# TT Revision Lectures on

# ELECTROMAGNETISM (CP2)

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- Electrostatics
- Magnetostatics
- Induction
- EM waves

… taken from previous years' Prelims questions

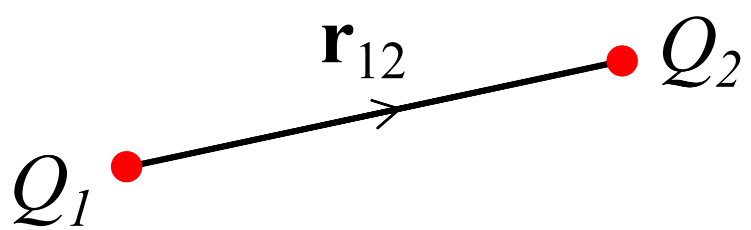
[1] with thanks to Profs Hans Kraus, Laura Hertz and Neville Harnew

# 1 Electrostatics

**1.1.** State Coulomb's Law for the force between two charges, $Q_1$ and $Q_2$. Hence show how the electric field $\mathbf{E}$ at a point $\mathbf{r}$ may be defined. What is meant by the statement that $\mathbf{E}$ is a conservative field? [4]

State Coulomb's Law. Show how **E** field may be defined. What is meant by **E** is a conservative field?

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2_{12}}\,\hat{\mathbf{r}}_{12}$$




Electric field due to single charge $Q$: force per unit charge

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}$$

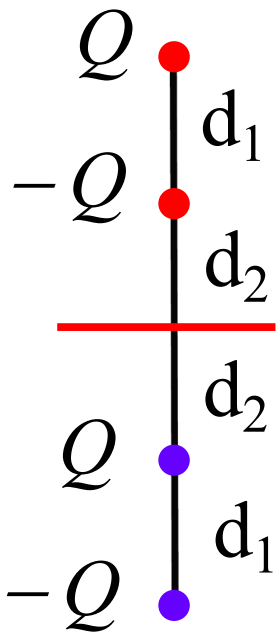
( $\hat{\mathbf{r}}$ points from $Q$ to point of observation )

Conservative field: $\nabla \times \mathbf{E} = 0$ and $\int \mathbf{E} \cdot \mathbf{dl}$ is path-independent. Therefore, a potential can be defined $\boldsymbol{E} = -\boldsymbol{\nabla} V$

**1.2.** A thundercloud and the ground below can be modelled as a charge of +40 As at a height of 10 km and a charge of −40 As at a height of 6 km above an infinite conducting plane. A person with an electrometer stands immediately below the thundercloud. What value of electric field do they measure, and what is its direction? [5]

A thundercloud with charges +40As at 10 km height and −40As at 6 km. Find the E-field on the ground.

Use method of image charges. Mirror the above to below the surface, with +40 As at depth 6 km and −40 As at depth 10 km.




$$E = \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{\left(10^4\,\mathrm{m}\right)^2}+\frac{1}{\left(6\times 10^3\,\mathrm{m}\right)^2}+\frac{1}{\left(6\times 10^3\,\mathrm{m}\right)^2}-\frac{1}{\left(10^4\,\mathrm{m}\right)^2}\right]$$

$$=\frac{2\cdot 40\mathrm{As}\quad \mathrm{Vm}}{4\pi\cdot 8.854\times 10^{-12}\,\mathrm{As}}\left[\frac{1}{3.6\times 10^7\,\mathrm{m}^2}-\frac{1}{10^8\,\mathrm{m}^2}\right]=12{,}780\frac{\mathrm{V}}{\mathrm{m}}$$

Field points upwards.

**1.3.** An array of localised charges $q_i$ experience potentials $V_i$ as a result of their mutual interaction. Show that their mutual electrostatic energy, $U$, is given by $U = \frac{1}{2}\sum_i q_i V_i$. [6]

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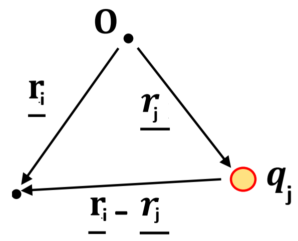
Potential energy of a single charge $q$ in potential $V$

$$U = -\int_\infty^r \mathbf{F}\cdot d\mathbf{l} = -q\cdot\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = q\cdot V(r)$$

Potential $V_i$ due to all other charges:

$$V_i = \sum_{j\neq i} \frac{q_j}{4\pi\varepsilon_0}\cdot\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$$




For total PE, sum over all charges. However, each charge appears twice:

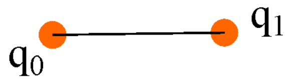
$$\frac{1}{2}\sum_i\sum_{j\neq i}\frac{q_i q_j}{4\pi\varepsilon_0}\cdot\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$$

$$U = \tfrac{1}{2}\sum_i q_i V_i$$

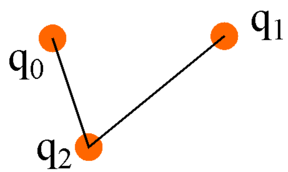
# Alternative: Assemble Charge Configuration explicitly

No penalty for charge $q_0$

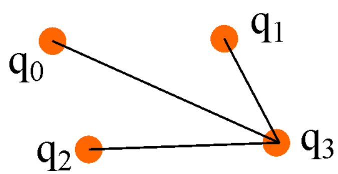

$q_1$ in potential due to $q_0$




$q_2$ in potential of $q_0$ and $q_1$

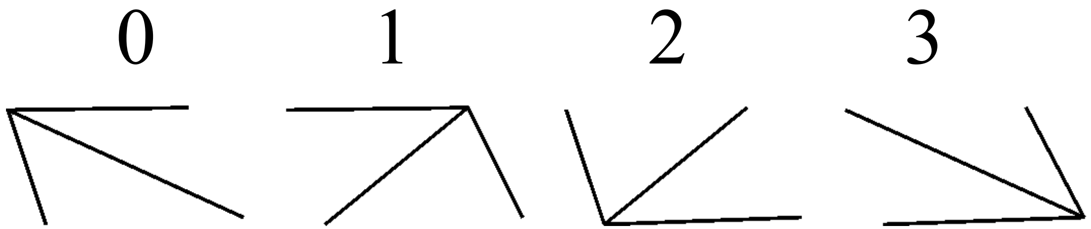



$q_3$ in pot. of $q_0$, $q_1$ and $q_2$




$$U = \frac{1}{4\pi\varepsilon_0}\Big[\, 0 + \frac{q_0 q_1}{r_{01}} + \frac{q_0 q_2}{r_{02}} + \frac{q_1 q_2}{r_{12}} + \frac{q_0 q_3}{r_{03}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \Big]$$

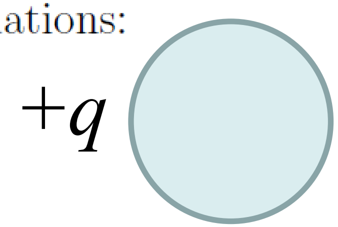
Half the links compared with:




Thus, as before:

$$U = \tfrac{1}{2}\sum_i q_i V_i$$

**1.4.** A sphere of radius $a$ is located at a large distance from its surroundings which define the zero of potential. It carries a total charge $q$. Determine the potential of its surface and the electrostatic energy of its charges in two separate situations:




(a) with the charge spread uniformly on its surface, [4]

(b) with the charge distributed uniformly within its volume. [10]

A sphere of radius $a$ is located at a large distance from its surroundings which define the zero of potential. It carries a total charge $q$. Determine the potential on its surface and the electrostatic energy : a) uniform q spread on surface.

$$V(r) = -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{r} = \frac{q}{4\pi\varepsilon_0 r}$$

Need to compute: $\int V dq$

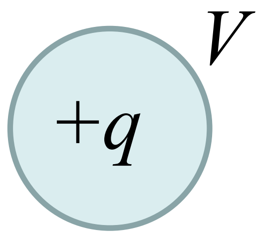
For shell:

$$V = \frac{q}{4\pi\varepsilon_0 a} \quad \text{and} \quad U = \int_0^q V(q')\mathrm{d}q' = \int_0^q \frac{q'\mathrm{d}q'}{4\pi\epsilon_0 a} = \frac{q^2}{8\pi\epsilon_0 a}$$

