

1 Problems 1

Three vectors are written in bold e.g. \mathbf{x} .
 Four vectors are written sans-serif, e.g. P .
 The metric is $\text{diag}(1, -1, -1, -1)$ so that with $X \cdot X = c^2 t^2 - \mathbf{x} \cdot \mathbf{x}$,
 time-like intervals have positive signs, and propagating particles satisfy $P \cdot P = +m^2$.

Core questions

1.1. a) What assumptions underlie the radioactivity law

$$\frac{dN}{dt} = -\Gamma N ?$$

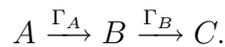
b) If some number N_0 of nuclei are present at time $t = 0$ how many are present at some later time t ?

c) Calculate the mean life τ of the species.

d) Relate the half-life $t_{\frac{1}{2}}$ to τ .

e) How do things change when the particle moves relativistically?

1.2. A sample consists originally of nucleus A only, but subsequently decays according to



Write down differential expressions for $\frac{dA}{dt}$, $\frac{dB}{dt}$ and $\frac{dC}{dt}$. Solve for, and then sketch, the fractions of, $A(t)$, $B(t)$ and $C(t)$. At what time is the decay rate of B maximum?

1.3. Consider Compton scattering $\gamma + e^- \rightarrow \gamma + e^-$ of a photon of energy E_γ (\sim MeV) from a stationary electron.

a) Show that the energy of the scattered photon is

$$E'_\gamma = \frac{m_e E_\gamma}{m_e + E_\gamma(1 - \cos \theta)},$$

where θ is the angle through which the photon is scattered.

b) If an incoming photon is scattered through an angle $\sim 180^\circ$ at the surface of a material, how much of the original photon's energy would you expect *not* to be deposited in the material?

1.4. Briefly explain the origin of each of the terms in the semi-empirical mass formula (SEMF)

$$M(N, Z) = Zm_p + Nm_n - \alpha A + \beta A^{\frac{2}{3}} + \gamma \frac{(N - Z)^2}{A} + \epsilon \frac{Z^2}{A^{\frac{1}{3}}} + \delta(N, Z)$$

and obtain a value for ϵ .

Show that we can include the gravitational interaction between the nucleons by adding a term to the SEMF of the form

$$-\zeta A^{5/3}$$

and find the value of ζ .

Use this modified SEMF to obtain a lower bound on the mass of a gravitationally-bound 'nucleus' consisting only of neutrons (a neutron star).

1.5. An analysis of a chart showing all stable ($t_{\frac{1}{2}} > 10^9$ years) nuclei shows that there are 177 even-even, 121 even-odd and 8 odd-odd stable nuclei and, for each A , only one, two or three stable isobars. Explain these observations qualitatively using the SEMF. Energetically $^{106}_{48}\text{Cd}$ could decay to $^{106}_{46}\text{Pd}$ with an energy release of greater than 2 MeV. Why does $^{106}_{48}\text{Cd}$ occur naturally?

1.6. The radius r of a nucleus with mass number A is given by $r = r_0 A^{\frac{1}{3}}$ with $r_0 = 1.2$ fm. What does this tell us about the nuclear force?

a) Use the Fermi gas model (assuming $N \approx Z$) to show that the energy ϵ_F of the Fermi level is given by

$$\epsilon_F = \frac{\hbar^2}{2mr_0^2} \left(\frac{9\pi}{8} \right)^{\frac{2}{3}}.$$

b) Estimate the total kinetic energy of the nucleons in an ^{16}O nucleus.

c) For a nucleus with neutron number N and proton number Z the asymmetry term in the semi-empirical mass formula is

$$\frac{\gamma(N - Z)^2}{A}.$$

Assuming that $(N - Z) \ll A$ use the Fermi gas model to justify this form and to estimate the value of γ . Comment on the value obtained.

1.7. Alpha-decay rates are determined by the probability of tunnelling through the Coulomb barrier. Draw a diagram of the potential energy $V(r)$ as a function of the distance r between the daughter nucleus and the α particle, and of the wave function $\langle r|\psi\rangle$.

The decay rate can be expressed as $\Gamma = fP$, where f is the frequency of attempts by the alpha particle to escape and $P = \exp(-2G)$ is the probability for the alpha particle to escape on any given attempt.

By using a one-dimensional Hamiltonian

$$H = \frac{p_r^2}{2m} + V,$$

with a 1D momentum operator $p_r = -i\hbar\frac{\partial}{\partial r}$, and by representing the wave function by

$$\langle r|\Psi\rangle = \exp[\eta(r)],$$

with $r = x$ show that

$$G = \frac{\sqrt{2m}}{\hbar} \int_a^b \sqrt{V(r) - Q} \, dr.$$

Integrate (a substitution $r = r_b \cos^2 \theta$ helps) to give

$$G = \frac{\pi}{2} Z z \alpha \sqrt{\frac{2mc^2}{Q}} \mathcal{F}(r_a/r_b)$$

where the dimensionless function

$$\mathcal{F}(r) = \frac{2}{\pi} \left(\cos^{-1} \sqrt{r} - \sqrt{r(1-r)} \right)$$

lies in the range between 0 and 1, and for small Q approaches 1.

[Hint: can you convince yourself that $\eta'' \ll (\eta')^2$?]

1.8. a) What are the basic assumptions of the Fermi theory of beta decay?

b) The Fermi theory predicts that in a beta decay the rate of electrons emitted with momentum between p and $p + dp$ is given by

$$\frac{d\Gamma}{dp_e} = \frac{2\pi}{\hbar} G^2 |M_{fi}^{\text{nuc}}|^2 \frac{1}{4\pi^4 \hbar^6 c^3} (E - Q)^2 p^2,$$

where E is the energy of the electron, and Q is the energy released in the reaction. Justify the form of this result.

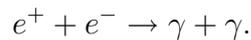
c) Show that for $Q \gg m_e c^2$ the total rate is proportional to Q^5

d) What spin states are allowed for the combined system of the electron + neutrino?

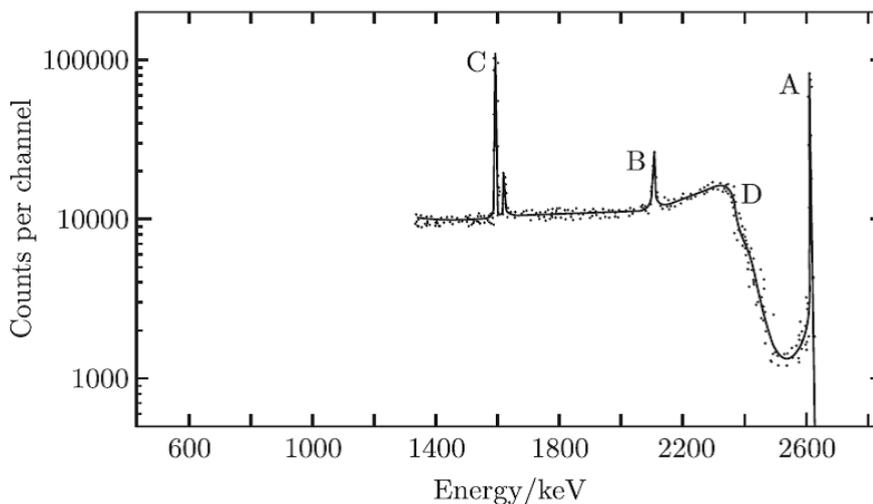
e) ‡ Why are transitions between initial and final nuclei with angular momenta differing by more than \hbar suppressed?

