higher order terms... $\hookrightarrow$ Equation (9)

Lets look at the first term in equation (9) on the previous page

$$i\int d^3x'\, d^3x\ \phi_f^*(x')\, G_0(x'-x)\, \phi_i(x)$$

if you integrate over x' then this is by definition of the Greens function (propagator).

$$\int d^3x\ \phi_f^*(x)\, \phi_i(x)$$

when we are calculating scattering amplitudes we are considering the following case:

free particle (plane wave) $\longrightarrow$ interaction $\longrightarrow$ free particle (plane-wave)

$$\therefore \quad \int d^3x\ \phi_f^*(x)\, \phi_i(x) = \delta^3(\vec{p}_f - \vec{p}_i)$$

by definition. (Scattering is elastic so $E_f = E_i$ $\therefore$ time dep. terms cancel out....).

$$\boxed{\text{in general} \quad \delta^3(p'-p) = \int \frac{d^3k}{(2\pi)^3} e^{ik(p'-p)}}$$

Now look at the second term in the series:

$$i\int d^3x \int d^4x_1 \underbrace{\int d^3x'\ \phi_f^*(x')\, G_0(x'-x_1)}\, V(x_1)\, \phi_i(x)\, G_0(x_1-x)$$

Now first integrate the bracketed terms over d^3x', using equation 8(b) on page 14.

$$\Rightarrow \underbrace{\int d^3x} \int d^4x_1\ \phi_f^*(x_1)\, V(x_1) \underbrace{G_0(x_1-x)\, \phi_i(x)}$$

Now integrate over these brackets and obtain using 8(a) (pg. 14)

$$(-i)\int d^4x_1\ \phi_f^*(x_1)\, V(x_1)\, \phi_i(x_1)$$

$\uparrow$ you'll pick up a $-i \Rightarrow$ division by i

The idea is to integrate out the first and last g_0 in each term, using the relations on page 14. For the third term, there will actually be a remaining g_0:

$$i\int d^3x\, d^3x'\, d^4x_1\, d^4x_2\, \phi_f^*(x')\, g_0(x'-x_2)\, V(x_2)\, g_0(x_2-x_1)\, V(x_1)$$
$$\times\, g_0(x_1-x)\, \phi_i(x)$$

integrate first over d^3x', we'll absorb the i:

$$\int d^3x\, d^4x_1\, d^4x_2\, \phi_f^*(x_2)\, V(x_2)\, g_0(x_2-x_1)\, V(x_1)\, g_0(x_1-x)\, \phi_i(x)$$

integrate over x, (you have to introduce a factor of $-i$ since its $i\int d^3x\, G_0(x_1-x)\,\phi_i(x)$ that equals $\phi_i(x_1)$).

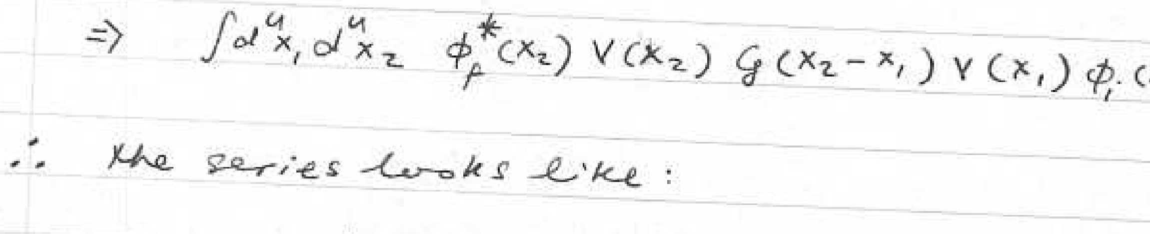
$$\Rightarrow \quad \int d^4x_1\, d^4x_2\;\; \phi_f^*(x_2)\, V(x_2)\, g(x_2-x_1)\, V(x_1)\, \phi_i(x_1)$$

$\therefore$ the series looks like:

$$S_{fi} = \delta^3(\vec{p}_f - \vec{p}_i) - i\int d^4x_1\, \phi_f^*(x_1) V(x_1) \phi_i(x_1) - i\int d^4x_1 d^4x_2\, \phi_f^*(x_2) V(x_2) \times g(x_2-x_1)\, V(x_1) \times \phi_i(x_1)$$
$$- i\int d^4x_1\, d^4x_2\, d^4x_3\, \phi_f^*(x_3)\, V(x_3)\, g_0(x_3-x_2)\, V(x_2)\, g(x_2-x_1)\, V(x_1)\, \phi_i(x_1)$$

+ higher order terms....