(alternatively use: $U = \frac{1}{2}\int \rho V \mathrm{d}^3 r$ or $U = \frac{1}{2}\epsilon_0 \int_{all\ space} E^2\,\mathrm{d}^3 r$ )

For part a), replace ρ with surface charge density σ and perform surface integral

## b) For uniformly distributed charged sphere:




Bring from infinity successive shells of thickness $dr$ to radius $r$ in potential $V$, and sum all contributions up to radius

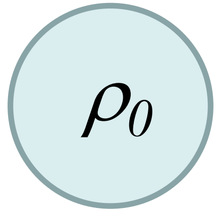
$$U = \int V(q')\mathrm{d}q' = \int V\rho\, \mathrm{d}^3 r$$

$$U = \int_0^a \frac{\frac{r^3}{a^3} q}{4\pi\varepsilon_0 r} \frac{q}{\frac{4\pi}{3} a^3} 4\pi r^2 \,\mathrm{d}r$$

[Potential @ radius r] × [Charge density] × [Volume element of shell]

$$U = 3\frac{q^2}{4\pi\varepsilon_0} \int_0^a \frac{r^4}{a^6} \mathrm{d}r = \frac{3}{5} \frac{q^2}{4\pi\varepsilon_0 a}$$

**1.5.** Calculate the electric field strength $E$ and the electrostatic potential $V$, as functions of radial distance $r$, for a sphere of uniform positive charge density $\rho_0$, of radius $R$, centred at the origin. Sketch graphs of $E$ and $V$ against $r$. [9]




Use Gauss' Law: $$\oiint \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\cdot\iiint \rho\, dV$$

$$\oiint E_r dS = E_r \cdot 4\pi r^2 = \tfrac{1}{\varepsilon_0}\cdot\begin{cases} \tfrac{4\pi}{3} R^3\rho_0 & \text{for} \quad r \geq R \\ \tfrac{4\pi}{3} r^3\rho_0 & \text{for} \quad r < R \end{cases}$$

$$E_r = \frac{\rho_0}{3\varepsilon_0}\cdot\frac{R^3}{r^2} \quad \text{for } r \geq R \qquad \text{and} \qquad E_r = \frac{\rho_0}{3\varepsilon_0}\cdot r \quad \text{for } r < R$$

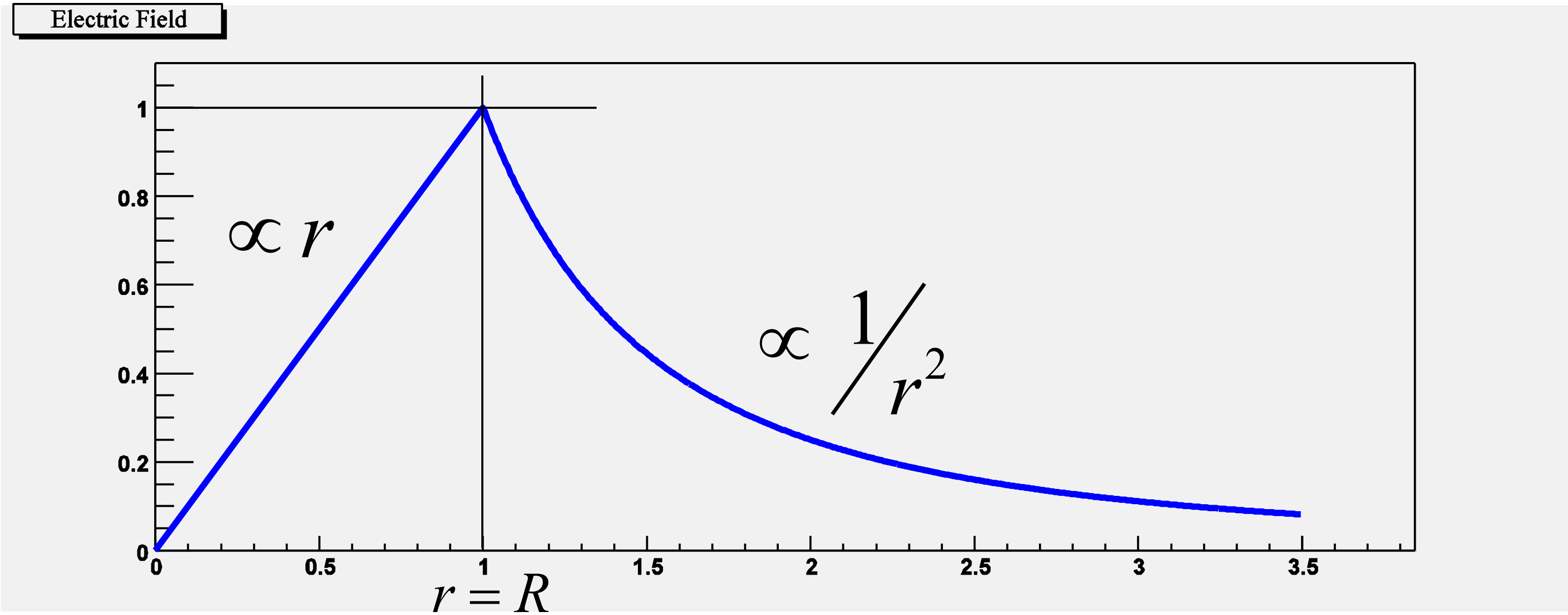
$$V_{out} = -\int_\infty^r E_r dr' = -\frac{\rho_0}{3\varepsilon_0}R^3\cdot\left[-\frac{1}{r'}\right]_\infty^r = \underline{\underline{\frac{\rho_0}{3\varepsilon_0}\cdot\frac{R^3}{r}}}$$

on sphere $(r = R)$: $$V_S = \frac{\rho_0}{3\varepsilon_0}R^2$$

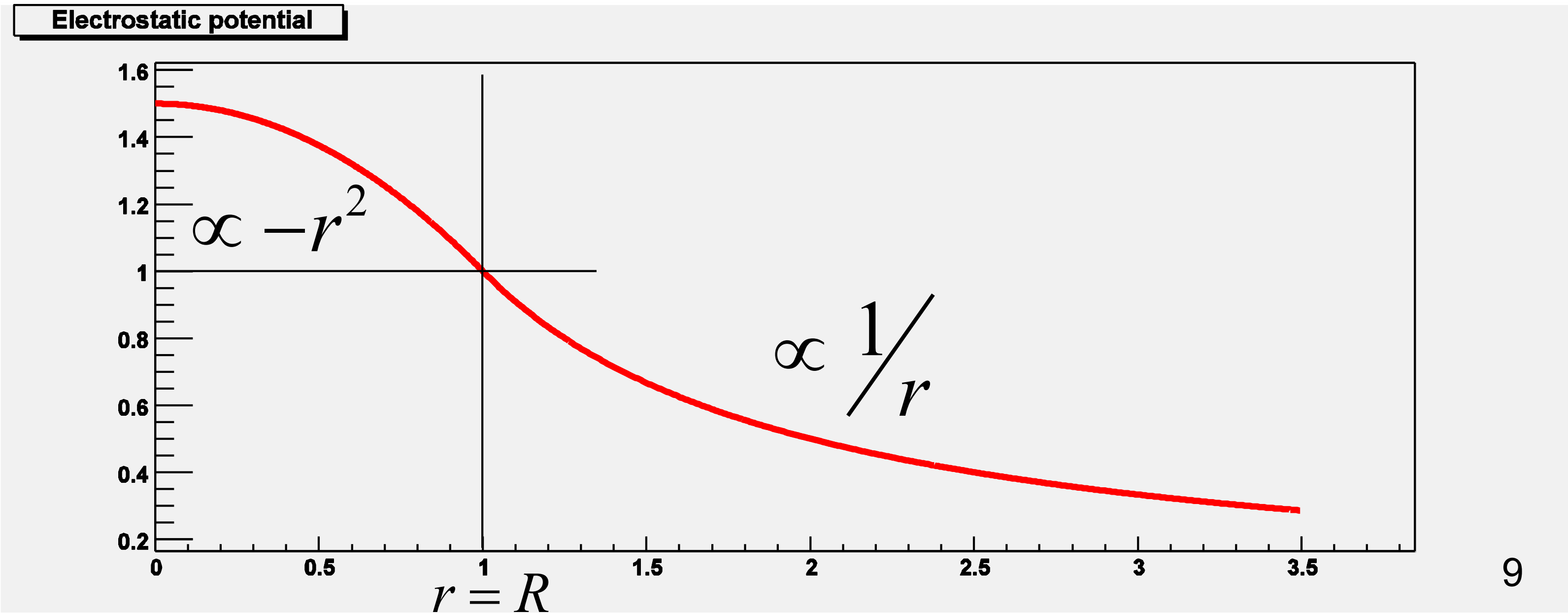
$$V_{ins} = V_S - \int_R^r E_r dr' = \frac{\rho_0}{3\varepsilon_0}\left[R^2 - \tfrac{1}{2}r^2 + \tfrac{1}{2}R^2\right] = \underline{\underline{\frac{\rho_0}{3\varepsilon_0}\left[\tfrac{3}{2}R^2 - \tfrac{1}{2}r^2\right]}}$$