Optional questions

1.9. A high energy photon can create an electron-positron pair within the material. When a positron comes to rest it will annihilate against an electron from the material

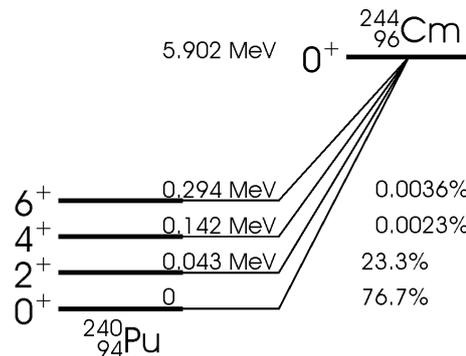


What will be the energy of these secondary photons?



The figure shows the energy spectrum from the gamma decay of ^{24}Na as measured in a small Ge(Li) detector. Suggest the origins of the peaks A, B, C and the edge D. For such a detector describe the stages by which gamma ray energy is converted into a measurable voltage pulse.

1.10. The figure shows the α -decay scheme of $^{244}_{96}\text{Cm}$ and $^{240}_{94}\text{Pu}$.



Show — either by (a) using a suitable approximation of \mathcal{F} calculated in a previous question or (b) by redoing the corresponding integral neglecting Q — that we would expect the rates to satisfy an equation of the form

$$\log \Gamma = A - \frac{BZ}{\sqrt{Q}}.$$

The Q value for the ground state to ground state transition is 5.902 MeV and for this transition $A = 132.8$ and $B = 3.97(\text{MeV})^{1/2}$ when Γ is in s^{-1} . The branching ratio for this transition is given in the figure. Calculate the mean life of ^{244}Cm .

Estimate the transition rate from the ground state of ^{244}Cm to the 6^+ level of ^{240}Pu using the same A and B and compare to the branching ratio given in the figure.

Suggest a reason for any discrepancy.

[Hint: what form does the Schrödinger equation take for angular momentum quantum number $l \neq 0$?]

1.11. Discuss the evidence for shell structure in the atomic nucleus. Indicate how closed shells for proton and neutron numbers of 2, 8, 20, 28, 50 can be explained. What are the other 'magic numbers'?

Deduce from the shell model the spins and parities of the ground states of the following nuclei, stating any assumptions you make: ${}^7_3\text{Li}$, ${}^{17}_8\text{O}$, ${}^{20}_{10}\text{Ne}$, ${}^{27}_{13}\text{Al}$, ${}^{14}_7\text{N}$, ${}^{39}_{19}\text{K}$, ${}^{41}_{21}\text{Sc}$.

2 Problems 2

Core questions

2.1. What is meant by the 'cross section' and the 'differential cross section'?

Consider classical Rutherford scattering of a particle with mass m and initial speed v_0 from a potential

$$V(r) = \frac{\alpha}{r}$$

a) Show from geometry that the change in momentum is given by

$$|\Delta\mathbf{p}| = 2p \sin(\Theta/2).$$

b) Considering the symmetry of the problem, show that

$$bv_0 = r^2 \frac{d\theta}{dt}$$

where b is the impact parameter, \mathbf{r} is the location of the particle from the origin and θ is the angle $\angle(\mathbf{r}, \mathbf{r}^*)$ where \mathbf{r}^* is the point of closest approach.

c) Starting from Newton's second law show that

$$|\Delta\mathbf{p}| = \frac{2\alpha}{v_0 b} \cos\left(\frac{\Theta}{2}\right).$$

d) Show that the scattering angle Θ is given by

$$\tan(\Theta/2) = \frac{\alpha}{2bT} \tag{1}$$

where b is the impact parameter (the closest distance of the projectile to the nucleus if it were to be undeflected) and $T = \frac{p^2}{2m}$ is the initial kinetic energy.

e) Calculate the Rutherford scattering cross section σ for scattering of projectiles by angles greater than Θ_{\min} .

f) Show that the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{16} \left(\frac{\alpha}{T}\right)^2 \frac{1}{\sin^4(\Theta/2)}$$

where T is the kinetic energy of the particle.

Why is it not possible to calculate the total cross section for this reaction?

2.2. The $J^P = \frac{3}{2}^+$ decuplet contains the following baryons:

$$\begin{array}{cccc} \Delta^- & \Delta^0 & \Delta^+ & \Delta^{++} \\ \Sigma^{*-} & \Sigma^{*0} & \Sigma^{*+} & \\ \Xi^{*-} & \Xi^{*0} & & \\ \Omega^- & & & \end{array}$$

What is the quark content of each of these baryons?

The constituent quarks have no relative orbital angular momentum. How does the Ω^- baryon state behave under exchange of any pair of quarks? Explain how this is achieved in terms of the space, spin, flavour and colour parts of the state vector.

Account for the absence of sss , ddd and uuu states in the $J^P = \frac{1}{2}^+$ octet.

What is meant by *quark confinement*?

2.3. Write down the valence quark content for each of the different particles in the reactions below and check that the conservation laws of electric charge, flavour, strangeness and baryon number are satisfied throughout.

$$\begin{array}{ll} (1) & \pi^- + p \rightarrow K^0 + \Lambda \\ (2) & K^- + p \rightarrow K^0 + \Xi^0 \\ (3) & \Xi^- + p \rightarrow \Lambda + \Lambda \\ (4) & K^- + p \rightarrow K^+ + K^0 + \Omega^- \end{array}$$

Draw a quark flow diagram for the last reaction.

2.4. Consider the decay of the ρ^0 meson ($J^P = 1^-$) in the following decay modes:

a) $\rho^0 \rightarrow \pi^0 + \gamma$

b) $\rho^0 \rightarrow \pi^+ + \pi^-$

c) $\rho^0 \rightarrow \pi^0 + \pi^0$

For case (b) and (c), draw a diagram to show the quark flow.

Consider the symmetry of the wave-function required for $\pi^0 + \pi^0$ and explain why this decay mode is forbidden.

From consideration of the relative strength of the different fundamental forces, determine which of the other two decay modes will dominate.

2.5. A wave function is modelled as the sum of the incoming plane wave and an outgoing (scattered) spherical wave,

$$\langle \mathbf{x} | \Psi^{(+)} \rangle = A \left[e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{r} f(\mathbf{k}', \mathbf{k}) \right].$$

Calculate the flux associated with the plane wave and the spherical wave separately. Hence show that the cross section into solid angle $d\Omega$ is

$$\frac{d\sigma}{d\Omega} = |f(\mathbf{k}', \mathbf{k})|^2.$$

justifying any assumptions you make.

[Hint: Remember that the flux is given by $\frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*)$.]

2.6. In the Born approximation, the scattering amplitude is given by

$$f^{(1)}(\mathbf{k}', \mathbf{k}) = -\frac{1}{4\pi} (2m)(2\pi)^3 \langle \mathbf{k}' | V | \mathbf{k} \rangle$$

Explain the terms in this equation, and state the conditions for which it is valid.

b) The Yukawa potential is given by

$$V(r) = \frac{g^2}{4\pi} \frac{e^{-\mu r}}{r}.$$

Show that for this potential

$$\langle \mathbf{k}' | V | \mathbf{k} \rangle = \frac{g^2}{4\pi} \frac{1}{(2\pi)^3} \int d^3x e^{-i\Delta\mathbf{k}\cdot\mathbf{x}} \frac{e^{-\mu r}}{r}$$

where $\Delta\mathbf{k} = \mathbf{k}' - \mathbf{k}$.

c) Hence show that

$$|\langle \mathbf{k}' | V | \mathbf{k} \rangle| = \frac{g^2}{(2\pi)^3} \frac{1}{q^2 + \mu^2}$$

where $q = |\Delta \mathbf{k}|$.

d) Show that within the Born approximation

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{(4\pi)^2} \frac{4m^2}{[2k^2(1 - \cos \Theta) + \mu^2]^2} \quad (2)$$

where Θ is the scattering angle.

e) Find the total cross section for the case when $\mu \neq 0$.

f) When $\mu \rightarrow 0$, $V(r) \propto 1/r$. Compare the differential cross section (2) to the classical Rutherford scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{16} \left(\frac{\alpha}{T} \right)^2 \frac{1}{\sin^4(\Theta/2)}.$$

What must be the relationship between g and α for Born approximation to reproduce the classical Rutherford formula for electron–proton scattering?