Diagrammatically one can draw this series:



(i) the $\rightarrow$ require the insertion of a propagator.
(ii) the crosses x require a V.
(iii) and we know <u>nothing</u> about the order of appearance of the points $x_1\ x_2\ x_3$ etc...

Well we went through the formalism and came up with a scheme independent of the kind of g_0 we were dealing with. Lets now come up with a g_0 for the Dirac equation and apply it to a "real" problem ⇒ We'll actually do the application later in a different batch of notes.

In fact the propagator (or Green's function) for the Dirac equation is not denoted by g_0 but by S_F, also known as the Feynmann propagator.

We seek a solution of equation (7) on page 11:

$$\hat{O}\, g(x'-x) = \delta^4(x'-x)$$

with $g = S_F(x'-x)$ (this is simply notation)
and $\hat{O} = (i\not{\partial}' - m) \rightarrow$ this is the Dirac equation

$$(i\not{\partial}' - m) S_F(x'-x) = \delta^4(x'-x)$$

We follow the standard "prescription" for solving such equations; i.e express the δ function and $S_F(x'-x)$ as a Fourier transform.

$$\therefore\ S_F(x'-x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x'-x)} S_F(p) \quad \text{and} \quad \delta^4(x'-x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x'-x)}$$

where the answer to the mystery is contained in $S_F(p)$ ..

So

$$\underset{\uparrow}{(i\not{\partial}' - m)} \int \frac{d^4p}{(2\pi)^4} \underset{\nwarrow}{e^{-ip\cdot(x'-x)}} S_F(p) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x'-x)}$$

operates on the exponential:

$$\int \frac{d^4p}{(2\pi)^4} (\not{p} - m) e^{-ip\cdot(x'-x)} S_F(p) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x'-x)}$$

at this point we can remove the integral on both sides of the equation:

$$(\not{p}-m)S_F(p) = 1$$

Multiply this on the left with $\not{p}+m$

$$\not{p}\not{p} + m\not{p} - m\not{p} - m^2$$

Now $\not{p}\not{p} = \gamma^\mu\gamma^\nu p_\mu p_\nu = (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)p_\mu p_\nu$

$\therefore \not{p}\not{p} = 2g^{\mu\nu}p_\mu p_\nu - \gamma^\nu\gamma^\mu p_\mu p_\nu = 2g^{\mu\nu}p_\mu p_\nu - \gamma^\nu\gamma^\mu p_\nu p_\mu$

$\not{p}\not{p} = 2p\cdot p - \not{p}\not{p} \quad \therefore 2\not{p}\not{p} = 2p\cdot p \Rightarrow \not{p}\not{p} = p\cdot p$

So the above equation becomes:

$$(\not{p}+m)(\not{p}-m)S_F(p) = (\not{p}+m)$$

$$S_F(p) = \frac{\not{p}+m}{p^2-m^2}$$

If you notice, the denominator is $p^2-m^2 = E^2 - \vec{p}\cdot\vec{p} - m^2$ which is $=0$ but in fact we have to consider the fact that the function of interest is

$$S_F(x'-x) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot(x'-x)} \frac{\not{p}+m}{p^2-m^2}$$

So you are in fact integrating over all p.

inorder to evaluate this integral, & examine this function we can re-write it as:

$$S_F(x'-x) = \lim_{\epsilon\to 0} \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot(x'-x)} \frac{\not{p}+m}{p^2-m^2+i\epsilon}$$

Now $p^2 - m^2 = p_0^{\,2} - \vec{p}\cdot\vec{p} - m^2$, $p_0 = E$

There are singularities in the complex p_0 plane at $p_0 = \pm\sqrt{\vec{p}\cdot\vec{p}+m^2} \mp i\epsilon$

