Lecture 5: QED

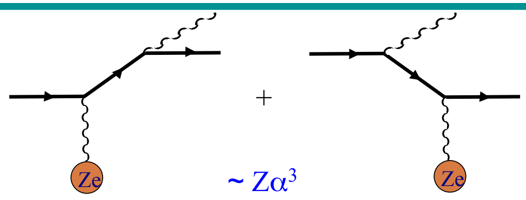
- The Bohr Magneton
- Off-Shell Electrons
- Vacuum Polarization
- Divergences, Running Coupling & Renormalization
- Yukawa Scattering and the Propagator

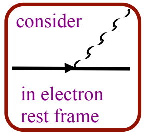
Useful Sections in Martin & Shaw:

Section 5.1, Section 5.2, Section 7.1.2

Bremsstrahlung

Internal electron line since real electron cannot emit a real photon and still conserve energy and momentum:





$m_e = m_e' + p_e'^2/2m_e' + E_\gamma$ (conservation of energy)

$|p'_e| = |p_\gamma| = E_\gamma$ (conservation of momentum)

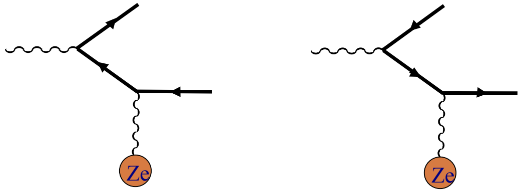
$\Rightarrow m_e - m_e' = E_\gamma^2/2m_e' + E_\gamma$

So, for a positive energy photon to be produced, the electron has to effectively "lose mass" $\Rightarrow$ virtual electron goes "off mass shell"

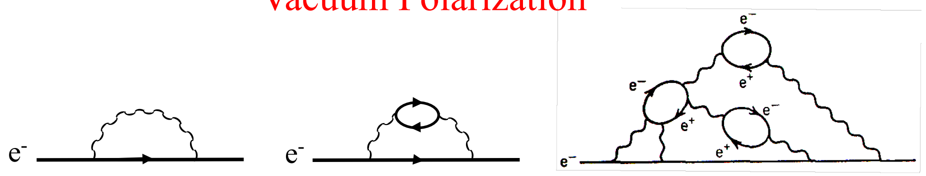
Pair Production

"twisted Bremsstrahlung"

Matrix element will be the same; Cross-section will only differ by a kinematic factor

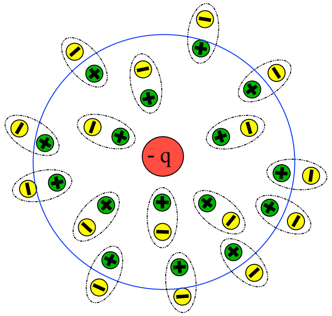


Vacuum Polarization



Thus, we never actually ever see a "bare" charge, only an effective charge shielded by polarized virtual electron/positron pairs. A larger charge **(or, equivalently, α)** will be seen in interactions involving a high momentum transfer as they probe closer to the central charge.

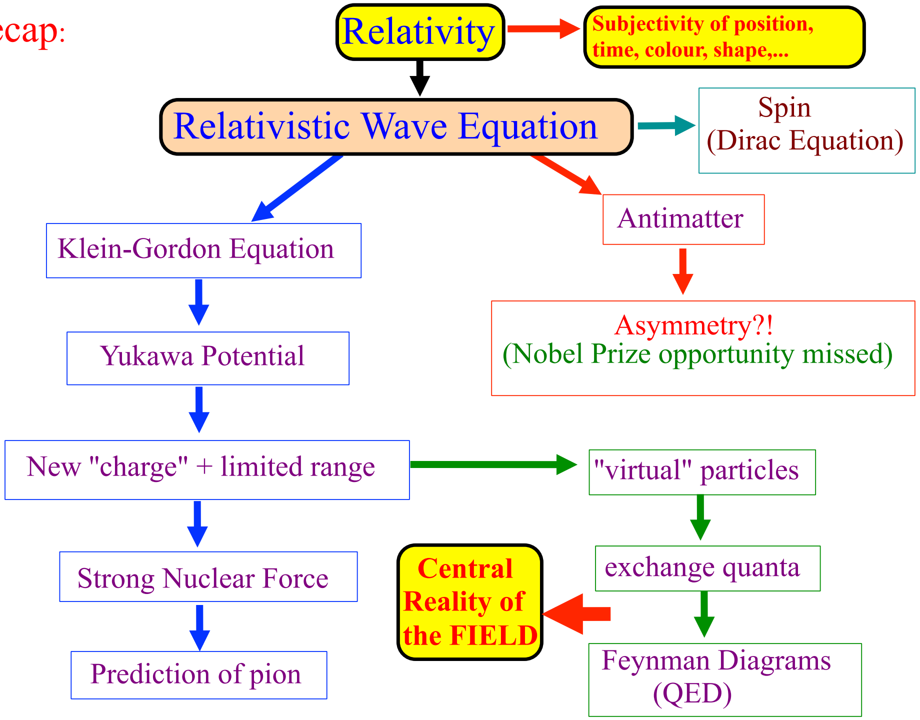
$\Rightarrow$ "running coupling constant"



