1 Thomas precession

1.1 As in the notes

As in section 5.7.1, let frame S'' be aligned with S' and move horizontally relative to S' at speed v, and let frame S' be aligned with S move vertically relative to S at speed u. Let point B be at the origin of S''. Let A be at the origin of S.

The components of velocity of B in S are given by the velocity addition formulae:

$$v_{Bx} = v_{B\perp} = \frac{v}{\gamma_u(1+0)} = v/\gamma_u$$
$$v_{By} = v_{B\parallel} = \frac{0+u}{1+0} = u$$

The angle θ of v_B relative to the x axis is given by

$$\tan\theta = \frac{v_{By}}{v_{Bx}} = \frac{u\gamma_u}{v} \tag{1}$$

The components of velocity of A in S'' are given by

$$v_{Ax} = v_{A\parallel} = \frac{0-v}{1+0} = -v$$
$$v_{Ay} = v_{A\perp} = \frac{-u}{\gamma_v(1+0)} = -u/\gamma_v$$

So the angle θ'' of v_A relative to the x'' axis is given by

$$\tan\theta'' = \frac{v_{Ay}}{v_{Ax}} = \frac{u}{v\gamma_v} \tag{2}$$
$$\neq \tan\theta$$

1.2 Perhaps clearer?

We can set up Thomas precession by making S the lab frame, S' the instantaneous rest frame of the particle at proper time τ and S'' the instantaneous rest frame at $\tau + \delta\tau$. The acceleration in the instantaneous rest frame gives that $\delta V_0 = a_0\delta\tau$. Then the speed of the particle $\vec{V_0}$ is vertical in S and $d\vec{V_0}$ is horizontal in S'. We can use the results (1) and (2) provided we see that

- the angles we want are the angles relative to the vertical (rather than the horizontal as above)

- we make the substitutions $v \to d\vec{V_0}$ and $u \to \vec{V_0}$

Now the angle of the relative velocity to the vertical in S is

$$\delta\theta = \frac{\delta V_0}{V_0 \gamma} \tag{3}$$

and the angle of that relative velocity to the vertical in S'' is (to leading order in δV_0)

$$\delta\theta'' = \frac{\delta V_0 \gamma_{\delta V_0}}{V_0} = \frac{\delta V_0}{V_0}$$

so

$$\delta\theta'' = \gamma \delta\theta$$

Let us define $\Delta\theta = \theta'' - \theta$, then the Thomas frequency is

$$\omega_T = \frac{\Delta\theta}{dt} = \frac{d\Delta\theta}{d\theta} \frac{d\theta}{d\tau} \frac{d\tau}{dt} = (\gamma - 1) \frac{a_0}{\gamma V_0} \frac{1}{\gamma} \tag{4}$$

Where we have used (3) and $a_0 = dV_0/d\tau$ to get $\frac{d\theta}{d\tau}$.

We want to translate a_0 into acceleration in the *lab* frame, which we can do by looking at the 4-vector for acceleration

$$\mathsf{A} = \left(\gamma\dot{\gamma}c, \gamma\dot{\gamma}\vec{u} + \gamma^2\vec{a}\right)$$

which shows that for $\vec{a}$ perpendicular to $\vec{v}$, the proper acceleration is related to the lab acceleration by

$$a_0 = \gamma^2 a.$$

Substituting this into (4) we have that

$$|\omega_T| = (\gamma_V - 1)\frac{a}{v}$$

which we can also write as

$$\vec{\omega}_T = \frac{\gamma^2}{1+\gamma} \frac{\vec{a}_{\times} \vec{v}}{c^2}$$

provided that we think carefully about the directions (as is done around Fig 5.7 of the notes).