# E-field and potential V as function of r

$$E_{max} = \frac{\rho_0}{3\varepsilon_0} R$$




$$V_S = \frac{\rho_0}{3\varepsilon_0} R^2$$



**1.6.** The electron charge density of a hydrogen atom in its ground state is given by

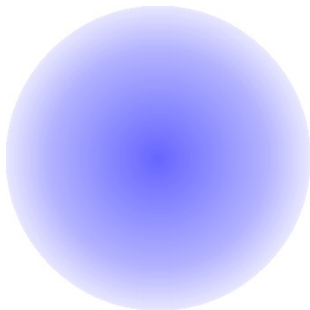
$$\rho(r) = -\frac{e}{\pi a_0^3}\exp[-2r/a_0],$$

where $a_0$ is the Bohr radius ($5.3 \times 10^{-11}$ m). Show that the electric field due to the electron cloud is given by

$$E(r) = \frac{e}{4\pi\epsilon_0}\left\{\frac{(e^{-2r/a_0} - 1)}{r^2} + \frac{2e^{-2r/a_0}}{a_0 r} + \frac{2e^{-2r/a_0}}{a_0^2}\right\}. \qquad [9]$$

$$\left[\int x^n e^{ax} dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx\right]$$

Electron cloud:



$$\rho(r) = -\frac{e}{\pi a_0^3}\cdot\exp\left(-\frac{2r}{a_0}\right)$$

Gauss' Law: $\oiint \mathbf{E}\cdot\mathbf{dS} = \frac{1}{\varepsilon_0}\cdot\iiint \rho dV$ ⇦ (charge enclosed)

$$E_r = \frac{1}{4\pi\varepsilon_0 r^2}\cdot\left[-\frac{e}{\pi a_0^3}\cdot\iiint \exp\left(-\frac{2r'}{a_0}\right) r'^2 \sin\theta d\theta d\varphi dr'\right]$$

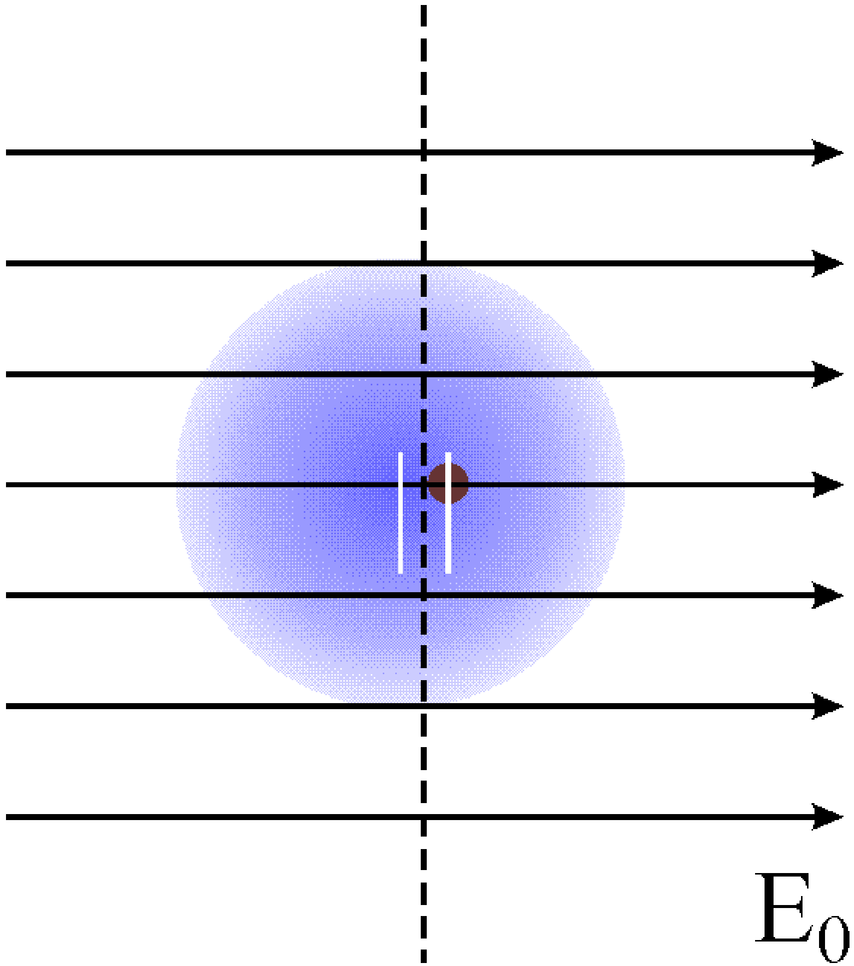
$$\int_0^r x^2 \exp(ax)dx = \tfrac{1}{a}x^2\, e^{ax}\Big|_0^r - \tfrac{2}{a^2}\, x\, e^{ax}\Big|_0^r + \tfrac{2}{a^3}\, e^{ax}\Big|_0^r$$

here: $a = -\frac{2}{a_0}$ and $\iint \sin\theta d\theta d\varphi = 4\pi$

$$E_r = \frac{e}{4\pi\varepsilon_0}\left\{\frac{\exp(-2r/a_0)-1}{r^2} + \frac{2\exp(-2r/a_0)}{a_0 r} + \frac{2\exp(-2r/a_0)}{a_0^2}\right\}$$

Explain qualitatively what happens when such an atom is placed in a steady, uniform electric field, of strength $E_0$. [4]

Explain qualitatively what happens when such an atom is placed in a steady, uniform electric field, of strength $E_0$.




Centres of gravity of the positive nucleus and the negative electron charge distribution shift.

Forces on charges due to $E_0$ balances the internal force of the dipole charges.

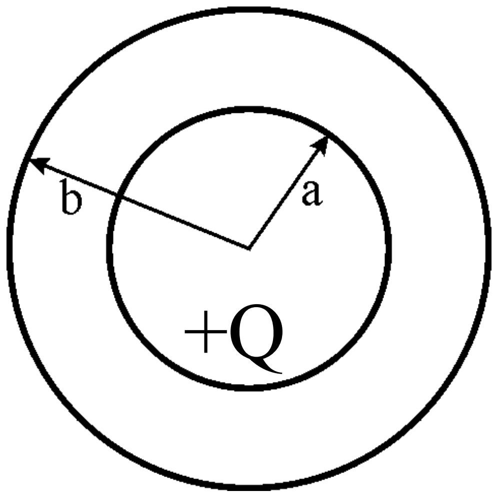
The atom exhibits an electric dipole moment.