2.7. Consider the scattering of an electron from a nucleus with extended spherical charge density $N(|\mathbf{x}''|)$ which is normalised such that $\int d^3x'' N(|\mathbf{x}''|) = 1$. The potential at any point is then

$$V(x') = zZ\alpha \int d^3x'' \frac{N(|\mathbf{x}''|)}{|\mathbf{x}' - \mathbf{x}''|},$$

where Z and z are the charges of the nucleus and the projectile respectively.

a) By expanding in the position basis, and defining a new variable $\mathbf{X} = \mathbf{x}' - \mathbf{x}''$, show that the Born approximation to the scattering amplitude can be expressed in the form

$$f(\mathbf{k}', \mathbf{k}) = f(\mathbf{k}', \mathbf{k})_{\text{point}} \times F_{\text{nucl}}(\Delta \mathbf{k})$$

where $f(\mathbf{k}', \mathbf{k})_{\text{point}}$ is the scattering amplitude from a point charge, the *nuclear form factor*

$$F_{\text{nucl}}(\Delta \mathbf{k}) = \int d^3x' e^{-i\Delta \mathbf{k} \cdot \mathbf{x}'} N(|\mathbf{x}'|)$$

is the 3D Fourier transform of the charge density distribution, and $\Delta \mathbf{k}$ is the change in momentum of the projectile.

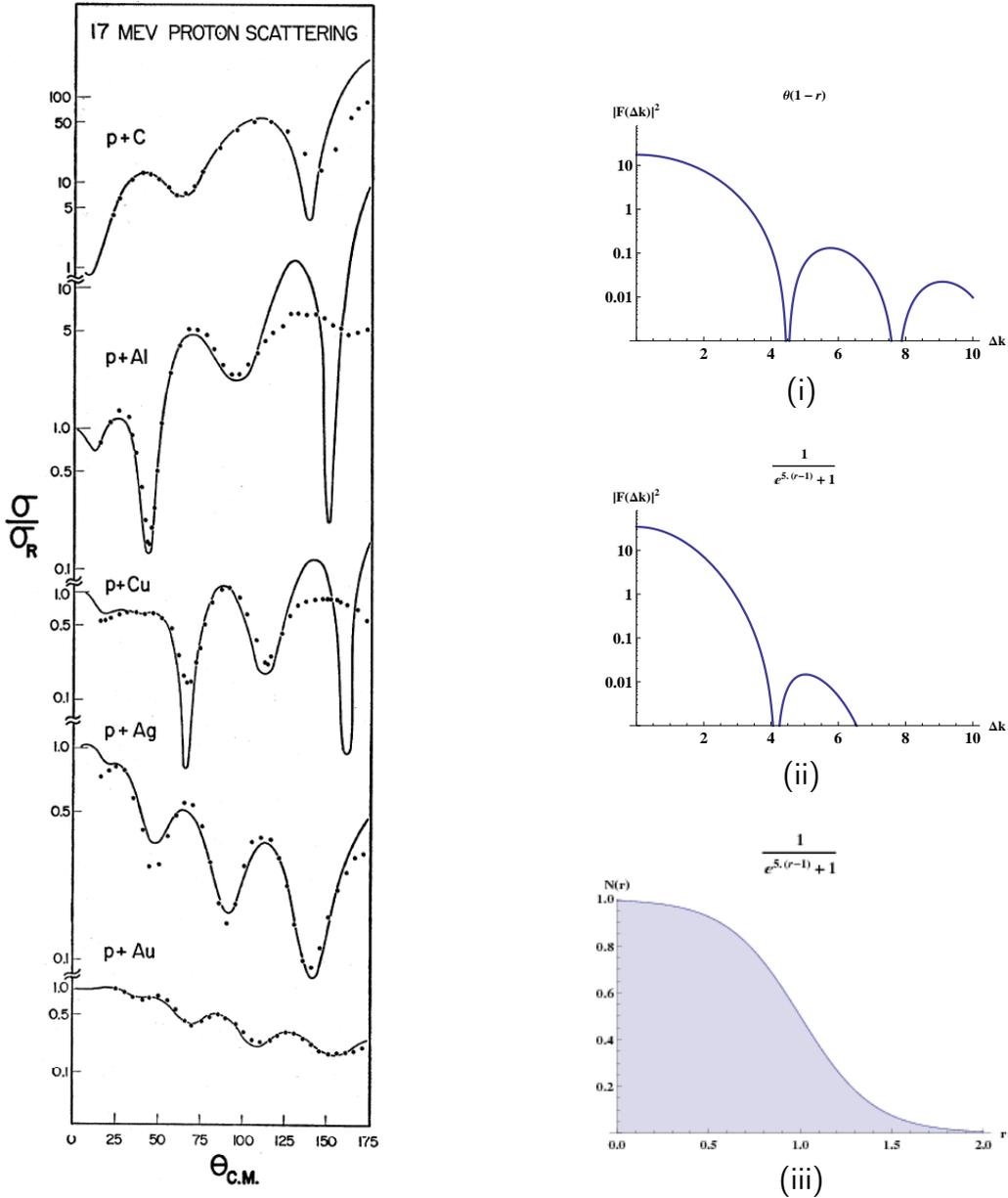


Figure 1: **(LHS)** Scattering cross section for K.E. = 17 MeV protons normalized to the Rutherford scattering cross section (from [Glassgold(1958)]). **(RHS)** Examples of nuclear form factors $|F(|\Delta \mathbf{k}|)|^2$ for different charge density functions: (i) uniform unit sphere; (ii) Saxon-Woods $\rho(r) \propto [1 + \exp((r - R)/a)]^{-1}$ with $R = 1$, $a = 0.2$. The corresponding Saxon-Woods charge density [Woods and Saxon(1954)] is shown in (iii).

Hence show that

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} |F_{\text{nucl}}(\Delta\mathbf{k})|^2.$$

b) Consider the example form factors in Figure 1 (i) and (ii). How would these form factors scale along the Δk -axis if the radius r of the corresponding sphere of charge was doubled? By relating the momentum transfer to the scattering angle, use the data to estimate the size of the silver nucleus. Compare to the expectation for an incompressible nucleus, $r = r_0 A^{\frac{1}{3}}$ with $r_0 = 1.25$ fm.

c) How might one accelerate protons to kinetic energy of 17 MeV, and subsequently detect the scattered protons experimentally?

2.8. The cross section for the production of γ -rays by neutrons incident on a certain nucleus (N, Z) is dominated by a resonance and given by the Breit-Wigner formula,

$$\sigma(n, \gamma) = \frac{\pi}{k^2} \frac{\Gamma_n \Gamma_\gamma}{(E - E_0)^2 + \Gamma^2/4}. \quad (3)$$

a) Define the symbols in this formula and explain the physical principles that underlie it, and the conditions under which it applies.

b) What are the mass and lifetime of the resonant state?

c) All spin effects have been ignored in (3). How would the formula differ if spins are included?

d) On the same plot draw how the inelastic (n, γ) and elastic (n, n) cross sections would behave close to the resonance, when $\Gamma_\gamma = 4\Gamma_n$, labeling important quantities including the peak cross section values.

