Feynman diagrams

Contents

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1 Aim of the game

To calculate the probabilities for *relativistic* scattering processes we need to find out the Lorentz-invariant scattering amplitude which connects an initial state $|\Psi_i\rangle$ containing some particles with well defined momenta to a final state $|\Psi_f\rangle$ containing other (often different) particles also with well defined momenta.

We make use of a graphical technique popularised by Richard Feynman[1]. Each graph – known as a **Feynman Diagram** – represents a contribution to $\mathcal{M}_{fi}$. This means that each diagram actually represents a **complex number** (more generally a complex function of the external momenta). The diagrams give a pictorial way to represent the contributions to the amplitude.

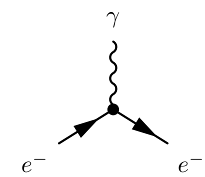
In Feynman diagrams, spin-$\frac{1}{2}$ particles such as electrons are indicated with a straight line with an arrow.

Fermion line

The arrow follows the direction of particle flow, in the same was as in quark-flow diagrams (§**??**).

Photon line

Diagrams consist of **lines** representing particles and **vertices** where particles are created or annihilated. I will place the incoming state on the left side and the outgoing state on the right side. Since the diagrams represent transitions between well-defined states in 4-momentum they already include the contributions from **all possible paths in both time and space** through which the intermediate particles might possibly have passed. This means that it is not meaningful to ask about the time-ordering of any of the internal events, since all possible time-orderings are necessarily included.



Vertex

2 Rules for calculating diagrams

It turns out that are simple rules for calculating the complex number represented by each diagram. These are called the **Feynman rules**. In quantum field theory we can derive these rules from the Lagrangian density, but in this course we will simply quote the rules relevant for the Standard Model.

[1] American physicist (1918-1988).

2.1 Vertices

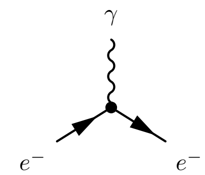
Vertices are places where particles are created or annihilated. In the case of the electromagnetic interaction there is only **one** basic vertex which couples a photon to a charged particle with strength proportional to its charge.

To calculate the contribution to $\mathcal{M}_{fi}$, for each vertex we associate a **vertex factor**. For interactions of photons with electrons the vertex factor is of size $\boxed{-g_{\rm EM}}$ where $g_{\rm EM}$ is a dimensionless charge or **coupling constant**.[2] The coupling constant is a number which represents the strength of the interaction between the particle and the force carrier at that vertex. For the electromagnetic force the coupling strength must be proportional to the electric charge of the particle. So for the electromagnetic vertex we need a dimensionless quantity proportional to the charge. Recall that for the electromagnetic fine structure constant:

$$\alpha_{\sf EM} \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

is dimensionless. It is convenient to choose $g_{\rm EM}$ such that

$$\alpha_{\sf EM} = \frac{g_{\rm EM}^2}{4\pi}.$$



The electromagnetic vertex. The vertex factor is $-g_{\rm EM}$.

In other words the coupling constant $g_{\rm EM}$ is a dimensionless measure of the $|e|$ where e is the charge of the electron. The size of the coupling between the photon and the electron is

$$-g_{\rm EM} = -\sqrt{4\pi\alpha_{\sf EM}}.$$

The electromagnetic vertex factor for any other charged particle f with charge $\mathcal{Q}_f$ times that of the proton is then

$$\boxed{g_{\rm EM}\mathcal{Q}_f}$$

So, for example, the electromagnetic vertex factor for an electron is of size $-g_{\rm EM}$ while for the up quark it is of size $+\frac{2}{3}g_{\rm EM}$.

2.2 Anti-particles

An anti-particle has the same mass as its corresponding particle cousin, but his charge is the opposite to that of the particle.[3] The Feynman diagram for an


Anti-fermion.

[2] We are simplifying the situation by ignoring the spin of the electron. If spin is included the vertex factor becomes $-g_{\rm EM}$ times a matrix, in fact a Dirac gamma matrix, allowing the spin direction of the electron as represented by a 4-component spinor. For now we will ignore this complication and for the purpose of Feynman diagrams treat all spin $\frac{1}{2}$ fermions, such as electrons, muons, or quarks, as spinless. The Dirac matrices also distinguish electrons from anti-electrons. The sign of the vertex factor is well defined when the Dirac representations are used for the particles.