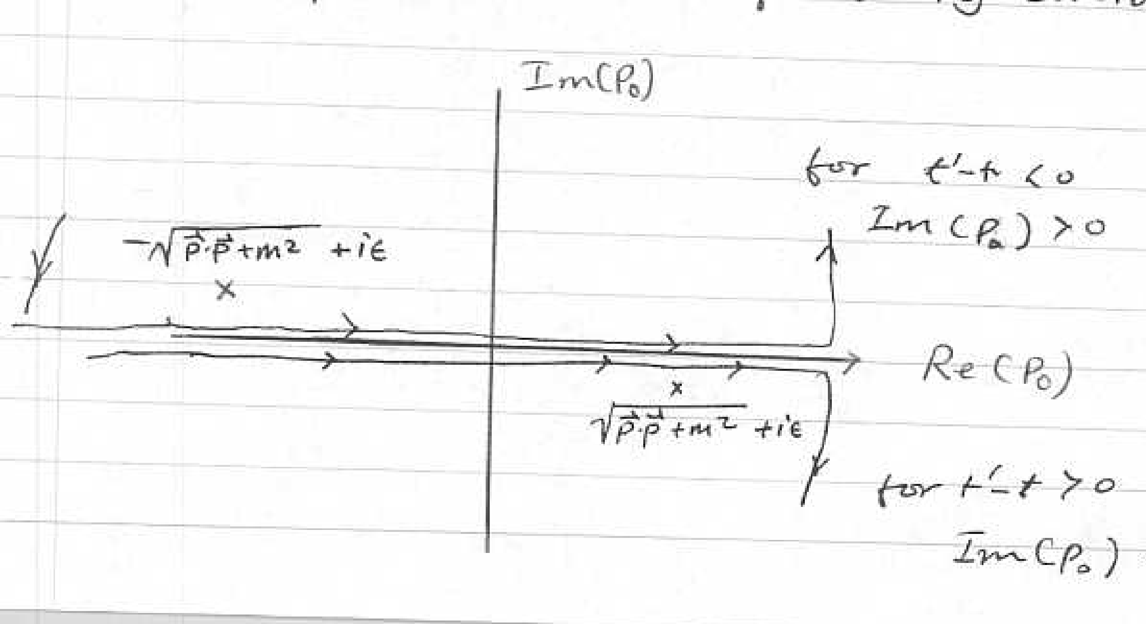
I can rewrite the integral on the previous page broken up into an integral over just p_0 and $d^3\vec{p}$ like so:

$$S_F(x'-x) = \lim_{\epsilon\to 0} \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{dp_0}{(2\pi)} \frac{e^{-ip_0\cdot(t'-t)} \cdot e^{+i\vec{p}\cdot(\vec{x}'-\vec{x})} (\gamma^0 p_0 + \vec{\gamma}\cdot\vec{p} + m)}{p_0^2 - \vec{p}\cdot\vec{p} - m^2}.$$

Note that if p_0 has an imaginary component then this can be $(p_{0R} + i p_{0I})$ re-written as:

$$\lim_{\epsilon\to 0} \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{dp_0}{(2\pi)} \frac{e^{-ip_{R0}(t'-t)} \cdot e^{+p_{0I}(t'-t)} \times e^{i\vec{p}\cdot(\vec{x}'-\vec{x})}(\gamma^0 p_0 + \vec{\gamma}\cdot\vec{p} + m)}{p_0^2 - \vec{p}\cdot\vec{p} - m^2}.$$

So if $t'-t > 0$ then this integral is exponentially damped as $|p_0| \to \infty$ (p_0 is complex) if p_{0I} is < 0, and we would require $p_{0I} > 0$ if $t'-t < 0$ for damping to occur that points us to the following contours:



$S_F(x'-x)$ can be expressed in somewhat more familiar terms:

If we write

$$\psi_{E>0,s}(x) = \sqrt{\frac{m}{E}}\,\frac{1}{(2\pi)^{3/2}}\,u_s(p)\,e^{-ip\cdot x}$$

$$\psi_{E<0,s}(x) = \sqrt{\frac{m}{E}}\,\frac{1}{(2\pi)^{3/2}}\,v_s(p)\,e^{+ip\cdot x}$$

then

$$S_F(x'-x) = -i\theta(t'-t)\int d^3p \sum_{s=1}^{2} \psi_{E>0,s}(x')\,\bar{\psi}_{E>0,s}(x)$$

$$+i\theta(t-t')\int d^3p \sum_{s=1}^{2} \psi_{E<0,s}(x')\,\bar{\psi}_{E<0,s}(x)$$