[In (d) you may assume that resonant scattering dominates both cross-sections, and that decays other than to n and γ are negligible.]

2.9. The cross section for the reaction $\pi^- p \rightarrow \pi^0 n$ shows a prominent peak when measured as a function of the π^- energy. The peak corresponds to the Δ resonance which has a mass of 1232 MeV, with $\Gamma = 120$ MeV. The partial widths for the incoming and outgoing states are $\Gamma_i = 40$ MeV, and $\Gamma_f = 80$ MeV respectively for this reaction.

At what pion beam energy will the cross section be maximal for a stationary proton?

Describe and explain the similarities and differences you would expect between the cross section for $\pi^- p \rightarrow \pi^0 n$ and the one for $\pi^- p \rightarrow \pi^- p$, for centre-of-mass energies not far from 1.2 GeV. Giving values for the variables in the Breit-Wigner formula where possible. Use quark-flow diagrams to explain what is happening.

By considering the quark content of the intermediate states, discuss whether you would expect similar peaks in the cross sections for the reactions (a) $K^- p \rightarrow$ products and (b) $K^+ p \rightarrow$ products.

Optional questions

2.10. Consider the Hamiltonian $H = H_0 + V$, where H_0 is the free-particle Hamiltonian, and V is some localised potential. Let the ket $|\phi\rangle$ represent an eigenstate of H_0 , and the ket $|\psi\rangle$ represent an eigenstate of H which shares the same energy eigenvalue as the $|\phi\rangle$ in the limit $\epsilon \rightarrow 0$.

Show that the Lippmann-Schwinger equation

$$|\psi^{(\pm)}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |\psi^{(\pm)}\rangle.$$

is consistent with the states defined above by multiplying it by the operator $(E - H_0 \pm i\epsilon)$ and taking the limit $\epsilon \rightarrow 0$.

2.11. Show that the Green's function

$$G_{\pm}(\mathbf{x}, \mathbf{x}') \equiv \frac{\hbar^2}{2m} \left\langle \mathbf{x} \left| \frac{1}{E - H_0 \pm i\epsilon} \right| \mathbf{x}' \right\rangle$$

can be written

$$G_{\pm}(\mathbf{x}, \mathbf{x}') = \frac{i}{4\pi^2 \Delta} \int_0^{\infty} dq q \frac{e^{iq\Delta} - e^{-iq\Delta}}{q^2 - k^2 \mp i\epsilon} \quad (4)$$

where $E = \frac{\hbar^2 k^2}{2m}$ and $\Delta = |\mathbf{x} - \mathbf{x}'|$.

[Hint: start by inserting identity operators $\int d^3 p' |\mathbf{p}'\rangle \langle \mathbf{p}'|$ and $\int d^3 p'' |\mathbf{p}''\rangle \langle \mathbf{p}''|$ on each side of the operator, and changing $\hat{H}_0 \rightarrow \hat{\mathbf{p}}^2/2m$.]

2.12. If you have done the course on functions of a complex variable, finish the integral in (4) using appropriate contour integrals to obtain

$$G_{\pm}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{e^{\pm ik\Delta}}{\Delta}.$$

2.13. The pions can be represented in an isospin triplet ($I = 1$) while the nucleons form an isospin doublet ($I = \frac{1}{2}$),

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p \\ n \end{pmatrix}$$

while the Δ series of resonances have $I = \frac{3}{2}$.

By assuming that the isospin operators I, I_3, I_{\pm} obey the same algebra as the quantum mechanical angular momentum operators J, J_z, J_{\pm} , explain why the ratio of $\Gamma_i/\Gamma_f \approx \frac{1}{2}$ was found in a previous question.

[Hint: you will need the Clebsch-Gordon coefficients for $\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle$ for $J = \frac{3}{2}$, $j_1 = 1$, $j_2 = \frac{1}{2}$, $M = -\frac{1}{2}$.]

2.14. A proton is travelling through a material and scattering the electrons in the material.

a) Express the scattering angle in terms of the impact parameter b , the reduced mass μ , the relative speed v , and the scattering angle in the ZMF. Hence show that the momentum transfer is

$$q = \frac{2\mu v}{\sqrt{1+z^2}}$$

where $z = b\mu v^2/\alpha$.

b) Write down the energy given to an electron for a collision for a given impact parameter b . Integrate this up with area element $2\pi b db$ to show that the average energy lost by the projectile per distance travelled is

$$-\left\langle \frac{dE}{dx} \right\rangle = \frac{4\pi n_e \alpha^2}{m_e v^2} \int_{z_{\min}}^{z_{\max}} \frac{z dz}{1+z^2},$$

where n_e is the number density of electrons.

[Hint: recycle results from the Rutherford scattering question.]

2.15. a) A projectile travels through a medium of thickness x with n targets per unit volume. Show that the fraction absorbed or deflected by the medium is

$$P_{\text{absorb}}(x) = 1 - e^{-n\sigma x},$$

where σ is the absorption cross section.

b) Estimate, stating any assumptions you make, the thickness of lead that would be required to have a 50% chance of stopping a 2.3 MeV neutrino coming from a solar nuclear fusion reaction.

The cross section for the scattering of a neutrino from a stationary target is approximately

$$\sigma_{\text{tot}} = 2\pi G_F^2 \frac{4\pi p_{\text{CM}}^2}{(2\pi)^3} \frac{dp_{\text{CM}}}{dE_{\text{CM}}}$$

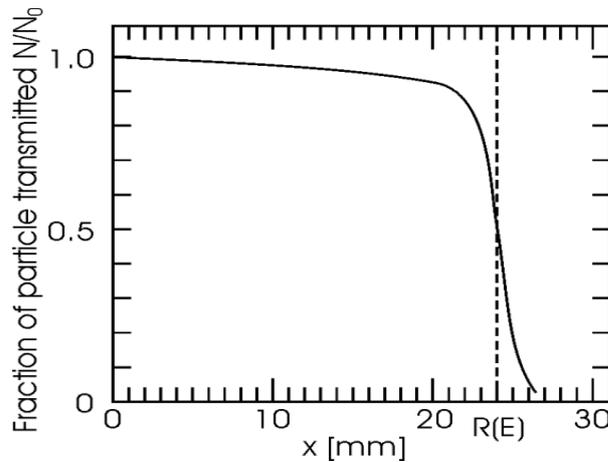
where E_{CM} is the centre-of-mass energy of the system, and p_{CM} is the momentum of the neutrino in the centre-of-mass frame.

c) Justify the form of this expression.

d) Explain how Figure 2 supports a model in which the proton contains point-like constituents.

[The density of lead is about 11.3 g cm^{-3} . Some data for the cross section of neutrinos ν scattering from nucleons – meaning protons or neutrons – are shown in Figure 2.]

2.16. The figure shows the fraction N/N_0 transmitted when protons of kinetic energy $E = 140 \text{ MeV}$ impinge as a collimated beam on sheets of copper of various thicknesses x . By considering the 2-body kinematics of proton–electron collisions and proton–nucleus collisions, account for the attenuation for values of x between 0 and 20 mm, and for the sudden change in behaviour around the value of x marked $R(E)$.