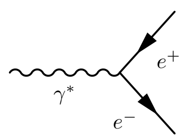
[3] In fact if the particle is charged under under more than one force then the anti-particle has the opposite values of all of those charges. For example an anti-quark, which has electromagnetic, strong and weak charges will have the opposite value of each of those compared to the corresponding quark.

anti-particle shows the arrow going the 'wrong' way (here right to left), since the particle flow is opposite to that of anti-particle.

The same basic electromagnetic vertex is responsible for many different reactions. Consider each of the partial reactions

$$\begin{aligned}
e^- &\rightarrow e^- + \gamma \\
e^- + \gamma &\rightarrow e^- \\
e^+ &\rightarrow e^+ + \gamma \\
e^+ + \gamma &\rightarrow e^+ \\
e^- + e^+ &\rightarrow \gamma \\
\gamma &\rightarrow e^- + e^+ .
\end{aligned} \tag{1}$$

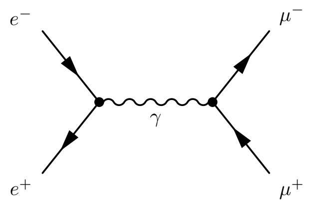
Each of these is just a different time ordering of the same fundamental vertex that couples an electron to a photon.

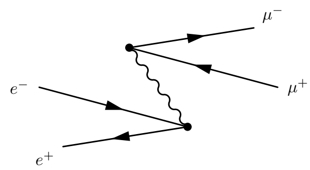


The electromagnetic vertex with particle-antiparticle final state

2.3 Distinct diagrams

A Feynman diagram represents all possible time orderings of the possible vertices, so the positions of the vertices within the graph are arbitrary. Consider the following two diagrams for $e^+ + e^- \rightarrow \mu^+ + \mu^-$:

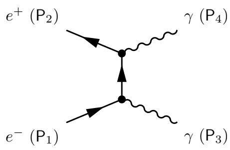


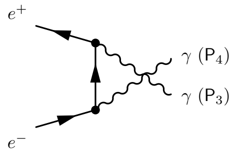


In the left diagram it appears that the incoming particles annihilated to form a virtual photon, which then split to produce the outgoing particles. On the right diagram it appears that the muons and the photon appeared out of the vacuum together, and that the photon subsequently collided with the electron and positron, leaving nothing. Changing the position of the internal vertices does not affect the Feynman diagram – it still represents the same contribution to the amplitude. The left side and right side just represent different time-orderings, so each is just a different way of writing the same Feynman diagram.

On the other hand, changing the way in which the lines in a diagram are connected to one another does however result in a new diagram. Consider for example the process $e^+ + e^- \rightarrow \gamma + \gamma$

In the two diagrams above the outgoing photons have been swapped. There is no way to move around the vertices in the second diagram so that it is the same as the first. The two diagrams therefore provide separate contributions to $\mathcal{M}_{fi}$, and must be added.





2.4 Relativistic propagators

For each internal line – that is each **virtual particle** – we associate a **propagator factor**. The propagator tells us about the contribution to the amplitude from a particle travelling through space and time (integrated over all space and time). For a particle with no spin, the **Feynman propagator** is a factor

$$\boxed{\frac{1}{\mathsf{Q}\cdot\mathsf{Q}-m^2}}$$

where $\mathsf{Q}\cdot\mathsf{Q} = E_{\mathsf{Q}}^2 - \mathbf{q}\cdot\mathbf{q}$ is the four-momentum-squared of the internal virtual particle[4].

These intermediate particles are called **virtual particles**. They do **not** satisfy the usual relativistic energy-momentum constraint $\mathsf{Q}\cdot\mathsf{Q} = m^2$. For an intermediate virtual particle,

$$\boxed{\mathsf{Q}\cdot\mathsf{Q} = E_{\mathsf{Q}}^2 - \mathbf{q}\cdot\mathbf{q} \neq \mathbf{m^2}.}$$

Such particles are said to be **off their mass-shell**.