Even from the expression on the last page in terms of the positive and negative energy projection operators one can see that $S_F(x'-x)$ propagates $\psi_{E>0}$ solutions forward in time and $\psi_{E<0}$ solutions backward in time

By treating $e\not{A}(x) = \gamma^\mu A_\mu(x) \times e$ as a perturbation (taking the place of V) we can write the transition amplitude as:

$$S_{fi} = \delta_{fi} - ie\int d^4x_1\,\bar{\psi}_f(x_1)\not{A}(x_1)\psi_i(x_i)$$

$$-ie^2\int d^4x_1\,d^4x_2\,\bar{\psi}_f(x_2)\not{A}(x_2)S_F(x_2-x_1)\not{A}(x_1)\psi_i(x_1)$$

$$-ie^3\int d^4x_1\,d^4x_2\,d^4x_3\,\bar{\psi}_f(x_3)\not{A}(x_3)S_F(x_3-x_2)\not{A}(x_2)\times S_F(x_2-x_1)\not{A}(x_1) \times\psi_i(x_1)$$

The n^{th} order correction to the terms is:

$$(-i)e^n\int d^4x_1\,d^4x_2\ldots d^4x_n\;\bar{\psi}_f(x_n)\not{A}(x_n)S_F(x_n-x_{n-1})\not{A}(x_{n-1}) \times S_F(x_{n-1}-x_{n-2})\not{A}(x_{n-2})S_F(x_{n-2}-x_{n-3})\cdots \cdots S_F(x_2-x_1)\not{A}(x_1)\psi_i(x_1)$$

Think of all the things $A^\mu(x)$ can be due to, an external electric field due to a nucleas a proton, another electron, a positron, don't forget free photons! Gives rise to a series of scattering problems...

Rewrite the integral as:

$$\lim_{\epsilon\to 0}\int\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dp_0}{(2\pi)}\frac{e^{-ip_0(t'-t)}}{\left[p_0-\sqrt{\vec{p}\cdot\vec{p}+m^2}\right]\left[p_0+\sqrt{\vec{p}\cdot\vec{p}+m^2}\right]+i\epsilon}$$

If $Im(p_0)<0$, then the exponential is damped and the integral well defined if $(t'-t)>0$, so we can then close the contour in the lower half plane, this is traversed in the clockwise direction

$\therefore$ $p_0=\sqrt{\vec{p}\cdot\vec{p}+m^2}$ it is a pole i.e $E_p=\sqrt{\vec{p}\cdot\vec{p}+m^2}$

$-2\pi i\sum$ Residues: (only one residue)

$$\lim_{\epsilon\to 0}(-2\pi i)\left(\frac{1}{2\pi}\right)\theta(t'-t)\frac{e^{-i(E_p-i\epsilon)(t'-t)}(E_p\gamma^0-\vec{p}\cdot\vec{\gamma})}{(p_0-E_p)\underbrace{(p_0+E_p)}_{=E_p}+i\epsilon}\times\cancel{(p_0-E_p)}$$

$\therefore$ $S_F(x'-x)$ for $t'>t$

$$=-i\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot(\vec{x}'-\vec{x})-iE_p(t'-t)}(E_p\gamma^0-\vec{\gamma}\cdot\vec{p})}{2E_p}\times\theta(t'-t)$$

Similarly it can be shown that for $t'>t$ we can close the contour in the upper half plane clock-wise and obtain:

$$-i\int\frac{d^3p}{(2\pi)^3}\frac{e^{iE_p(t'-t')}\wedge e^{i\vec{p}\cdot(\vec{x}'-\vec{x})}\left[-E_p\gamma^0-\vec{p}\,\vec{\gamma}\right]}{2E_p}\theta(t-t')$$

if you recall the definition of the Λ_+ and Λ_- positive and negative energy projection operators this is simply:

$$S_F(x'-x)=-i\int\frac{d^3p}{(2\pi)^3}\left\{\left(\frac{m}{E_p}\right)\Lambda^+e^{-ip\cdot(x'-x)}\theta(t'-t)+\left(\frac{m}{E_p}\right)\Lambda^-e^{ip\cdot(x'-x)}\theta(t-t')\right\}$$