**1.7.** The space between two concentric spheres, of radii $a$ and $b$ ($b > a$), is filled with air which has a relative permittivity of 1. Show that the capacitance $C$ of the combination is given by

$$C = 4\pi\epsilon_0 \left(\frac{ab}{b-a}\right).$$

[7]

Gauss's Theorem in vacuo: $$\oiint \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\cdot\iiint \rho dV$$




Calculate the capacitance for a spherical capacitor: $$C = \frac{Q}{V}$$

Between $a$ and $b$ : $E_r \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\cdot Q$

$$V = -\int_b^a E_r dr = \frac{Q}{4\pi\varepsilon_0}\cdot\left[\frac{1}{r}\right]_b^a = \frac{Q\cdot(b-a)}{4\pi\varepsilon_0 ab}$$

$$\underline{\underline{C = 4\pi\varepsilon_0\cdot\frac{ab}{b-a}}}$$

The inner sphere is raised to a potential $V$ and then isolated, the outer sphere being earthed. The outer sphere is then removed. Show that the resulting potential $V'$ of the remaining sphere is given by

$$V' = \frac{bV}{b-a}.$$ [5]

The inner sphere is raised to a potential V and then isolated, the outer sphere being earthed. The outer sphere is then removed. Find the resulting potential of the remaining sphere.

Before (and after) removal, charge stored on inner sphere:

$$Q = 4\pi\varepsilon_0 \cdot \frac{ab}{b-a} \cdot V \quad [1]$$

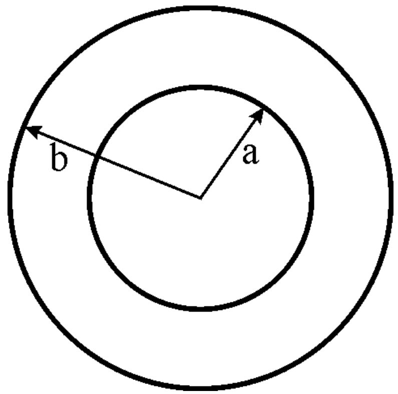
After removal, field of remaining sphere:

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\underline{\underline{V'}} = -\int_{\infty}^{a} E_r dr = -\frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{a}\frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0 a} = \underline{\underline{\frac{b}{b-a}\cdot V}}$$

If the values of $a$ and $b$ are 0.9 m and 1.0 m respectively, and given that air cannot sustain an electric field greater than 3000 V mm$^{-1}$, calculate the maximum potential to which the inner sphere can be initially charged. [8]

Now back to the original configuration:




$$E_{\max} = 3000\text{V/mm} \qquad a = 0.9\text{m} \qquad b = 1.0\text{m}$$

E is at a maximum when r is at its smallest → consider $E$(a)

From [1] ( $Q = 4\pi\varepsilon_0 \cdot \frac{ab}{b-a} \cdot V$ ) and Gauss' Law ( $E_r = \frac{Q}{4\pi\varepsilon_0 r^2}$ )

$$E_r(a) = \frac{ab}{b-a} \cdot V \cdot \frac{1}{a^2} = V \cdot \frac{b}{a} \cdot \frac{1}{b-a}$$

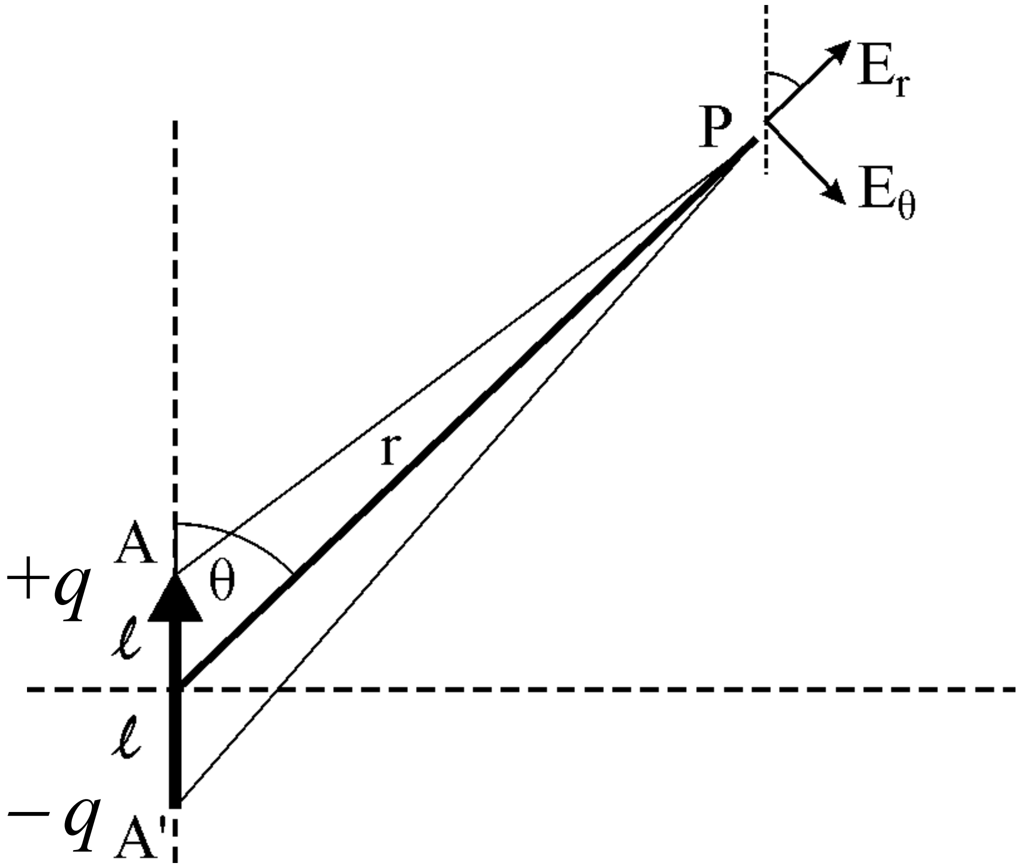
$$\underline{\underline{V_{\max}}} = E_{\max} \cdot \frac{a(b-a)}{b} = 3 \cdot 10^6 \frac{\text{V}}{\text{m}} \cdot \frac{0.9\text{m} \cdot 0.1\text{m}}{1\text{m}} = \underline{\underline{2.7 \cdot 10^5\,\text{V}}}$$

**1.8.** An *electric dipole* consists of charges $-q$ and $+q$ separated by a distance $2l$, the resulting dipole moment $\mathbf{p}$ being of magnitude $2ql$ and with direction from $-q$ to $+q$. At a point $(r, \theta)$ relative to the centre and the direction of the dipole axis, derive from first principles, in the case where $r \gg l$,

(a) the electrostatic potential, [4]

(b) the radial and tangential components of the electric field, [4]

(c) the torque exerted on such a dipole by a uniform electric field $\mathbf{E}$. [6]