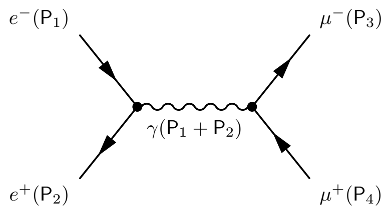
If this inequality worries you, it might help you if you consider that their energy and momentum cannot be measured without hitting something against them. So you will never "see" off-mass-shell particles, you will only see the effect they have on other objects they interact with.

External particles in Feynman diagrams do always individually satisfy the relativistic energy-momentum constraint $E^2 - p^2 = m^2$, and for these particles we should therefor not include any propagator factor. The external lines are included in the diagram purely to show which kinds of particles are in the initial and final states.

2.4.1 Propagator example

Consider the annihilation-creation process $e^+ + e^- \to \gamma^* \to \mu^+ + \mu^-$ proceeding via a virtual photon γ^*. (The star on the particle name can be added to help remind us that it is off mass shell and virtual). We will ignore the spin of all the particles, so that we can concentrate on the vertex factors and propagators. The Feynman diagram is:

[4] This propagator is the relativistic equivalent of the non-relativistic version of the Lippmann-Schwinger propagator $(E - H + i\epsilon)^{-1}$ that we found in non-relativistic scattering theory. Why are the forms different? Non-relativistic propagators are Greens functions for integration over all space. Relativistic propagators by contrast are Greens functions for integrations over both space and time.



where we have labelled the four-momenta of the external legs. The diagram shows two vertices, and requires one propagator for the internal photon line. We can calculate the photon's energy-momentum four-vector Q_γ from that of the electron P_1 and the positron P_2. Four momentum is conserved **at each vertex** so the photon four-vector is $\mathsf{Q}_\gamma = \mathsf{P}_1 + \mathsf{P}_2$. Calculating the momentum components in the zero momentum frame:

$$\mathsf{P}_1 = (E, \mathbf{p}), \quad \mathsf{P}_2 = (E, -\mathbf{p}). \tag{2}$$

Conserving energy and momentum at the first vertex, the energy-momentum vector of the internal photon is

$$\mathsf{Q}_\gamma = (2E, \mathbf{0}).$$

So this *virtual* photon has more energy than momentum.

The propagator factor for the photon in this example is then

$$\frac{1}{(2E)^2 - m_\gamma^2} = \frac{1}{4E^2}.$$

The contribution to $\mathcal{M}_{fi}$ from this diagram is obtained my multiplying this propagator by two vectex factors each of size g_{EM}. The modulus-squared of the matrix element is then

$$|\mathcal{M}_{fi}|^2 = \left|\frac{g_{\mathrm{EM}}^2}{4E_e^2}\right|^2.$$

We can get the differential scattering cross section by inserting this $|\mathcal{M}_{fi}|^2$ into Fermi's Golden Rule with the appropriate density of states

$$\frac{dN}{dp_\mu} = \frac{p_\mu^2 d\Omega}{(2\pi)^3},$$

and divide by an incoming flux factor $2v_e$. The differential cross section is then

$$d\sigma = \frac{1}{2v_e}\ 2\pi\ |\mathcal{M}_{fi}|^2\ \frac{p_\mu^2}{(2\pi)^3}\frac{dp_\mu}{d(E_0)}\ d\Omega.$$

A little care is necessary in evaluating the density of states. Overall momentum conservation means that only one of the two outgoing particles is free to contribute to the density of states. The muon energy in the ZMF, for $E_\mu \gg m_\mu$ is $E_\mu = \frac{1}{2}E_0$, so

$$\frac{dp_\mu}{dE_0} = \frac{1}{2}\frac{dp_\mu}{dE_\mu},$$

where p_μ and E_μ are the momentum and energy of one of the outgoing muons. Since those muons are external legs they are on-shell so that

$$p_\mu^2 + m_\mu^2 = E_\mu^2.$$

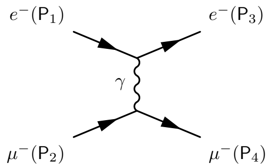
Taking a derivative $p_\mu \, dp_\mu = E_\mu \, dE_\mu$. Inserting this into the F.G.R. we get

$$\frac{dp_\mu}{dE_0} = \frac{1}{2}\frac{dp_\mu}{dE_\mu} = \frac{1}{2}\frac{E_\mu}{p_\mu} = \frac{1}{2}\frac{1}{v_\mu} \approx \frac{1}{2}.$$