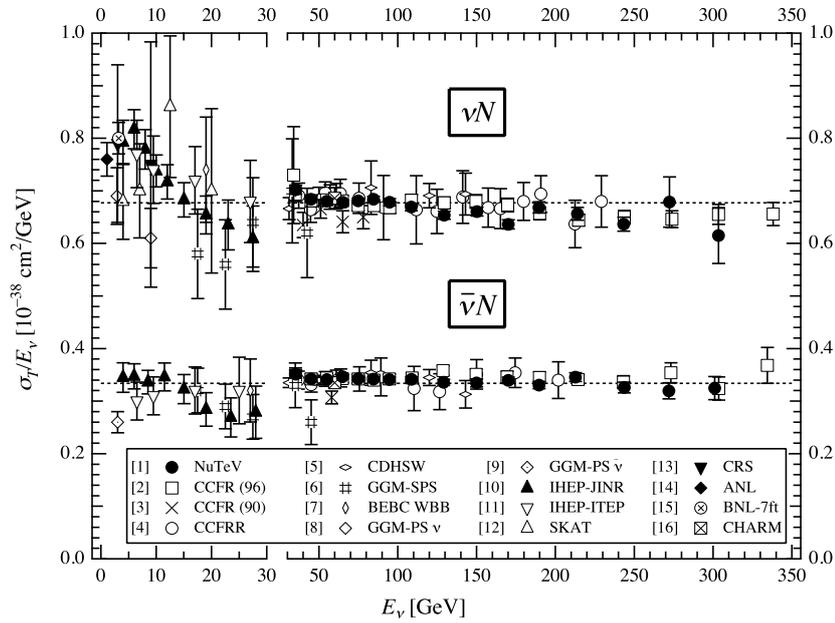


Figure 2: $\sigma_{\text{tot}}/E_{\nu}$ for neutrino-nucleon νN (and anti-neutrino-nucleon $\bar{\nu} N$) interactions as a function of neutrino energy. From [PDG(2008)].

Results similar to the figure were obtained for protons of $E = 100 \text{ MeV}$, except that in this case a value of $R(E) = 14 \text{ mm}$ was obtained. Offer a brief explanation for the change in $R(E)$. What is the relative size of the nuclear scattering cross section σ_{Nucel} in copper compared to the geometric cross section?

[The density of copper is 8.9 g cm^{-3} , and it has relative atomic mass 63.5. You may assume that the nuclear radius is given by $r = r_0 A^{1/3}$ with $r_0 = 1.25 \text{ fm}$]

3 Problems 3

Core questions

3.1. The Klein-Gordon equation,

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \varphi(\mathbf{r}, t) = 0. \quad (5)$$

is the relativistic wave equation for spin-0 particles.

a) Show that

$$\varphi = \frac{e^{-\mu r}}{r}$$

is a valid static-field solution.

b) Show that another possible solution to the Klein-Gordon equation is:

$$\phi(X) = A \exp [iP \cdot X]$$

Where P and X are the momentum and position four-vectors respectively. What restrictions does (5) place on the components of P?

What are the physical interpretation of these solutions?

3.2. Draw all the lowest order electromagnetic Feynman diagram(s) for the following processes:

- a) $e^- + e^+ \longrightarrow e^- + e^+$
- b) $e^- + e^- \longrightarrow e^- + e^-$
- c) $e^- + e^- \longrightarrow e^- + e^- + \mu^+ + \mu^-$
- d) $\gamma \longrightarrow e^+e^-$ in the presence of matter
- d) $\gamma + \gamma \longrightarrow \gamma + \gamma$

3.3. Why does the ratio

$$\frac{\sigma(e^+ + e^- \rightarrow \mu^+ + \mu^-)}{\sigma(e^+ + e^- \rightarrow \tau^+ + \tau^-)}$$

tend to unity at high energies? Would you expect the same to be true for

$$\frac{\sigma(e^+ + e^- \rightarrow \mu^+ + \mu^-)}{\sigma(e^+ + e^- \rightarrow e^+ + e^-)} ?$$

3.4. Draw leading order electromagnetic Feynman diagrams for the processes

$$e^+ + e^- \rightarrow \mu^+ + \mu^- \quad \text{and} \quad e^+ + e^- \rightarrow q + \bar{q}.$$

How do the vertex and propagator factors compare?

Figure 3 shows the ratio of the cross sections for the process of electron–positron annihilation to hadrons, and the corresponding cross section to the muon–antimuon final state as a function of \sqrt{s} , the centre-of-mass energy.

Considering the number of quarks that can be created at particular centre-of-mass energy, what values of R would you expect for centre-of-mass energy in the range $2 \text{ GeV} < \sqrt{s} < 20 \text{ GeV}$? How do your predictions match the data? How do these measurements support the existence of quark colour?

What is causing the sharp peaks in σ and R at centre-of-mass energy of $\approx 3 \text{ GeV}$, 10 GeV , and 100 GeV ?

3.5. At the HERA collider 27 GeV positrons collided with 920 GeV protons. Why can these collisions can be considered to be due to positrons scattering off the quarks in the protons?

For these collisions draw **one** example of a Feynman diagram for **each** of the cases of weak charged-current, weak neutral-current and electromagnetic interaction.

Calculate the center-of-mass energy of the quark–positron system assuming that the 4-momentum of the quark P_q can be represented as a fixed fraction f of the proton 4-momentum P_p , in the approximation where both particles are massless.

What is the highest-mass particle that can be produced in such a collision in the approximation that a quark carries about $\frac{1}{3}$ of the proton momentum?

How does the propagator for the weak charged current and electromagnetic interactions vary with 4-momentum transfer P^2 ? Hence explain the fact that at low values of the momentum transfer it is found that the ratio of weak interactions to electromagnetic interactions is very small whereas at very high values it is found that the ratio is of the order of unity.

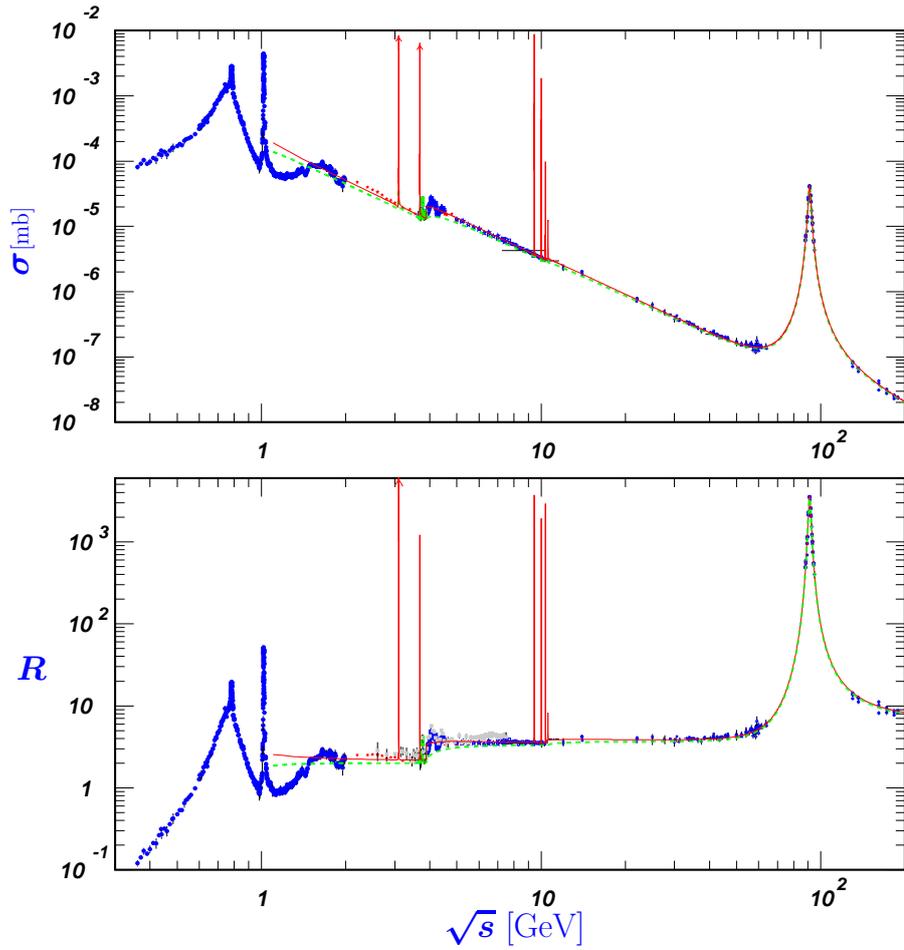


Figure 3: The cross section $\sigma(e^+e^- \rightarrow \text{hadrons})$ and the ratio of cross sections $R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$ as a function of the center of mass energy \sqrt{s} .