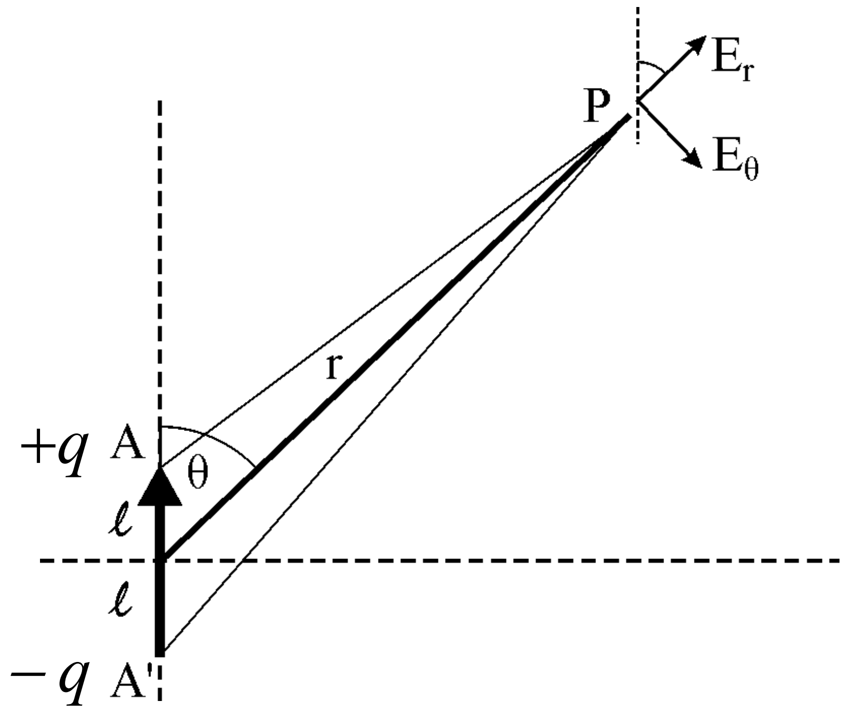


# The electrostatic potential of a dipole:

Charges +q at A and –q at A’

$$\overline{\mathrm{AP}}^2 = r^2 + \ell^2 - 2r\ell\cos\theta$$

$$\overline{\mathrm{A'P}}^2 = r^2 + \ell^2 + 2r\ell\cos\theta$$




$$V_P = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{\overline{\mathrm{AP}}} - \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{\overline{\mathrm{A'P}}}$$

Binomial expansion

$$\frac{1}{\overline{\mathrm{AP}}} = \frac{1}{r}\cdot\left[1+\left(\ell/r\right)^2 - 2\,\ell/r\cos\theta\right]^{-1/2} \approx \frac{1}{r}\cdot\left[1+\ell/r\cos\theta+\ldots\right]$$

$$\frac{1}{\overline{\mathrm{A'P}}} \approx \frac{1}{r}\cdot\left[1-\ell/r\cos\theta+\ldots\right]$$

so: $$V_P = \frac{q}{4\pi\varepsilon_0 r}\cdot\left[1+{\ell}/{r}\cos\theta - 1 + {\ell}/{r}\cos\theta\right] = \frac{2q\ell}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\cos\theta$$

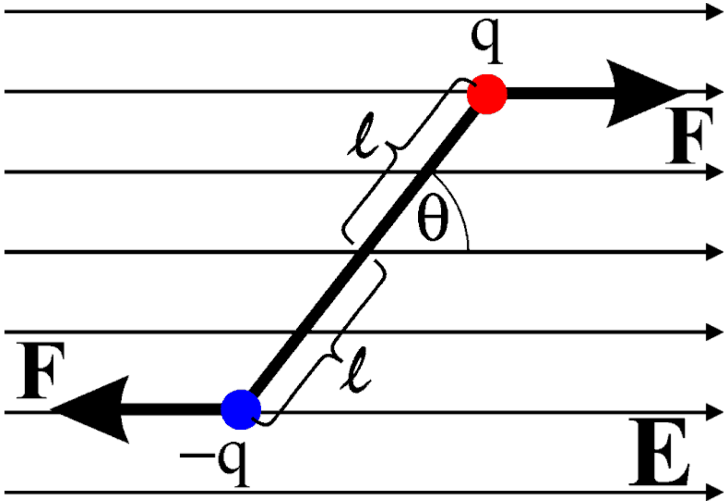
$$\underline{\underline{V_P = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}}$$

The radial and tangential components of the E-field:

$$\mathbf{E} = -grad\left(V_P\right); \qquad E_r = -\frac{\partial V_P}{\partial r} \quad \text{and} \quad E_\theta = -\frac{1}{r}\cdot\frac{\partial V_P}{\partial\theta}$$

$$\underline{\underline{E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}}} \qquad \text{and} \qquad \underline{\underline{E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}}}$$

## Show that the torque exerted on a dipole by a uniform electric field **E** is **p** x **E**




Torque (couple) on the dipole:

$$\underline{\mathbf{T}} = \sum_{\mathrm{i}} \underline{\mathbf{r}}_{\mathrm{i}} \times \underline{\mathbf{F}}_{\mathrm{i}}$$

$$\begin{aligned} |\underline{\mathbf{T}}| &= 2\,(l\,F\sin\theta) \\ &= 2\,l\,q\,E\sin\theta \\ &= p\,E\sin\theta \end{aligned}$$

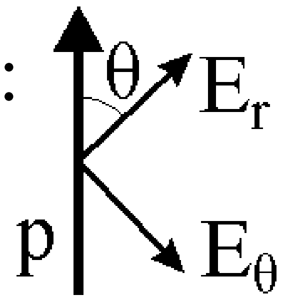
(with $p = 2ql$)

( $\underline{\mathbf{T}} = \underline{\mathbf{p}} \times \underline{\mathbf{E}}$ in vector notation )

Using these results find the angle $\theta$ for which the resultant electric field $\mathbf{E}$ at the point $(r, \theta)$ is in a direction *normal* to the axis of the dipole. [11]

Find the angle θ for which $\mathbf{E}(r, \theta)$ at point P is in a direction normal to the axis of the dipole.

Take the dipole moment p to be along the *z*-axis :




Find angle for which $\mathbf{p} \cdot \mathbf{E} = p_z \cdot E_z = 0$

$$E_z = E_r \cdot \cos\theta - E_\theta \cdot \sin\theta = 0 \quad \text{thus} \quad \frac{2p\cos^2\theta}{4\pi\varepsilon_0 r^3} - \frac{p\sin^2\theta}{4\pi\varepsilon_0 r^3} = 0$$

$$2\cos^2\theta = \sin^2\theta \qquad \text{and} \qquad \tan\theta = \pm\sqrt{2} \quad \text{or} \quad \theta = \pm 54.73^o$$

**1.9.** Show that the work $W$ done in bringing a dipole of equal magnitude from infinity to a point at distance $r$ from the first, along the normal to its axis, is given by

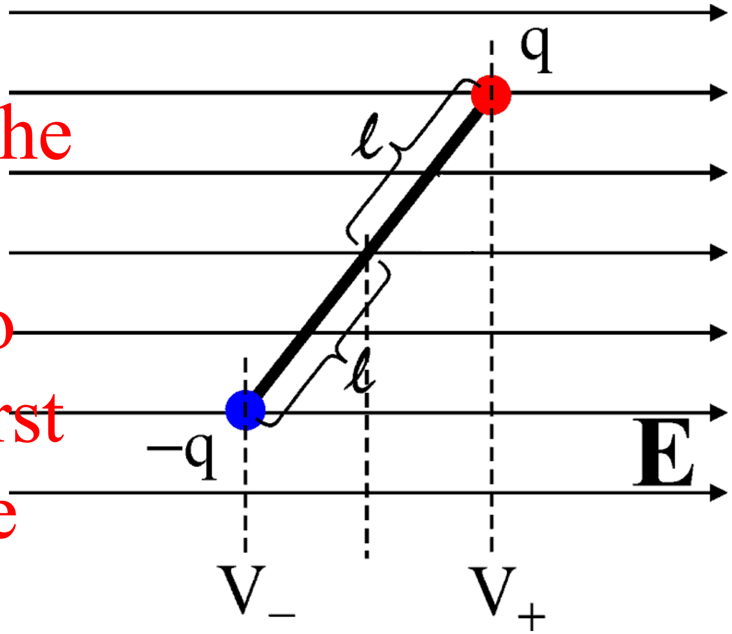
$$W = \frac{4q^2l^2}{4\pi\epsilon_0 r^3}\cos\theta,$$

where $\theta$ is the angle between the axes of the dipoles. [12]

[This is not the best worded question !]