We then integrate over all possible outgoing angles to gain a factor of 4π and note that $g^2/4\pi = \alpha$, and that $\frac{p_\mu}{E_\mu} = v_\mu$. Gathering all the parts together, and taking the limit $v \to c$ we find we have a total cross-section for $e^+ + e^- \to \mu^+ + \mu^-$ of [5]

$$\sigma = \pi \frac{\alpha^2}{s}$$

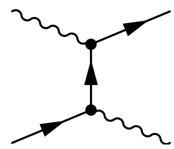
where $s = (2E)^2$ is the square of the center-of-mass energy.



A Feynman diagram for electron–muon elastic scattering, via photon (γ) exchange.

A quick check of dimensions is in order. The dimensions of s are $[E]^2$, while those of σ should be $[L]^2 = [E]^{-2}$. The fine structure constant α is dimensionless, so the equation is dimensionally consistent.

2.4.2 Other propagator examples



Feynman diagram for Compton scattering, with a virtual internal electron

In the previous example the virtual photon's four-momentum vector $(E, \mathbf{0})$ was time-like.

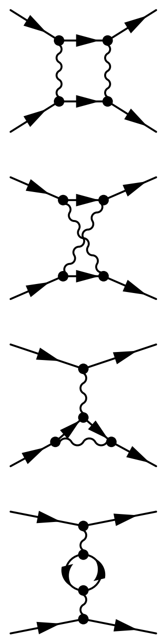
In the electron–muon scattering case $e^- + \mu^- \to e^- + \mu^-$ the virtual photon (γ^*) is exchanged between the electron and the muon. The virtual photon carries momentum and not energy, so the propagator is space-like.

To see this, transform to in the zero-momentum frame. In the ZMF the electron is kicked out with the same energy as it came in with, so it has received no energy from the photon, and conserving energy at the vertex $E_\gamma = 0$. The direction of the electron momentum vector *has* changed so it *has* received momentum from the photon, $\mathbf{p}_\gamma \neq 0$. Therefore $E_\gamma^2 - |\mathbf{p}|_\gamma^2 < 0$ and the propagator is space-like.

An internal line requires a propagator regardless of the type of particle. An example of a process in which an electron is the virtual particle is the Compton process in which an electron scatters a photon

$$e^- + \gamma \to e^- + \gamma.$$

2.5 Trees and loops



Some of the more complicated loop diagrams for electron–muon scattering.

In principle to calculate $|\mathcal{M}_{fi}|$ we are supposed to draw and calculate all of the infinite number of possible Feynman diagrams. Then we have to add up all those complex numbers to get the total amplitude $\mathcal{M}_{fi}$.

[5] Neglecting spin and relativistic normalization and flux factor issues – see 'caveats'.

However in practice we can get away with just summing the simplest diagram(s). To see why, we first note that the electromagnetic fine structure constant is small ($\alpha_{\mathsf{EM}} \ll 1$)

The simplest "tree level" scattering diagram has two vertices so contains two factors of $g_{\rm EM}$. The diagrams with the loops contain four vertices and hence four factors of $g_{\rm EM}$. Since $g_{\rm EM}^2/4\pi = \alpha_{\mathsf{EM}} \ll 1$, we can see that the more complicated diagrams with with more vertices will (all other things being equal) contribute much less to the amplitude than the simplest ones since they contain higher powers of α_{EM}. This process of truncating the sum of diagrams is a form of perturbation theory.

In general tree diagrams are those without closed loops. Loop diagrams – those with internal closed loops – tend to have larger powers of the coupling constant. A good approximation to $\mathcal{M}_{fi}$ can usually be obtained from the sum of the amplitudes for the 'leading order' diagrams – those with the smallest power of α_{EM} that make a non-zero contribution to $\mathcal{M}_{fi}$.

