

### III. ELECTROMAGNETIC WAVES

#### A. PLANE WAVES IN DIELECTRICS

1. Maxwell's equations in dielectrics
2. wave equation
3. plane wave solutions
  - (a) waves are transverse
  - (b) polarization
  - (c) dispersion relation
  - (d) ratio of amplitudes of  $\vec{E}$  and  $\vec{B}$  fields (impedance)

#### B. PLANE WAVES IN CONDUCTORS

1. Maxwell's equations
2. good and poor conductors
  - (a) decay time of charge distribution vs period of em wave
  - (b) conduction vs displacement current
  - (c) numbers
3. wave equation
4. plane wave solutions
  - (a) waves are transverse
  - (b) polarization
  - (c) dispersion relation
  - (d) ratio of amplitudes of  $\vec{E}$  and  $\vec{B}$  fields (impedance)
  - (f) good and poor conductor limits; skin depth

### 3. Electromagnetic Waves

#### A. Plane Waves in Dielectrics

Maxwell's  
equations

$$\begin{aligned} \operatorname{div} \underline{D} &= \rho \\ \operatorname{div} \underline{B} &= 0 \\ \operatorname{curl} \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \operatorname{curl} \underline{H} &= \underline{J} + \frac{\partial \underline{D}}{\partial t} \end{aligned}$$

(1) Maxwell's equations in dielectrics (insulators) <sup>ie.</sup>

for dielectrics: no free current, no free charge;  
material linear, isotropic, homogeneous

$$\begin{aligned} \underline{B} &= \mu \mu_0 \underline{H} \\ \underline{D} &= \epsilon \epsilon_0 \underline{E} \end{aligned}$$

$\mu, \epsilon$  independent  
of direction

$\mu, \epsilon$  independent  
of position

Maxwell's equations become

$$\operatorname{div} \underline{E} = 0 \quad (1)$$

$$\operatorname{div} \underline{B} = 0 \quad (2)$$

$$\operatorname{curl} \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad (3)$$

$$\operatorname{curl} \underline{B} = \mu \mu_0 \epsilon \epsilon_0 \frac{\partial \underline{E}}{\partial t} \quad (4)$$

(2) Wave equation

curl (3)

$$\operatorname{curl} \operatorname{curl} \underline{E} \equiv \operatorname{grad} \operatorname{div} \underline{E} - \nabla^2 \underline{E} = -\frac{\partial}{\partial t} \operatorname{curl} \underline{B}$$

↓  
zero using (1)

↓ using (4)

$$-\mu \mu_0 \epsilon \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

$$\nabla^2 \underline{E} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

(taking curl (4) gives an identical equation for  $\underline{B}$ )

wave equation: solutions have a speed of propagation

$$v = \frac{1}{(\mu_0 \epsilon_0)^{1/2}}$$

in free space  $v \equiv c = \frac{1}{(\mu_0 \epsilon_0)^{1/2}}$

↑  
velocity of light, which is part of the e.m. spectrum

refractive index  $n$  defined by

$$n = \frac{c}{v} = (\mu \epsilon)^{1/2}$$

(NB in general  $n, \epsilon$  depend on  $\omega$ )

### (3) plane wave solutions

choose  $\hat{z}$  to define direction of propagation

plane wave means  $\frac{\partial}{\partial x} = 0, \frac{\partial}{\partial y} = 0$

#### (a) plane waves are transverse

$$\textcircled{1} \Rightarrow \frac{\partial E_z}{\partial z} = 0$$

$$\textcircled{2} \Rightarrow \frac{\partial B_z}{\partial z} = 0$$

$$\textcircled{3} \Rightarrow \frac{\partial B_z}{\partial t} = 0$$

$$\textcircled{4} \Rightarrow \frac{\partial E_z}{\partial t} = 0$$

$\therefore$  all derivatives of  $E_z, B_z$  zero  $\therefore E_z, B_z$  constant and not part of any wave motion  $\therefore$  wave transverse

(b) polarization

x and y components of (3)

$$-\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad (5a) \quad \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad (5b)$$

x and y components of (4)

$$-\frac{\partial B_y}{\partial z} = \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \quad (6a) \quad \frac{\partial B_x}{\partial z} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t} \quad (6b)$$

solving (5a), (6b) will give a sol<sup>n</sup> for  $B_x, E_y$

" (5b), (6a) " " " " "  $B_y, E_x$

two independent solutions with  $\underline{B}, \underline{E}$  and the direction of propagation ( $\hat{z}$ ) mutually orthogonal

The direction of the  $\underline{E}$ -field is taken to define the polarization of the wave. Here we have two plane polarised solutions.