Calculate the work done in bringing a dipole of equal magnitude from infinity to a distance $r$ from the first along the **normal** to its axis

**E** is now the field due to the first dipole




$$U_E = q\cdot V_+ + (-q)\cdot V_- = q\cdot(V_+ - V_-)$$

where:

$$V_+ - V_- = -\mathbf{E}\cdot(\mathbf{r}_{q+} - \mathbf{r}_{q-})$$

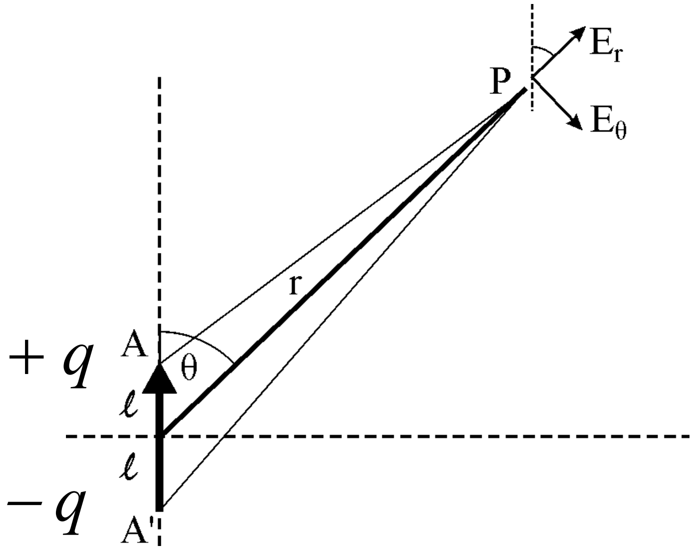
$$U_E = -q\mathbf{E}\cdot(2\mathbf{l}) = -\mathbf{p}\cdot\mathbf{E}$$

When 1st dipole is at ∞, **E**=0 and $U_E$=0, and is then brought in:

$$\underline{\underline{U_E}} = -\mathbf{p}\cdot\mathbf{E} = +pE\cos\theta = 2q\ell\cdot\frac{2q\ell}{4\pi\varepsilon_0 r^3}\cdot\cos\theta = \underline{\underline{\frac{4q^2\ell^2}{4\pi\varepsilon_0 r^3}\cos\theta}}$$

Note the direction of **E** and definition of $\theta$

**1.10.** If a second dipole, free to rotate, is placed firstly along the line $\theta = 0$, and secondly in the plane $\theta = \pi/2$, in what direction will it point relative to the first? [4]




Second dipole placed at $\theta = 0$ and then at $\theta = \pi/2$, free to rotate :

| | | | |
|---|---|---|---|
| $\theta = 0$ | $E_r = \dfrac{2p}{4\pi\varepsilon_0 r^3}$ | $E_\theta = 0$ | $p_2$ Parallel |
| $\theta = \dfrac{\pi}{2}$ | $E_r = 0$ | $E_\theta = \dfrac{p}{4\pi\varepsilon_0 r^3}$ | $p_2$ Anti-parallel |

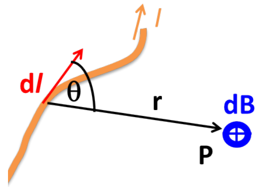
# 2 Magnetostatics

**2.1.** State the Biot-Savart Law which describes the magnetic flux density $d\mathbf{B}$ at a distance $\mathbf{r}$ from a current element $I d\mathbf{l}$. [4]

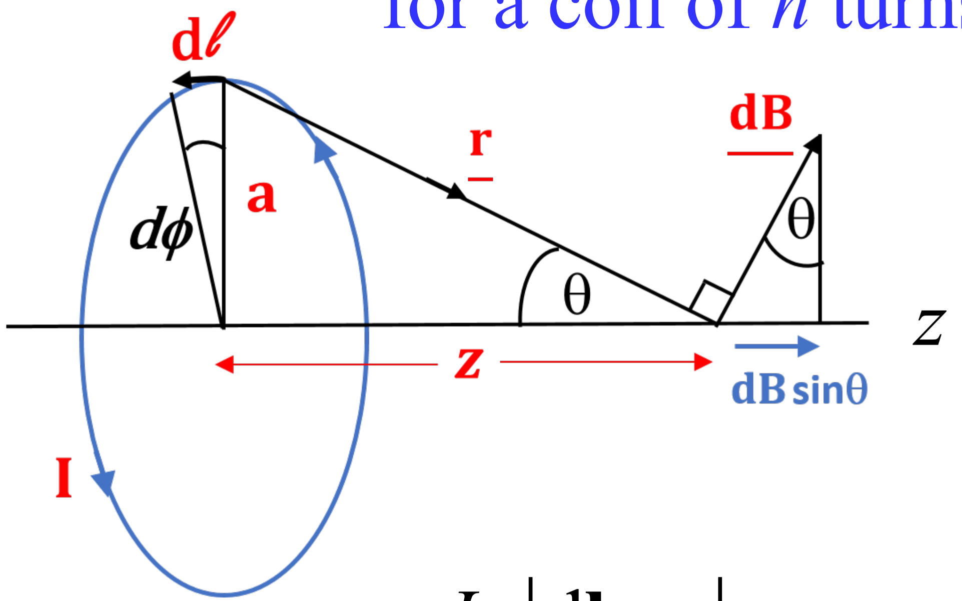
Find the magnitude $B$ of the magnetic flux density on the axis of a plane coil of $n$ turns and radius $a$ for a current $I$ in the coil and at a distance $z$ from the plane of the coil. [7]

State the Biot-Savart law :

$$\mathbf{dB} = \mu_0 I \cdot \frac{\mathbf{dl}\times\hat{\underline{\mathbf{r}}}}{4\pi r^2}$$




Find the magnitude of **B** on axis for a coil of *n* turns.




Symmetry:

d**B** has *z*-component only.

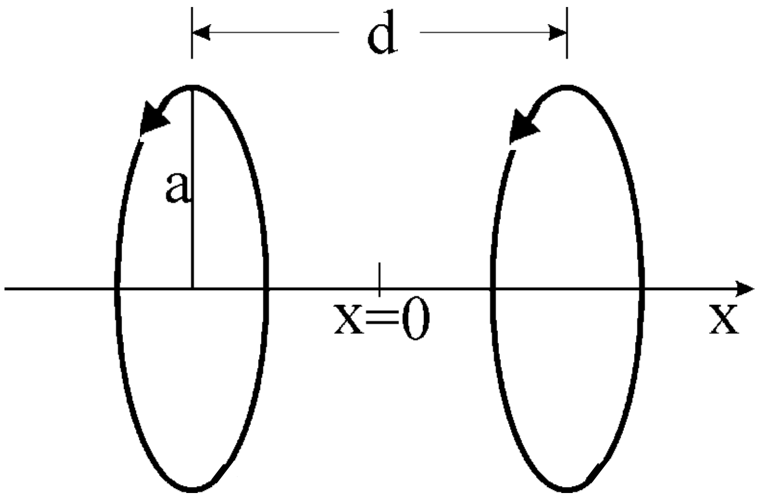
Perp. components cancel.

And also: **dl** is perp. to **r**

$$\underline{\underline{B}} = B_z = \int \frac{\mu_0 n I}{4\pi r^2} \cdot \underbrace{\left|\frac{\mathbf{dl}\times\mathbf{r}}{r}\right|}_{=|\,\mathbf{dl}\,|\,=\,a\,d\varphi} \cdot \underbrace{\sin\theta}_{a/r} = \int_0^{2\pi} \frac{\mu_0 n I a^2 d\varphi}{4\pi r^3} = \underline{\underline{\frac{\mu_0 n I a^2}{2\left(z^2+a^2\right)^{3/2}}}}$$

Two such coils are placed a distance $d$ apart on the same axis. They are connected in series in such a way as to produce fields on the axis in the same direction. Write down an expression for the magnitude of the net field $B'$ on the axis at a distance $x$ from the point midway between the coils. [3]

Two such coils are placed a distance *d* apart on the same axis. Find B as function of x.




$$B'(x) = \frac{\mu_0 n I a^2}{2} \cdot \left[ \frac{1}{\left(a^2 + \left(\frac{d}{2} + x\right)^2\right)^{3/2}} + \frac{1}{\left(a^2 + \left(\frac{d}{2} - x\right)^2\right)^{3/2}} \right]$$