In QED, the bare charge of an electron is actually infinite !!!

Note: due to the field-energy near an infinite charge, the bare mass of the electron ($E=mc^2$) is also infinite, but the effective mass is brought back into line by the virtual pairs again !!

Recap:



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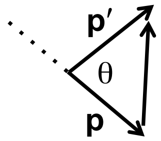
Consider the scattering of one nucleon by another via the Yukawa potential: $V(r) = \frac{-g^2}{4\pi r} e^{-Mcr/\hbar}$

In the CM, both nucleons have equal energy and equal & opposite momenta, $\mathbf{p}$*. Take the incoming state of the 1st nucleon (normalised per unit volume) to be:* $\Psi_o = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$

After scattering, the final state will have a new momentum vector, $\mathbf{p}'$*, where* $|\mathbf{p}| = |\mathbf{p}'|$ *:* $\Psi_f = e^{i\mathbf{p}'\cdot\mathbf{r}/\hbar}$

If θ *is the angle between* $\mathbf{p}$ *and* $\mathbf{p}'$*, write an expression for the momentum transfer,* $\mathbf{q}$ *, in terms of* $\mathbf{p}$ *.*



$\mathbf{q} = \mathbf{p}' - \mathbf{p}$ magnitude of q = 2p sin(θ/2)

Compute the matrix element for the transition.

$$<\Psi_f|V(r)|\Psi_o> = \int e^{-i\mathbf{p}'\cdot\mathbf{r}/\hbar}\left(\frac{-g^2}{4\pi r} e^{-Mcr/\hbar}\right) e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\; r^2\, dr\, d\phi\, d(\cos\theta)$$

$$= \frac{-g^2}{2}\int e^{-i\mathbf{q}\cdot\mathbf{r}/\hbar}\, e^{-Mcr/\hbar}\; r\, dr\, d(\cos\theta)$$

$$= \frac{-g^2}{2}\int e^{-iqr\cos\theta/\hbar}\, e^{-Mcr/\hbar}\; r\, dr\, d(\cos\theta)$$

$$= \frac{-i\hbar g^2}{2q}\int \left(e^{-(iq+Mc)r/\hbar} - e^{(iq-Mc)r/\hbar}\right) dr$$

$$= \frac{i\hbar^2 g^2}{2q}\left[\frac{e^{-(iq+Mc)r/\hbar}}{iq+Mc} + \frac{e^{(iq-Mc)r/\hbar}}{iq-Mc}\right]_{r=0}^{r=\infty}$$

$$= \frac{i\hbar^2 g^2}{2q}\left[\frac{e^{-(iq+Mc)r/\hbar}}{iq+Mc} + \frac{e^{(iq-Mc)r/\hbar}}{iq-Mc}\right]_{r=0}^{r=\infty}$$

This will go to zero as $r \to \infty$, so we're left with:

$$<\Psi_f|V(r)|\Psi_o> = \frac{-i\hbar^2 g^2}{2q}\left[\frac{1}{iq+Mc} + \frac{1}{iq-Mc}\right]$$

$$= \frac{-i\hbar^2 g^2}{2q}\left[\frac{iq-Mc \; + \; iq+Mc}{(iq+Mc)(iq-Mc)}\right]$$

$$= \frac{\hbar^2 g^2}{(q^2+M^2c^2)} \sim \frac{1}{q^2+M^2} \text{ (natural units)}$$

NOTE:

This is all really a semi-relativistic approximation!