(c) dispersion relation

For plane waves of a single frequency  $\omega$ , with  $\underline{E}$  polarised along  $\hat{z}$

$$\underline{E}(z, t) = E_0 e^{j(\omega t - kz)}$$

$$\underline{B}(z, t) = B_0 e^{j(\omega t + kz)}$$

wave travelling towards +ve  $\hat{z}$

wave travelling towards -ve  $\hat{z}$

$$\textcircled{5b} \Rightarrow \mp k E_0 = -\omega B_0 \quad \textcircled{7}$$

$$\textcircled{6a} \Rightarrow \pm k B_0 = \mu_0 \epsilon_0 \omega E_0 \quad \textcircled{8}$$

these equations are consistent if

$$\frac{\omega^2}{k^2} = \frac{1}{\epsilon_0 \mu_0} \equiv v^2$$

speed  $\therefore v = \frac{1}{(\mu_0 \epsilon_0)^{1/2}}$

(as we already knew; from properties of the wave equation)

(d) ratio of the amplitudes of  $\underline{E}$  and  $\underline{B}$  fields

from  $\textcircled{7}$  or  $\textcircled{8}$

$$\frac{E_0}{B_0} = \pm \frac{\omega}{k} = \pm \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Impedance

$$Z = \left| \frac{E_0}{H_0} \right| = \left| \frac{\mu_0 E_0}{B_0} \right| = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

Impedance of free space  $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \Omega$

## B. Plane Waves in Conductors

### (1) Maxwell's equations

no free charge

linear, isotropic, homogeneous material  $\therefore \underline{D} = \epsilon\epsilon_0 \underline{E}$ ;  $\underline{B} = \mu\mu_0 \underline{H}$

$$\underline{J} = \sigma \underline{E} \quad \text{Ohm's law}$$

↑ conductivity (material property)

Maxwell's equations become

$$\text{div } \underline{E} = 0 \quad (1)$$

$$\text{div } \underline{B} = 0 \quad (2)$$

$$\text{curl } \underline{E} = - \frac{\partial \underline{B}}{\partial t} \quad (3)$$

$$\text{curl } \underline{B} = \mu\mu_0 \sigma \underline{E} + \mu\mu_0 \epsilon\epsilon_0 \frac{\partial \underline{E}}{\partial t} \quad (4)$$

(2)

Good and poor conductors(a) decay of an inhomogeneous charge distribution vs period of e.m. wave

If there is a charge distribution  $\rho_f(0)$  in a conductor at  $t=0$ , how quickly does it decay to zero?

continuity equation

$$\text{div } \underline{J} = - \frac{\partial \rho}{\partial t}$$

↙ Ohm's law

$$\sigma \text{ div } \underline{E}$$

↓ Gauss' law

$$\frac{\sigma \rho}{\epsilon \epsilon_0} = - \frac{\partial \rho}{\partial t}$$

$$\therefore \rho(t) = \rho(0) e^{-\frac{\sigma t}{\epsilon \epsilon_0}}$$

characteristic decay time  $\tau = \frac{\epsilon \epsilon_0}{\sigma}$

$$\sigma \text{ large} \Rightarrow \tau \text{ small} \quad \checkmark$$

$$\epsilon \text{ large} \Rightarrow \text{field gradients smaller} \Rightarrow \tau \text{ larger} \quad \checkmark$$

if an e.m. wave of frequency  $\omega$  is passing through the conductor the other time scale is  $\omega^{-1}$

charge distribution  $\omega^{-1} \gg \tau$   
can keep up with fields

$$\frac{\sigma}{\omega \epsilon \epsilon_0} \gg 1$$

good conductor

charge distribution  $\omega^{-1} \ll \tau$   
can't keep up with fields

$$\frac{\sigma}{\omega \epsilon \epsilon_0} \ll 1$$

poor conductor

(b) conduction vs displacement currents  
comparing to Maxwell (4)

$$\therefore \text{curl } \underline{B} = \mu_0 \sigma \underline{E} + \mu_0 \epsilon \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$\uparrow$                        $\uparrow$   
 conduction          displacement  
 current                current

for a wave  $\sim e^{j\omega t}$

$$\frac{\text{conduction current}}{\text{displacement current}} \sim \frac{\sigma}{\epsilon \epsilon_0 \omega}$$

$\therefore$  good conductor    conduction current  $\gg$  displacement current  
 poor conductor        "                                       $\ll$                                       "