Show that the derivative of $B'$ with respect to $x$ is zero when $x = 0$. Find the value of $d$ for which the second derivative of $B'$ with respect to $x$ is also zero at $x = 0$. Under these conditions, show that the variation of $B'$ between $x = 0$ and $x = d/2$ is less than 6 percent. [11]

$$B'(x) = \frac{\mu_0 n I a^2}{2} \cdot \left[ \frac{1}{\left(a^2 + \left(\frac{d}{2} + x\right)^2\right)^{3/2}} + \frac{1}{\left(a^2 + \left(\frac{d}{2} - x\right)^2\right)^{3/2}} \right]$$

## Show that the derivative of B' is 0 for x=0

$$\left(a^2 + \left(\tfrac{d}{2} \pm x\right)^2\right)^{-3/2} \xrightarrow{\frac{d}{dx}} -\frac{3}{2}\left(a^2 + \left(\tfrac{d}{2} \pm x\right)^2\right)^{-5/2} \cdot 2\left(\tfrac{d}{2} \pm x\right) \cdot (\pm 1)$$

which is $\pm$ the same, when $x = 0$, hence: $\underline{\underline{\frac{dB'}{dx}(0) = 0}}$

## Find the value of d for which the second derivative of B'(0) is 0.

$$\partial_x B' \propto -3\left(a^2+\left(\tfrac{d}{2}+x\right)^2\right)^{-5/2}\left(\tfrac{d}{2}+x\right)+3\left(a^2+\left(\tfrac{d}{2}-x\right)^2\right)^{-5/2}\left(\tfrac{d}{2}-x\right)$$

$$\begin{aligned}\partial_x^2 B' \propto\ & -3\left(a^2+\left(\tfrac{d}{2}+x\right)^2\right)^{-5/2} + 15\left(a^2+\left(\tfrac{d}{2}+x\right)^2\right)^{-7/2}\left(\tfrac{d}{2}+x\right)^2 \\ & -3\left(a^2+\left(\tfrac{d}{2}-x\right)^2\right)^{-5/2} + 15\left(a^2+\left(\tfrac{d}{2}-x\right)^2\right)^{-7/2}\left(\tfrac{d}{2}-x\right)^2\end{aligned}$$

$$\partial_x^2 B'(0) \propto -3\cdot\frac{2}{\left(a^2+\left(\frac{d}{2}\right)^2\right)^{7/2}}\cdot\left[\left(a^2+\left(\tfrac{d}{2}\right)^2\right)-5\left(\tfrac{d}{2}\right)^2\right] \quad = 0$$

$$a^2-4\left(\tfrac{d}{2}\right)^2=0 \qquad \underline{\underline{d=a}}$$

## When $a = d$, show that the variation of B' between x=0 and x=d/2 is <6%

$$B'(x) = \frac{\mu_0 n I}{2a} \cdot \left[ \frac{1}{\left(1 + \left(\frac{1}{2} + \frac{x}{d}\right)^2\right)^{3/2}} + \frac{1}{\left(1 + \left(\frac{1}{2} - \frac{x}{d}\right)^2\right)^{3/2}} \right]$$

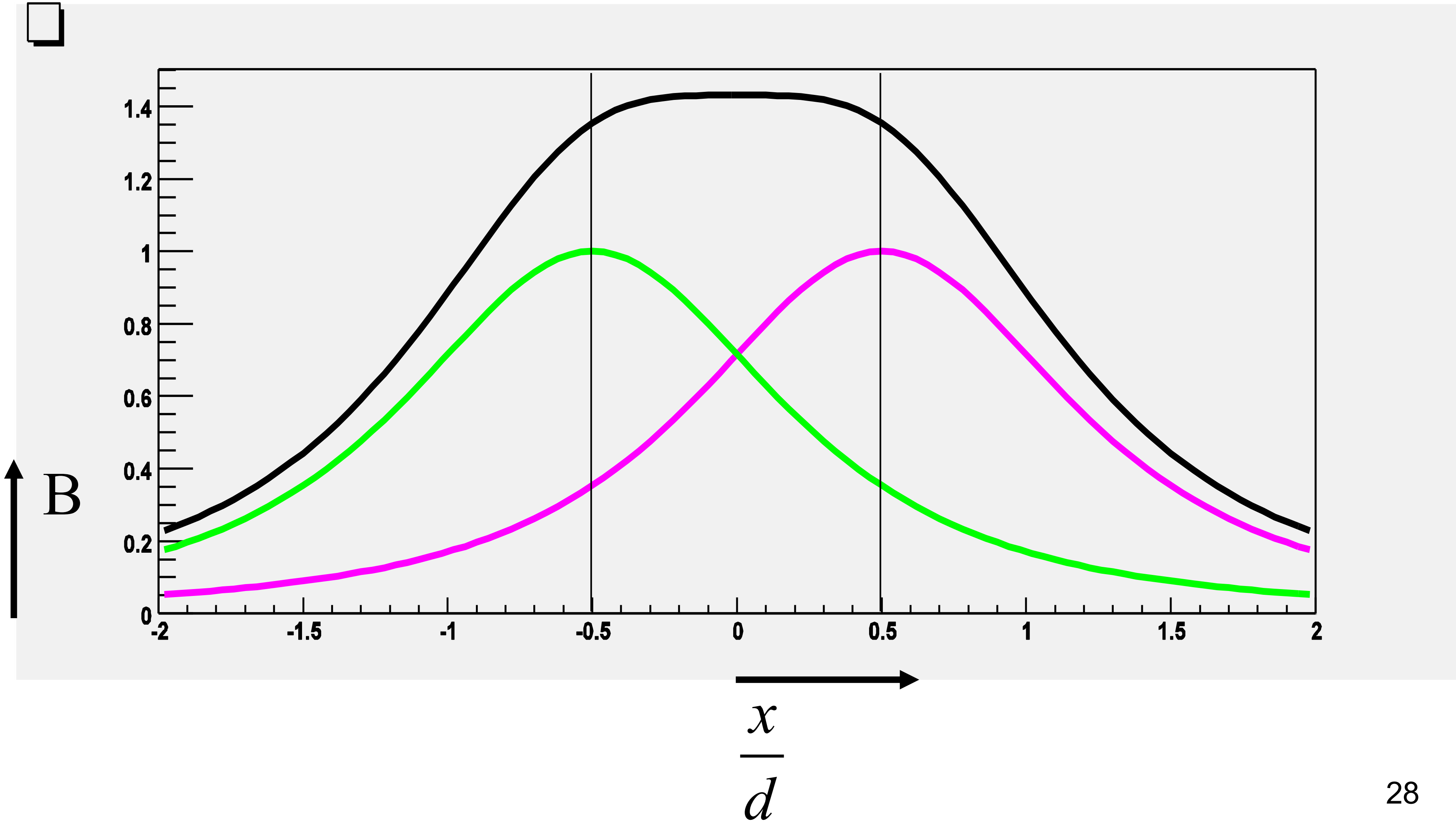
$$B'(0) = \frac{\mu_0 n I}{2a} \cdot \frac{2}{\left(5/4\right)^{3/2}} \qquad B'\left(\tfrac{d}{2}\right) = \frac{\mu_0 n I}{2a} \cdot \left[ \frac{1}{\left(1+1\right)^{3/2}} + \frac{1}{1} \right]$$

$$B'(0) = B_0 \cdot 1.43108 \qquad B'\left(\tfrac{d}{2}\right) = B_0 \cdot 1.35355$$

$$\underline{\underline{\Delta B' / \overline{B'} = 5.57\%}}$$

# Sketch of the field of a pair of Helmholtz coils

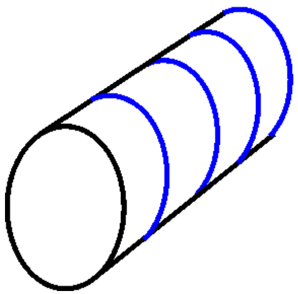
B in units of $\dfrac{\mu_0 n I}{2a}$

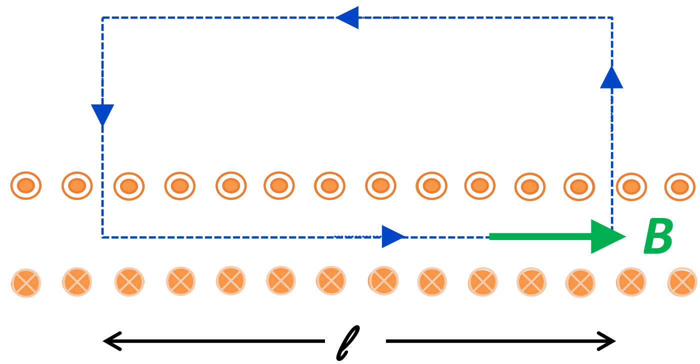


**2.2.** A very long cylindrical solenoid has radius $R$ and is wound with $N$ turns of wire per unit length. If the winding carries a current $I$, show that the magnetic induction $B$ inside the coil is radially uniform and give an expression for its value. [8]

