Relativity & Introductory Particle Physics

Problem Set 4, SDB

1. The energy levels and Bohr radius of a hydrogen-like system are given by

$$E_n = -\frac{\alpha^2 \mu}{2n^2}, \quad a_0 = \frac{1}{\alpha\mu}$$

where μ is the reduced mass. Restore the factors of $\hbar$ and c by dimensional analysis and check against the standard expressions for these quantities that you found in the quantum mechanics course. Calculate the splitting between the $n = 1$ and $n = 2$ states of positronium in eV.

In the ground state the spatial part of the wave-function is given by

$$\psi = N e^{-r/a_0}$$

where N is a normalization constant. Show that the expectation value of the kinetic energy is given by

$$\left\langle \frac{\hat{p}^2}{2\mu} \right\rangle = \frac{\alpha^2 \mu}{2}$$

Hence explain why to a first approximation we can treat positronium as a non-relativistic system.

2. To a first approximation the potential in a $q\overline{q}$ system is (in natural units)

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \kappa r$$

The Coulomb-like piece has the same form as the electromagnetic potential except that α is replaced by $\frac{4}{3}\alpha_s$; the linear piece is there to ensure that the quarks are confined (i.e. cannot separate to infinite distance).

(a) Plot $V(r)$ against r. Show that for $r << r_0 = \sqrt{\alpha_s/\kappa}$ the Coulomb piece dominates.

(b) Now let's assume we can ignore the linear piece. Calculate values of α_s (different in i. and ii.) corresponding to the information:

i. the splitting between the $n = 2$ and $n = 1$ states in the J/ψ ($c\bar{c}$) is 588 MeV, and $m_c = 1870$ MeV;

ii. the splitting between the $n = 2$ and $n = 1$ states in the Υ ($b\bar{b}$) is 563 MeV, and $m_b = 5280$ MeV.

(c) Still ignoring the linear piece, compute the values of a_0 for the J/ψ and the Υ. The value of $\sqrt{\kappa}$ is about 400 MeV (justify the units); is it a reasonable approximation to ignore the linear term in $V(r)$?

(d) Calculate the expectation value of the kinetic energy in the $n = 1$ states of the J/ψ and the Υ. Comment on whether we are justified in using a non-relativistic approximation for these two states; are relativistic corrections more significant in positronium or in these $q\overline{q}$ states?

3. Consider a massless $q\overline{q}$ pair linked by a rotating string with ends moving at the speed of light. At rest, the string stores energy κ per unit length and we assume no transverse oscillations on the string. This configuration has the maximum angular momentum for a given mass and all of both reside in the string - the quarks have none. Consider one little bit of string at a distance r from the middle, with the quarks located at fixed distances R. Accounting for the varying velocity as a function of radial position, calculate both the mass, M, and angular momentum, J, as a function of κ and R. Thus, from the 2 equations derive the relation (in natural units)

$$J = \frac{M^2}{2\pi\kappa}$$

From experimental measurements of J versus M ("Regge trajectories") it is found that $\kappa \sim 0.18 GeV^2$ when expressed in natural units. Convert this to an equivalent number of tonnes.

Now consider the "colour charge" contained within a Gaussian surface centred around a quarks and cutting through a flux tube of cross sectional area A. By computing an effective "field strength" (in analogy to electromagnetism), derive an expression for the energy density of the string (*i.e.* κ) in terms of the colour charge and the area A. Thus show that

$$\alpha_s = \frac{q^2}{4\pi\hbar c} \sim \frac{\kappa A}{2\pi\hbar c}$$

Hence, estimate the value of the strong coupling constant α_s.

4. Explain briefly the concept that the fundamental forces of nature are due to the exchange of virtual particles or force quanta. Identify the force quanta for the electromagnetic and weak interactions.

Draw the lowest-order diagrams for the process $e^+e^- \to \mu^+\mu^-$. Well above threshold, the expression for the total cross-section is

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi}{3}\frac{(\alpha\hbar c)^2}{E^2} \text{ for } E \ll M_Z c^2$$

where α is the electromagnetic fine-structure constant, M_Z is the mass of the Z^0 boson and E is the centre of mass energy. Discuss whether the same formula may also be used for the processes $e^+e^- \to e^+e^-$ and $e^+e^- \to \tau^+\tau^-$. Starting from the equation derive an expression for $\sigma(e^+e^- \to \text{hadrons})$, stating clearly any assumptions that you make.

