31/10/2002

<Advanced Quantum Mechanics>
Problem Set # 1

1. Evaluate $g_{\mu\nu} g^{\mu\nu}$ where $g_{\mu\nu} = g^{\mu\nu}$ is the metric tensor: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

2. Consider the collision of two particles of mass M. Assume that this collision is viewed in a frame where it is head-on with <u>equal</u> and <u>opposite</u> momenta (direction)

(i) In this frame what is the 4 momentum of one particle if the 4-momentum of the other is $\left(\frac{E}{c}, \vec{P}\right)$
[this frame is known as the centre of mass frame]

(ii) Denote the 4-momenta of the particles by P_1 & P_2 and calculate the Lorentz invariant $(P_1 + P_2) \cdot (P_1 + P_2)$ $= (P_1 + P_2)_\mu (P_1 + P_2)^\mu$. What is the <u>total</u> energy in this frame?

(iii) Assume now that this collision is viewed in a frame where one of the particles is at rest, [this is known as the Laboratory frame], show that the energy E' of the other [moving] particle is:
$E' = \frac{E^2_{T,CM}}{2Mc^2} - Mc^2$ where $E_{T,CM}$ is the total available energy in the centre of mass frame from (ii). Note E' is often referred to as E_{lab}.

3. Consider the Dirac equation:
$(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi$
operate on this on the left with
$(i\hbar\not{\partial} - mc)$ and show that each component
of ψ satisfies the Klein-Gordon equation
$\left(\Box + \frac{mc^2}{\hbar^2}\right)\psi_i = 0$.

Note: $(i\hbar\not{\partial} - mc)$ will have to be expressed
as $(i\hbar\gamma^{\nu}\partial_{\nu} - mc)$ a <u>different</u> index than
the μ used before. So you want to do:
$(i\hbar\gamma^{\nu}\partial_{\nu} - mc)(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi$
for all possible ν & μ and use the
anticommutation relations in the notes.
⟨<u>N.B.</u> you don't need to know what the
components of ψ actually are, I hope that's
clear, if not just ask⟩

4. Solve the Dirac equation for the
simple case of a stationary particle.
Use the definitions of the γ matrices
supplied. You should end up with
4 solutions:

$$\begin{pmatrix} e^{-\frac{imc^2t}{\hbar}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ e^{-\frac{imc^2t}{\hbar}} \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ e^{+\frac{imc^2t}{\hbar}} \\ 0 \end{pmatrix} \;\&\; \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{+\frac{imc^2t}{\hbar}} \end{pmatrix}$$

Note that the first two are positive energy
solutions and the second two are negative
energy solutions.

5. Now pick your favourite spinor and
factor out the exponential, apply a boost
of β, by considering a series of infinitesimal
boosts, following the approach in class along $+x$.

6. Proove that for the S, $\epsilon^{\mu\nu}$ and $\sigma^{\mu\nu}$ as defined in class, and in the notes,
$$S^{-1}\gamma^{\alpha}S = \gamma^{\beta}\Lambda^{\alpha}_{\beta}$$
where the Λ^{α}_{β} are the coefficients of an infinitesimal Lorentz boost i.e $\Lambda^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + \epsilon^{\alpha}_{\beta}$ and $S = 1 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}$.

7. Show that $(i\hbar\gamma^{\mu}\partial_{\mu} - mc)u = 0$
can be written as $(\not{p} - mc)u = 0 \quad -(i)$
$\not{p} = \gamma_0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3 \qquad p^1 = p_x,\ p^2 = p_y,\ p^3 = p_z$
Show that if v is a spinor representing a solution with $E < 0$ then we might as well write:
$$(\not{p} + mc)v = 0$$

8. By operating $\Lambda^{-} = \frac{-\not{p} + mc}{2mc}$ on a negative energy solution of the Dirac equation, v, show that
$$\Lambda^{-}v = v \qquad (\text{use } (\not{p} + mc)v = 0)$$

9. By using the spinors in Batch $\overline{\text{VII}}$ page 5, derive explicitly $\Lambda^{-} = -(v^1\bar{v}^1 + v^2\bar{v}^2)$.
verify that this is $\frac{-\not{p} + mc}{2mc}$