3.6. The J/Ψ has mass 3097 MeV, width 87 keV and equal branching ratios of 6% to $e^+ + e^-$ and $\mu^+ + \mu^-$ final states. What would you expect for these branching ratios if the J/Ψ decayed only electromagnetically? What does this tell you about the “strength” of the strong interaction in this decay? For comparison, the Ψ'' has mass 3770 MeV, width 24 MeV, but branching ratio to $e^+ + e^-$ of 10^{-5} .

Draw diagrams for the decays $D^0 \rightarrow K^- + \pi^+$ and $D^0 \rightarrow K^- + e^+ + \nu_e$. Disregarding the differences in the 2-body and 3-body density of states factors, what do you expect for the relative rates of these decays?

Optional questions

3.7. By conserving momentum at each vertex in the centre-of-mass frame (or otherwise) determine whether the propagator momentum is space-like ($P^2 < m^2$) or time-like ($P^2 > m^2$) for (i) $e^- + \mu^- \rightarrow e^- + \mu^-$ and for (ii) $e^+e^- \rightarrow \mu^+ + \mu^-$.

3.8. Draw Feynman diagrams showing a significant decay mode of each of the following particles:

- a) π^0 meson
- b) π^+ meson
- c) μ^-
- d) τ^- to a final state containing hadrons
- e) K^0
- f) top quark

4 Problems 4

Core questions

4.1. a) Draw Feynman diagrams for the production of W^\pm bosons being produced in a $p\bar{p}$ collider. If the W^+ boson is close to its Breit-Wigner peak, what possible decays may it have? (Which final states are kinematically accessible?)

b) What fraction of W^+ decays would you expect to produce positrons?

c) Suggest why the W was discovered in the leptonic rather than hadronic decay channels.

d) How could the outgoing (anti-)electron momentum be determined? How might the components of the neutrino momentum perpendicular to the beam be determined?

4.2. Write down Feynman diagrams for the decays of the muon and the tau lepton. Are hadronic decays possible? By considering the propagator factor in each case explain why one might expect on dimensional grounds that lifetimes should be in the ratio

$$\frac{\Gamma(\tau^- \rightarrow e^- + \nu + \bar{\nu})}{\Gamma(\mu^- \rightarrow e^- + \nu + \bar{\nu})} = \left(\frac{m_\tau}{m_\mu}\right)^5.$$

Using the following data

$$\begin{aligned} m_\tau &= 1777.0 \text{ MeV} & \tau_\tau &= 2.91 \times 10^{-13} \text{ s} \\ m_\mu &= 105.66 \text{ MeV} & \tau_\mu &= 2.197 \times 10^{-6} \text{ s} \\ BR(\tau^- \rightarrow e^- \nu + \bar{\nu}) &= 17.8\% \end{aligned}$$

test this prediction.

4.3. Which of the Standard Model fermions couple to the Z^0 boson? To which final states may a Z^0 boson decay?

Explain why for the Z^0 the sum of the partial widths to the observed states (e^+e^- , $\mu^+\mu^-$, $\tau^+\tau^-$, hadrons) does not equal the FWHM of the Breit-Wigner.

By referring to the properties of the Breit-Wigner formula, suggest how the LEP e^+e^- collider operating at centre-of-mass energies in the range 80 GeV to 100 GeV could have inferred that there are three neutrino species with $m_\nu < m_Z/2$, even though the detectors were unable to detect those neutrinos.

4.4. Consider a model with two neutrino mass eigenstates ν_2 and ν_3 with masses m_2 and m_3 and energies E_2 and E_3 , mixed so that

$$\begin{aligned} |\nu_\mu\rangle &= |\nu_2\rangle \cos\theta + |\nu_3\rangle \sin\theta \\ |\nu_\tau\rangle &= -|\nu_2\rangle \sin\theta + |\nu_3\rangle \cos\theta. \end{aligned}$$

Consider a beam of neutrinos created from from $\pi^- \rightarrow \mu^- \nu$ decays. Show that the observed flux of muon neutrinos observed at a distance L from such a source is

$$J(L) = J(L=0) \times \left[1 - \sin^2(2\theta) \sin^2 \left\{ \left(\frac{E_3 - E_2}{2\hbar} \right) \frac{L}{c} \right\} \right].$$

If m_2 and m_3 are much less than the neutrino momentum, $|\mathbf{p}|$, show that

$$|\nu_\mu(L)|^2 \approx |\nu_\mu(0)|^2 \times \left[1 - \sin^2(2\theta) \sin^2 \left\{ A (m_2^2 - m_3^2) \frac{L}{|\mathbf{p}|} \right\} \right].$$

What is the first length L^* at which the ν_μ detection rate is at a minimum?

If a range of neutrino energies are present what will be the ratio of the rate (per neutrino) of $\nu + n \rightarrow \mu^- + p$ for $L \ll L^*$ and $L \gg L^*$.

What would be the corresponding ratio for neutral-current scattering?

Solar neutrinos emitted in $p-p$ fusion have been detected via the processes

$$\begin{aligned} \nu_e + d &\rightarrow p + p + e^- \quad \text{and,} \\ \nu_x + d &\rightarrow p + n + \nu_x. \end{aligned}$$

Suggest why the charged-current reaction showed only a third of the neutrino flux of the neutral-current reaction.

4.5. Draw all leading Feynman diagrams for the following processes:

a) $\nu_\mu + n \rightarrow p + \mu^-$

b) $\nu_\mu + e^- \rightarrow \nu_\mu + e^-$

c) $\bar{\nu}_e + e^- \rightarrow \bar{\nu}_e + e^-$

d) $\bar{\nu}_e + p \rightarrow e^+ + n$

For d) the cross section takes the form

$$\sigma = \frac{2\pi}{\hbar} \frac{1}{c} G_F^2 \frac{4\pi}{(2\pi\hbar)^3} \frac{E_\nu^2}{c^3}$$

Justify this expression in terms of the Golden rule and the Fermi four-fermion theory.

Antineutrinos are incident on a stationary proton target. At what $\bar{\nu}_e$ energy would you expect the above formula to break down?

4.6. Write brief notes on:

- a) The evidence that there are three and only three families of quarks and leptons.
- b) The Cabibbo angle and quark mixing.
- c) The evidence for confinement of quarks in hadrons.

4.7. How does helicity of a state change on application of the parity operator (which reverses the coordinate axes: $\mathbf{x} \xrightarrow{\mathbb{P}} -\mathbf{x}$)?