An e^+e^- storage ring operates at a centre of mass energy of 30GeV. Enumerate the lepton pair and $q\overline{q}$ states that could be produced via the annihilation channel at this energy. Given that the luminosity of the ring is $3\times10^{35} m^{-2}s^{-1}$, calculate the rate of production of hadronic events.

Describe the essential features of a detector that would allow one to identify the lepton pair and hadronic final states. What fraction of hadronic events originate from the production of $b\overline{b}$ pairs? Is it possible to identify such events?

5. Calculate the momenta above which pions, kaons and protons begin to emit Cerenkov radiation when traveling through air. Over what range of momenta would an air-filled threshold Cerenkov counter be useful in discriminating pions from Kaons? Particle identification may also be performed by time-of-flight (ToF) measurements. What timing resolution would you need to separate π, K and p of momentum 1 GeV/c over a flight path of 2 m ?

6. The reaction $\overline{\nu}_e + p \to e^+ + n$ is the inverse of neutron β decay. For $\overline{\nu}_e$ energies significantly above 2 MeV the total cross section for this reaction is

$$\sigma \simeq \frac{2\pi}{\hbar c} g^2 \frac{4\pi}{(2\pi\hbar)^3} \frac{E_\nu^2}{c^3}$$

where g is the weak four-fermion coupling constant. Justify this expression in terms of the Golden Rule and the assumptions of the simple Fermi theory. Evaluate this cross section for an anti-neutrino energy of 10 MeV.

Another recipient of a Nobel Prize in 2002 was Masatoshi Koshiba, partly for the detection of a burst of 12 neutrinos from supernova 1987A by the Kamiokande II experiment. Most if not all of these interactions were

$$\overline{\nu}_e + p \to e^+ + n$$

on hydrogen nuclei in 2.1 kilotonnes of water. SN1987A was located in the Large Magellanic Cloud, at a distance of $\sim 150,000$ light years.

Estimate the total energy radiated in the form of neutrinos by SN1987A.

Compare your result with the gravitational energy released by the collapse of one solar mass to nuclear density.

7. Estimate the flux of pp chain neutrinos at the earth given the solar luminosity of $L_\odot = 3.8 \times 10^{26} J/s$, the fact that 2 neutrinos are produced in every conversion of 4 protons to one α-particle and that this conversion yields 26.72 MeV. If $\sim 2\%$ of the energy is lost this way, what is the average neutrino energy?

   The Sudbury Neutrino Observatory (SNO) uses 1 kilotonne of heavy water (D_2O) located $\sim$2 km below ground in a Canadian nickel mine to measure solar neutrinos from the ^8B reaction in three different ways:

	Reaction	Typical Cross-Section (cm^2)
Charged Current (CC):	$\nu_e + d \to p + p + e^-$	1×10^{-42}
Neutral Current (NC):	$\nu_x + d \to p + n + \nu_x$	4×10^{-43}
Elastic Scattering (ES):	$\nu_x + e^- \to \nu_x + e^-$	6×10^{-44}

   where d is the deuteron and x refers to any of the 3 neutrino flavours. Draw the leading order Feynman diagrams for each of these reactions. Although the ES reaction is sensitive to neutrinos of all flavours, it is $\sim$6.5 times more sensitive to ν_e than ν_μ or ν_τ. Assuming that the relevant diagrams add constructively and that the coupling to the Z^0 is weaker than the coupling to the W, does this value of the relative sensitivity make sense?

   The flux of higher energy neutrinos from the ^8B branch of the pp chain is $\sim 10^4$ times smaller than the total solar neutrino flux which is obtained in the simple calculation above. An analysis threshold of $\sim$6 MeV was used by SNO in their first paper, published in 2001. This further reduced the observable number of CC events by about a factor of two. How many CC events would you expect to have been seen in SNO during one year?

   The longstanding "Solar Neutrino Problem" consists of the fact that various experiments have consistently measured fewer neutrinos from the sun than predicted by detailed solar models. Given that only electron-type neutrinos can be produced in the sun, it had been hypothesized that a possible explanation for this puzzle was that some fraction of these neutrinos are "oscillating" to other flavours as they travel to us from the sun. One of the unique achievements of SNO has been to cleanly measure the pure ν_e component in an energy range where the contribution of neutrinos of other flavours can also be inferred to provide a direct test of this hypothesis. In their first paper, SNO have observed 35% of the expected number from CC interactions. The Super-Kamiokande experiment in Japan had previously measured the ES flux in the same energy range to be 45% of the expected value. Show that this is almost exactly what is expected for neutrino oscillation with nearly maximal mixing.