The other forces also have coupling constants, which have different strengths. The **strong force** is so-called because it has a fine structure constant close to 1 which is about a hundred times larger than α_{EM}. In fact the **weak force** actually has a larger coupling constant $\approx 1/29$ than the electromagnetic force $\approx 1/137$. The reason why this force appears weak is because the force is transmitted by very heavy particles (the W and Z bosons) so it is very short-range.

3 Key concepts

- Feynman (momentum-space) diagrams help us calculate relativistic, Lorentz-invariant scattering amplitudes.
- Vertices are associated with dimensionless **coupling constants** g with vertex factors that depend on the charge $\mathcal{Q}g$
- Internal lines are integrated over **all time and space** so include all internal time orderings.
- Intermediate/virtual/off-mass-shell particles have $\boxed{\mathsf{Q}^2 \neq m^2}$ and have propagators $\boxed{\dfrac{1}{\mathsf{Q}^2 - m^2}}$.
- For fermions, arrows show the sense of particle flow. Anti-particles have arrows pointing the "wrong way".

Caveats

- Sometimes you will see books define a propagator with a plus sign on the bottom line: $1/(q^2 + m^2)$. One of two things is going on. Either (a) q^2

is their notation for a four-vector squared, but they have defined the metric $(-,+,+,+)$ in the opposite sense to us so that $q^2 = -m^2$ is their condition for being on-mass-shell *or* (b) q^2 is acually intended to mean the three-momentum squared. A bit of context may be necessary, but regardless of the convention used the propagator should diverge in the case when the virtual particle approaches its mass-shell.

- We have not attempted to consider what the effects of spins would be. This is done in the fourth year after the introduction of the Dirac equation – the relativistic wave equation for spin-half particles. The full treatment is done in e.g. Griffiths Chs. 6 & 7.

- We have played fast and loose with phase factors (at vertices and overall phase factors). You can see that this will not be a problem so long as only one diagram is contributing to $\mathcal{M}_{fi}$, but clearly relative phases become important when adding diagrams together.

- Extra rules are needed for diagrams containing loops, because the momenta in the loops are not fully constrained. In fact one must integrate over all possible momenta for such diagrams. We will not need to consider such diagrams in this course.

- The normalization of the incoming and ougtoing states needs to be considered more carefully. The statement "*I normalize to one particle per unit volume*" is **not** Lorentz invariant. The volume of any box at rest will compress by a factor of $1/\gamma$ due to length contraction along the boost axis when we Lorentz transform it. For relativistic problems we want to normalize to a Lorentz invariant number of particles per unit volume. To achieve this we conventionally normalize to $1/(2E)$ particles per unit volume. Since $1/(2E)$ also scales like $1/\gamma$ it transforms in the same manner as V. Therefore the statement "*I normalize to $1/(2E)$ particles per unit volume*" **is** Lorentz invariant.

Terminology

$\mathcal{M}_{fi}$	Lorentz invariant amplitude for $\lvert\Psi_i\rangle \to \lvert\Psi_f\rangle$ transition
Feynman diagram ...	Graphical representation of part of the scattering amplitude
Vertex	Point where lines join together on such a graph
Constant coupling (g)	Dimensionless measure of strength of the force
Vertex factor ($\mathcal{Q}g$) ..	The contribution of the vertex to the diagram
Propagator	Factor of $1/(\mathsf{Q}\cdot\mathsf{Q}-m^2)$ associated with an internal line
Tree level / leading order	Simplest diagrams for any process with the smallest number of g factors. Contain no closed loops.

References and further reading

- *"Introduction to Elementary Particles"* D. Griffiths Chapters 6 and 7 does the full relativistic treatment, including spins, relativistic normalization and relativistic flux factor.
- *"Femptophysics"*, M.G. Bowler – contains a nice description of the connection between Feynman propagators and non-relativistic propagators.
- *"Quarks and Leptons"*, Halzen and Martin – introduction to the Dirac equation and full Feynman rules for QED including spin.
- *"QED - The Strange Theory of Light and Matter"*, Richard Feynman. Popular book with almost no maths. Even a PPE student could understand it – if you explained it slowly to him. In fact it has a lot to recommend it, not least that you can buy it for about five points.