If $|\Phi\rangle$ is an eigenstate of the parity operator, what can be said about the parity of the state $(1 + a \mathbf{S} \cdot \mathbf{p})|\Phi\rangle$?

^{60}Co nuclei ($J^P = 5^+$) are polarised by immersing them at low temperature in a magnetic field. When these nuclei β decay to ^{60}Ni (4^+) more electrons are emitted opposite to the aligning B field than along it¹. Explain carefully why this demonstrates parity violation in the weak interaction.

Justify the direction of the parity-violating effect.

4.8. Neutrino and anti-neutrino states have only ever been observed with the following eigenvalues of the helicity operator respectively:

$$\nu : -\frac{1}{2}\hbar \quad \bar{\nu} : +\frac{1}{2}\hbar$$

¹Reported in [Wu et al.(1957)].

What values of the projection operators

$$\mathcal{P}_{\pm} = \frac{1}{2} \left(1 \pm \boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} \right)$$

must be present in weak processes for (anti-)neutrinos reactions?

What is the implication for parity in the weak interaction?

4.9. Write down Feynman diagrams and for the processes $\pi^+ \rightarrow e^+ \nu_e$ and $\pi^+ \rightarrow \mu^+ \nu_{\mu}$. By considering the helicities of the final state particles, suggest why the π^+ ($J^P = 0^-$) decays dominantly to $\mu^+ \nu_{\mu}$.

Optional questions

4.10. The Large Hadron Collider has been designed to accelerate counter-rotating beams of protons to energies of 7 TeV, and to collide those beams at a small number of interaction points.

a) Use dimensional analysis to estimate the smallest length scale which this machine could be used to resolve. How does this compare to the size of e.g. atoms, nuclei and protons?

b) The LHC beam pipe is evacuated to reduce loss of beam from collisions with gas molecules. If less than 5% of the beam protons are to be lost from collisions with gas nuclei over a ten hour run, estimate the maximum permissible number density of H gas atoms in the beam pipe.

c) The machine collides counter-rotating bunches of protons, each bunch having circular cross section with radius $17 \mu\text{m}$ (in the direction perpendicular to travel). How many protons are required in a bunch to have an average of ten interactions per bunch crossing?

d) If such bunches collide every 25 ns , what is the *luminosity* of the machine? (Express your answer in units of $\text{cm}^{-2} \text{s}^{-1}$.)

e) If the cross section for producing a Higgs Boson is 50 pb , how many will be made each second?

f) What is the kinetic energy of each bunch of protons in the LHC?

[Some data for proton-proton cross sections can be found in Figure 4.]

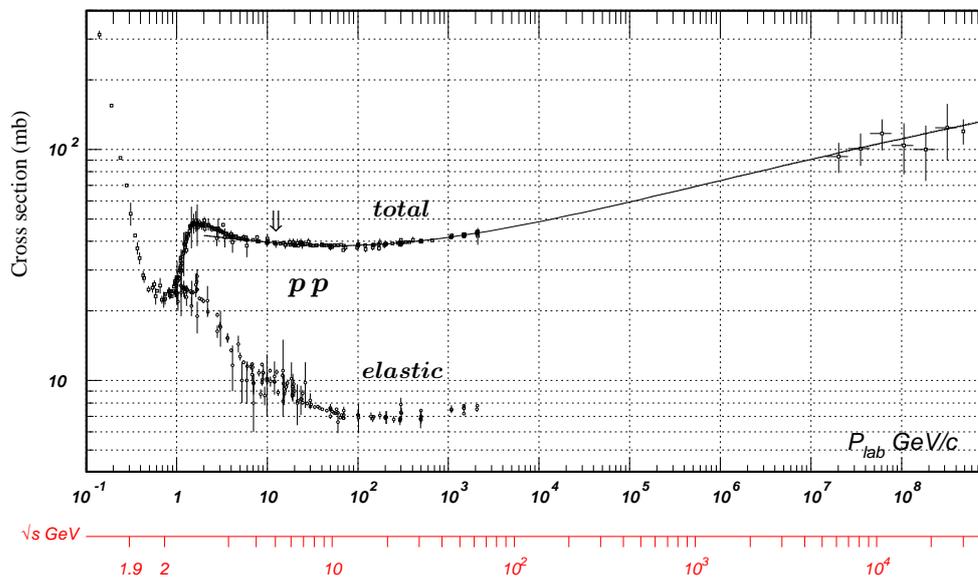


Figure 4: Proton–proton scattering cross section data as a function of laboratory momentum (upper scale) and of the centre-of-mass energy (lower scale). From [PDG(2008)].

5 Problems 5

Core questions

5.1. a) Why does ^{235}U fission with thermal neutrons whereas ^{238}U requires neutrons with energies of order MeV?

b) The fission of ^{235}U by thermal neutrons is asymmetric, the most probable mass numbers of fission fragments being 93 and 140. Use the semi-empirical mass formula to estimate the energy released in fission of $^{235}_{92}\text{U}$ and hence the mass of $^{235}_{92}\text{U}$ consumed each second in a 1 GW reactor.

c) In almost all uranium ores, the proportion of ^{235}U to ^{238}U is 0.0072. However, in certain samples from Oklo in the Gabon the proportion is 0.0044. Assuming that a natural fission reactor operated in the Gabon 2×10^9 years ago, estimate the total energy released from 1 kg of the then naturally occurring uranium. How might the hypothesis that ^{235}U was depleted by fission be tested?

[$t_{1/2}(^{238}\text{U}) = 4.5 \times 10^9$ years, $t_{1/2}(^{235}\text{U}) = 7.0 \times 10^8$ years.]

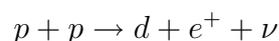
5.2. Write down the semi empirical mass formula. Which terms are responsible for the existence of a viable chain reaction of thermal-neutron-induced uranium fission? What distinguishes the isotopes of uranium that support such a reaction?

In the construction of a nuclear fission reactor an important role is often played by water, heavy water or graphite. Describe this role and explain why are these materials are suitable.

Why is the fissile material not completely mixed up with the moderator?

5.3. A neutron produced in a fission reaction is emitted with considerable energy. Discuss how the design of the reactor determines the competition between i) neutron absorption by sharp resonances; ii) neutron decay; iii) neutron energy transfer to the reactor media (and thence to turbines); iv) neutron absorption by resonances with high branching ratios to further fission. Most of these processes happen very fast indeed. How is it possible to control the reactor flux with a response time of seconds to minutes, or even longer?

5.4. a) Draw a diagram showing how the fusion process

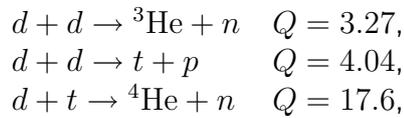


is related to the Fermi theory of beta decay.

b) Find the approximate height \mathcal{B} of the Coulomb barrier for pp fusion. To approximately what temperature would one need to heat hydrogen for pp fusion to overcome the Coulomb barrier?

c) If plasma at this temperature is to be magnetically confined what will be a typical Larmor radius (gyro-radius) for the deuterium ions in the magnetic field?

d) Given the following reactions and energy release (in MeV)



suggest *two* reasons why the artificial fusion reactors depend largely on the $d + t$ reaction.

e) Tritium has a half-life of about 12 years, and must be generated through reactions with both ${}^6\text{Li}$ and ${}^7\text{Li}$. Write down the form of these reactions, and explain why the ${}^7\text{Li}$ reaction is helpful even though it is endothermic.

[d means ${}^2\text{H}$ and t means ${}^3\text{H}$. You may assume the magnetic field strength is 13.5 Tesla, which is what has been proposed for the ITER Tokamak.]