Ampere's law in its integral form: $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$ ($I$ enclosed)

$N$ turns of wire per unit length.
Winding carries a current $I$.
Find B and show it is radially uniform inside the coil.






$$B \cdot \ell = \mu_0 \cdot N' \cdot I \qquad \text{with} \quad N = \frac{N'}{\ell}, \quad \text{thus} \quad \underline{\underline{B = \mu_0 \cdot N \cdot I}}$$

For infinite solenoid, B constant within it (and zero outside)
→ radially uniform field; symmetry means no azimuthal dependence

Calculate the self-inductance per unit length of the solenoid. [7]

## Calculate the self-inductance per unit length.

$$L = \frac{\Phi_{tot}}{I} = \frac{B \cdot area}{I} \cdot turns = \frac{\mu_0 N I \cdot \pi R^2}{I} \cdot N\ell = \mu_0 N^2 \pi R^2 \ell$$

... and per length: $\underline{\underline{L/\ell = \mu_0 \pi R^2 N^2}}$

A superconducting solenoid has radius 0.5 m, length 7 m and consists of 1000 turns. Calculate the magnetic induction in the solenoid, and the energy stored in it when it carries a current of 5000 A. You may approximate its behaviour to that of a very long solenoid. [7]

## Calculate the magnetic induction and the energy stored.

$$R = 0.5\text{m}, \quad \ell = 7\text{m}, \quad N' = 1000 \quad \Rightarrow \quad N = 142.86\text{m}^{-1}$$

$$\underline{\underline{B}} = \mu_0 N I = 4\pi \cdot 10^{-7} \frac{\text{Vs}}{\text{Am}} \cdot 142.86 \frac{1}{\text{m}} \cdot 5000\text{A} = \underline{\underline{0.897\text{T}}}$$

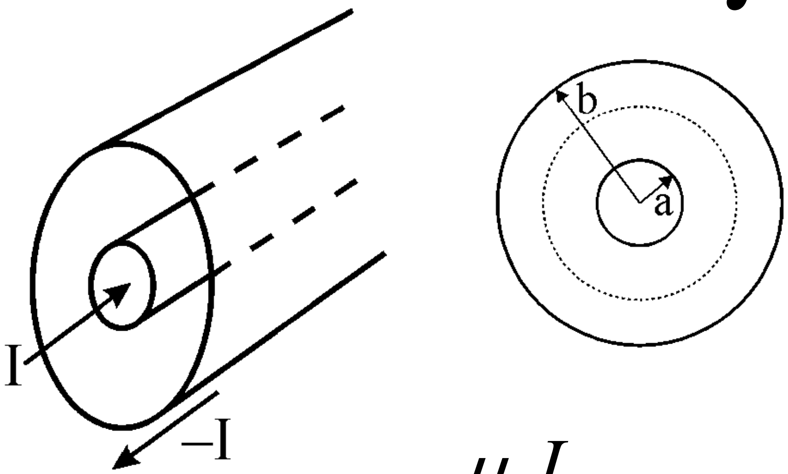
$$\underline{\underline{U_M}} = \tfrac{1}{2} L I^2 = \tfrac{1}{2} \mu_0 N^2 \pi R^2 \ell \cdot I^2 = \underline{\underline{1.76 \cdot 10^6\,\text{J}}}$$

**2.3.** A long coaxial cable consists of two thin-walled coaxial cylinders of radii $a$ and $b$. The space between the cylinders is maintained as a vacuum and a current $I$ flows down the inner and returns along the outer cylinder. Calculate the magnetic field at a distance $r$ from the axis when

(i) $b > r > a$, (ii) $r > b$ and (iii) $r < a$. [10]

## Calculate magnetic field inside a pair of co-axial cylinders due to current I flowing as shown.

Ampere's Law: $$\oint \mathbf{B}\cdot \mathrm{d}\mathbf{l} = \iint \mathbf{J}\cdot \mathrm{d}\mathbf{A} = \mu_0 I$$




$$b > r > a: \quad 2\pi r B_\theta = \mu_0 I$$

$$r > b: \quad 2\pi r B_\theta = \mu_0 (I - I) = 0$$

$$r < a: \quad 2\pi r B_\theta = 0$$

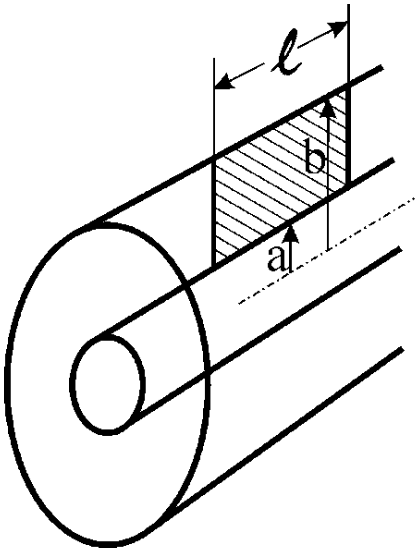
$$\underline{\underline{B_\theta = \frac{\mu_0 I}{2\pi r}}}$$ only for $b > r > a$

(direction azimuthal : cf. RH screw)

Hence show that the *self inductance* of a length $l$ of this cable is $L = \frac{\mu_0 l}{2\pi}\ln(b/a)$. [8]

## Calculate the self-inductance:

(surface dS= r.dl )




$$\Phi = \int \mathbf{B}\cdot d\mathbf{S} = \int_a^b \frac{\mu_0 I}{2\pi r}\, dr \cdot \ell = \frac{\mu_0 I}{2\pi}\ln\left(\frac{b}{a}\right)\cdot \ell$$

$$L = \frac{\Phi}{I} = \underline{\underline{\frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)\cdot \ell}}$$

Alternatively, use: $U_M = \frac{1}{2\mu_0}\int_{all\ space} B^2\, dV$

Here: $U_M = \frac{1}{2\mu_0}\int_a^b \left(\frac{\mu_0 I}{2\pi r}\right)^2 2\pi r\, dr \cdot \ell = \frac{1}{2}\frac{\mu_0 I^2}{2\pi}\ln\left(\frac{b}{a}\right)\cdot \ell$

Also, since: $U_M = \frac{1}{2}LI^2 \;\Rightarrow\; L = \underline{\underline{\frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)\cdot \ell}}$

For the case where the inner cylinder is replaced by a solid wire, also of radius $a$, throughout which the current is uniformly distributed, sketch the variation of the magnitude of magnetic field with $r$ over the range $r = 0$ to $r = 2b$. [7]

## Sketch the magnitude of B when the inner cylinder is replaced by a solid wire

for $r > a$: see before

(ratio of areas)

for $r < a$: $2\pi r B_\theta = \mu_0 I \cdot \dfrac{\pi r^2}{\pi a^2}$ thus $\underline{\underline{B_\theta = \dfrac{\mu_0 I}{2\pi a} \cdot \dfrac{r}{a}}}$