1 Hyperbolic motion

Hyperbolic motion holds an similar place in special relativity to that which circular motion holds in Newtonian mechanics.

	Circular motion	**Hyperbolic motion**
Coordinate vectors	$\vec{x} = (x, y)^T$	$\mathsf{X} = (ct, x)^T$
Parameter	$\theta = \omega t$	$\rho = a\tau$
Parametric equations of particle	$x = r_0 \cos\theta$ $y = r_0 \sin\theta$	$ct = x_0 \sinh\rho$ $x = x_0 \cosh\rho$
Ratio	$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = -\tan\theta$	$v/c = \frac{dx}{d\theta}\frac{d\theta}{d(ct)} = \tanh\rho$
Transformation matrix	$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	$L_\rho = \begin{pmatrix} \cosh\rho & -\sinh\rho \\ -\sinh\rho & \cosh\rho \end{pmatrix}$
Primed coordinates of particle	$\vec{x}' = R_\theta \vec{x} = \begin{pmatrix} r_0 \\ 0 \end{pmatrix}$	$\mathsf{X}' = L_\rho \mathsf{X} = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$

where the rapidity $\rho = \tanh^{-1}(v/c)$ plays an analogous role to the angle.

2 Non-constant acceleration

Even if the acceleration is not constant (in which case the path is not hyperbolic), rephrasing the problem in terms of rapidity can be helpful.

Example A rocket emits fuel at relativistic speed u relative to its instantaneous rest frame. Calculate the speed when fraction α of the mass remains.

In the instantaneous rest frame consider conservation laws when the ship loses a small amount of mass–energy δm which is emitted at speed u. Writing the new rest mass as $m - \delta m$, and being careful to remember that for relativistic problems the mass ejected is *not* the same as the change in mass of the ship, conservation of energy gives

$$mc^2 = E_u + \gamma_{\delta v}(m + \delta m)c^2$$

while conservation of momentum gives

$$0 = -p_u + \gamma_{\delta v}(m + \delta m)(\delta v).$$

To solve, recall that $p_u c = (u/c)E_u$, spot that to leading order in v the Lorentz factor $\gamma_{\delta v} \approx 1$, and ignore the quatratically small term $\delta m\, \delta v$, leading to

$$(-dm)u = m\, dv \tag{1}$$

It is most convenient to solve this in terms of the *rapidity*[1] $\rho = \tanh^{-1}(\beta)$, where $\beta = v/c$. To do so we need to calculate the change of variables in the local rest frame

$$\left.\frac{d\beta}{d\rho}\right|_{\beta=0} = \operatorname{sech}^2 \beta\big|_{\beta=0} = 1.$$

Thus the differential equation (1) becomes

$$-\frac{u}{c}\frac{dm}{m} = d\rho.$$

This is easily integrated giving

$$\rho = -\frac{u}{c}\log\alpha \tag{2}$$

The speed is given by

$$v/c = \tanh\rho = \frac{e^{\rho} - e^{-\rho}}{e^{\rho} + e^{-\rho}} = \frac{\alpha^{-u/c} - \alpha^{u/c}}{\alpha^{-u/c} + \alpha^{u/c}}$$

where in the second step we have used (2). The inverse hyperbolic tangent can be written

$$\rho = \tanh^{-1}(v/c) = \frac{1}{2}\log\frac{c+v}{c-v}$$

which can be equated with (2) giving

$$\alpha = \left(\frac{c-v}{c+v}\right)^{c/2u}.$$

[1]As we would solve a circular motion problem in terms of the angle