Really, we want to consider a 4-momentum transfer

Recall that $\mathbf{P}^2 = E^2 - p^2$

so, $\frac{1}{q^2+M^2} \rightarrow \frac{1}{q_4^2 - M^2}$ (choosing appropriate sign convention)

Known as the "propagator" associated with the field characterised by an exchange particle of mass M

Show that the relation $dp/dE = 1/v$, where v is the velocity, holds for both relativistic and non-relativistic limits.

Classically:

$$E = p^2/2m$$
$$dE = (p/m)\,dp$$
$$dp/dE = (m/mv) = 1/v$$

Relativistically:

$$E^2 = p^2 + m^2$$
$$dp/dE = E/p = (\gamma m/\gamma mv) = 1/v$$

Also show that, for the present case: $p^2 d(\cos\theta) = -\tfrac{1}{2}\, dq^2$

$$q = 2p\sin(\theta/2)$$
$$q^2 = 4p^2\sin^2(\theta/2)$$

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = 1 - 2\sin^2(\theta/2)$$
$$\sin^2(\theta/2) = (1-\cos\theta)/2$$

$$q^2 = 2p^2(1-\cos\theta)$$
$$dq^2 = -2p^2\, d\cos\theta$$
$$p^2 d(\cos\theta) = -\tfrac{1}{2}\, dq^2$$

Now, from the definition of cross-section in terms of a rate, the relative velocities of the nucleons in the CM, and using Fermi's Golden Rule, derive the differential cross-section $d\sigma/d\Omega$.

$$\text{Rate} = \frac{v_{beam} N_{beam} N_{target}\, \sigma}{\text{Volume}}$$

in our case

$N_{beam} = N_{target} = 1$

$v_{beam} = 2v$ (relative velocity in CM)

Normalised Volume (= 1)

$$\text{so,}\quad d\sigma = \frac{1}{2v}\, d\text{Rate}$$

← given by FGR

$$d\sigma = \frac{1}{2v}\,\frac{2\pi}{\hbar}\,|\langle\Psi_f|V(r)|\Psi_o\rangle|^2\,\frac{1}{(2\pi\hbar)^3}\,\frac{p^2 dp\, d\phi\, d\cos\theta}{dE_{tot}}$$

$$= \frac{1}{2v}\,\frac{2\pi}{\hbar}\left(\frac{\hbar^2 g^2}{(q^2+M^2c^2)}\right)^2 \frac{1}{(2\pi\hbar)^3}\,\frac{1}{2v}\left(-\tfrac{1}{2}\, dq^2\right) d\phi$$

$$= \frac{\pi}{4v^2\hbar}\left(\frac{\hbar^2 g^2}{(q^2+M^2c^2)}\right)^2 \frac{dq^2 d\phi}{(2\pi\hbar)^3}$$

$$d\sigma = \frac{\pi}{4v^2\hbar}\left(\frac{\hbar^2 g^2}{(q^2+M^2c^2)}\right)^2 \frac{dq^2 d\phi}{(2\pi\hbar)^3}$$

$$= \frac{\pi}{4v^2\hbar}\,\frac{\hbar^4 g^4}{(q^2+M^2c^2)^2}\,\frac{dq^2 d\phi}{8\pi^3\hbar^3} = \frac{1}{(4\pi)^2}\,\frac{g^4}{2v^2}\,\frac{dq^2 d\phi}{(q^2+M^2c^2)^2}$$

Integrate this expression and take the limit as $v \to c$

$$\sigma = \frac{g^4}{2v^2(4\pi)^2}\int \frac{dq^2 d\phi}{(q^2+M^2c^2)^2} = \frac{\pi g^4}{v^2(4\pi)^2}\int \frac{dq^2}{(q^2+M^2c^2)^2}$$

$$= \frac{\pi g^4}{v^2(4\pi)^2}\left[-\frac{1}{(q^2+M^2c^2)}\right]_{q^2=\infty}^{q^2=0}$$

since integral was from $\cos\theta = -1$ to 1 (where 1 = zero momentum transfer)

$$\sigma = \frac{\pi g^4}{Mc^2(4\pi)^2} = \pi\left(\frac{g^2}{4\pi\hbar c}\right)^2\left(\frac{\hbar}{Mc}\right)^2$$

taking $v \sim c$

strong coupling constant (~1)

"range" of Yukawa potential (~1fm)

$\sigma = \pi \times 10^{-30}\ m^2$

~ 30 mb

(within a factor of 2-3)