5.5. Show that for a star in a state of hydrostatic equilibrium (with pressure balancing gravity), the pressure gradient is given by

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$

where m is the mass contained within the sphere of radius r . Hence show that the pressure at the center of a star satisfies

$$P_c = \int_0^M \frac{Gm \, dm}{4\pi r^4} > \int_0^M \frac{Gm \, dm}{4\pi R^4},$$

where R is the radius of the star.

Estimate the pressure and temperature at the the center of the sun.

5.6. The rate of thermonuclear fusion reactions is approximately proportional to

$$\exp\left(-\frac{2\pi Z_1 Z_2 \alpha c}{v}\right) \exp\left(-\frac{mv^2}{2kT}\right).$$

Sketch the form of this curve and explain the origin of these two terms.

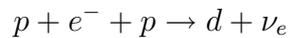
Find the value of v at which the rate is maximal. At what temperature should one run a Tokamak?

5.7. a) Assuming that the energy for the sun's luminosity is provided by the conversion of $4H \rightarrow {}^4\text{He}$, and that the neutrinos carry off only about 3 percent of the energy liberated how many neutrinos are liberated each second from the sun?

b) What neutrino flux would you expect to find at the Earth?

c) By what sequence of reactions do the above conversions dominantly proceed?

d) Why might the alternative rare process



be of interest when studying solar neutrinos from the earth?

[$4M({}^1\text{H}) - M({}^4\text{He}) = 26.73 \text{ MeV}$. The earth is on average about $1.50 \times 10^{11} \text{ m}$ from the sun, and is subject to a radiation flux of about 1.3 kW m^{-2} .]

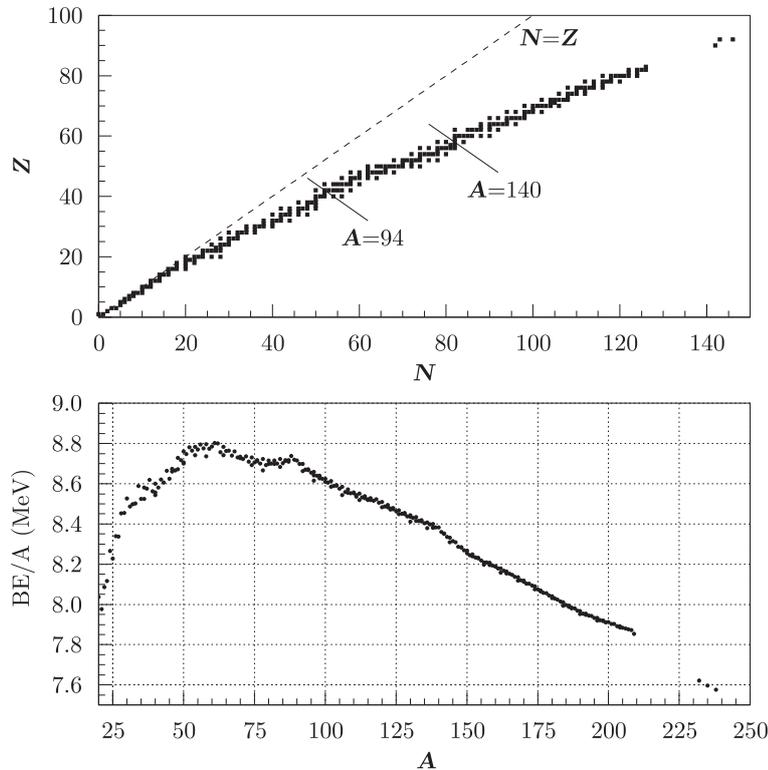
5.8. Give brief accounts of the methods of synthesis of:

a) ${}^{12}\text{C}$

b) ${}^{28}\text{Si}$

c) ${}^{56}\text{Fe}$

d) ${}^{238}\text{U}$



Optional questions

5.9. The two figures show properties of the ‘valley of stability’ of nuclei in the N - Z plane and the binding energy per nucleon versus mass number A , for nuclei with lifetimes greater than 10^8 years.

Using the data in the figures, estimate the energy released in the thermal-neutron induced fission of ^{235}U , given that the daughter nuclei tend to cluster asymmetrically around $A = 140$ and 94 . Where are the daughter nuclei in relation to the valley of stability and what happens to them subsequently? Compare this with the ^{238}U decay chain, which comprises eight α decays and six β decays to ^{206}Pb with a total release of 48.6 MeV and a lifetime of 2×10^{17} s.

There is a flow of heat from the Earth’s interior amounting to a total of the order of 35 TW. Much of this may be accounted for by decay of radioactive elements. Using the model above for a typical fission, estimate the rate of fissions needed to produce such a heat flow and the associated flux of neutrinos.

In a certain model of the Earth it is postulated that there is a self-sustaining fission reactor at the Earth's centre, fuelled by ^{235}U , contributing as much as 5 TW to the overall heat flow. If the 'geo-neutrino' flux could be measured, how might the 'core reactor' model be tested?

5.10. The CNO cycle proceeds in the following steps

	Q / MeV	rate	lifetime
$^{12}\text{C} + p \rightarrow ^{13}\text{N} + \gamma$	1.944	r_{12}	
$^{13}\text{N} \rightarrow ^{13}\text{C} + e^+ + \nu$	2.221		τ_N
$^{13}\text{C} + p \rightarrow ^{14}\text{N} + \gamma$	7.550	r_{13}	
$^{14}\text{N} + p \rightarrow ^{15}\text{O} + \gamma$	7.293	r_{14}	
$^{15}\text{O} \rightarrow ^{15}\text{N} + e^+ + \nu$	2.761		τ_O
$^{15}\text{N} + p \rightarrow ^{12}\text{C} + ^4\text{He}$	4.965	'fast'	

How much energy is released per cycle? Estimate the fraction of that energy in neutrinos.

The beta decay time constants are of the order of minutes. The shortest proton capture time is for ^{15}N which is of the order of years, whereas the other capture timescales are significantly longer. Write the coupled linear differential equations of (^{12}C , ^{13}C , ^{14}N) in the form

$$\frac{d}{dt}\mathbf{U} = \mathbf{M}\mathbf{U}$$

where \mathbf{M} depends on the r_x but not the τ_x .

Show that this set of coupled differential equations admits a solution

$$\mathbf{U}(t) = \sum_{i=1,3} a_i e^{\lambda_i t} \mathbf{u}_i$$

with

$$\lambda_1 = 0 \quad \lambda_{2,3} = \frac{1}{2}(-\Sigma \pm \Delta)$$

and find Σ and Δ . Why must the elements of $\mathbf{u}_{2,3}$ sum to zero?

Express the relative equilibrium abundances of ^{12}C , ^{13}C and ^{14}N in terms of the r_x , and show that the equilibrium fractions of the beta decaying isotopes satisfy equations of the form

$$B = \tau_B r_A A.$$

References

- [Glassgold(1958)] A. E. Glassgold, "Nuclear density distributions from proton scattering", *Rev. Mod. Phys.* **30** Apr (1958) 419–423.
- [Woods and Saxon(1954)] R. D. Woods and D. S. Saxon, "Diffuse surface optical model for nucleon-nuclei scattering", *Phys. Rev.* **95** Jul (1954) 577–578.
- [PDG(2008)] **Particle Data Group** Collaboration, PDG, "Review of particle physics", *Phys. Lett.* **B667** (2008) 1.
- [Wu et al.(1957)] C. S. Wu *et al.*, "Experimental test of parity conservation in beta decay", *Phys. Rev.* **105** Feb (1957) 1413–1415.