(c) numbers

	$\sigma$ (S/m)	$\epsilon$	$\omega^* \sim \frac{\sigma}{\epsilon \epsilon_0}$ $\omega^*$ (s <sup>-1</sup> )
typical metal	$5 \times 10^7$	$\sim 1$	$10^{19}$
graphite	$7 \times 10^4$	$\sim 1$	$10^{16}$
salt water	20	80	$10^{10}$
silicon	$4 \times 10^{-4}$	12	$10^7$
pure water	$4 \times 10^{-6}$	80	$10^4$
typical insulator (e.g. wood, glass)	$10^{-10}$	5	10





$$\therefore \underline{E}(z, t) = E_0 e^{-\alpha z} e^{j(\omega t - kz)} \hat{z} \quad (6)$$

$$\underline{B}(z, t) = B_0 e^{-\alpha z} e^{j(\omega t - kz)} \hat{y}$$

definitions:  $v = \frac{\omega}{k}$

refractive index  $n = \frac{ck}{\omega}$

complex refractive index  $\tilde{n} = \frac{ck}{\omega}$

skin depth  $\delta = \alpha^{-1}$  (length over which amplitude decays by a factor  $e^{-1}$ )

subst. (6) into the wave eq<sup>n</sup> (5)

$$(k + j\alpha)^2 = j\mu_0\sigma\omega - \mu_0\epsilon\epsilon_0\omega^2$$

$$\therefore k^2 - \alpha^2 = -\mu_0\epsilon\epsilon_0\omega^2 \quad (7) \quad 2k\alpha = \mu_0\sigma\omega \quad (8)$$

eliminating  $k$   
subst. (8) into (7) to get a quadratic for  $k^2$ ; solve and take  $\sqrt{\quad}$  to give

$$k = \sqrt{\frac{\mu_0\epsilon\epsilon_0\omega^2}{2} \left\{ \sqrt{1 + \left(\frac{\sigma}{\epsilon\epsilon_0\omega}\right)^2} - 1 \right\}^{\frac{1}{2}}} \quad (9)$$

$$k = \sqrt{\frac{\mu_0\epsilon\epsilon_0\omega^2}{2} \left\{ \sqrt{1 + \left(\frac{\sigma}{\epsilon\epsilon_0\omega}\right)^2} + 1 \right\}^{\frac{1}{2}}}$$

N.B. always take  $\sqrt{\quad}$  so that  $k$  +ve

N.B. take  $+$   $\sqrt{\quad}$  for wave travelling to  $+\hat{z}$

(d) ratio of amplitudes of  $\underline{E}$  and  $\underline{B}$ -fields

Maxwell ③, taking y-component of curl,

$$\frac{\partial E_{zc}}{\partial z} = - \frac{\partial B_y}{\partial t}$$

substituting in the plane wave sol<sup>n</sup> ⑥

$$-(k + jk) E_0 = -j\omega B_0$$

$$\therefore B_0 = \frac{(k + jk) E_0}{j\omega}$$

$$\underline{Z} = \frac{\mu\mu_0 E_0}{B_0} = \frac{\mu\mu_0 \omega}{k - jk} \equiv Z_0 e^{+j\phi}$$

complex  
impedance

$$\text{where } Z_0 = \mu\mu_0 \left| \frac{E_0}{B_0} \right| = \frac{\mu\mu_0 \omega}{(k^2 + k^2)^{1/2}}$$

$$\therefore \tan \phi = \frac{x}{k} = \left\{ \frac{\sqrt{1 + \left(\frac{\sigma}{\epsilon\epsilon_0 \omega}\right)^2} - 1}{\sqrt{1 + \left(\frac{\sigma}{\epsilon\epsilon_0 \omega}\right)^2} + 1} \right\}^{1/2}$$

(e) Limits(i) 'good' conductor  $\frac{\sigma}{\epsilon \epsilon_0 \omega} \gg 1$ 

$$\therefore k = \kappa = \sqrt{\frac{\mu_0 \epsilon \epsilon_0 \omega^2 \sigma}{2 \epsilon \epsilon_0 \omega}} = \sqrt{\frac{\mu_0 \omega \sigma}{2}}$$

## IMPORTANT N.B!

To work out the dispersion relation or skin depth of a good conductor can derive the general formula<sup>(9)</sup> for  $\kappa$ ,  $k$  and then approximate

but unless you need the general formula for some other reason it is much easier to approximate earlier e.g. in (5)

$$(5) \Rightarrow \nabla^2 \underline{E} = \mu_0 \sigma \frac{\partial \underline{E}}{\partial t} + \mu_0 \epsilon \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

↑  
negligible for a good conductor

substituting in the plane wave solution (6)

$$\underline{E}(z, t) = E_0 e^{-kz} e^{j(\omega t - kz)} \hat{x}$$

gives the good conductor dispersion relation

$$(k + j\kappa)^2 = j\mu_0 \sigma \omega$$

equating real and imaginary parts

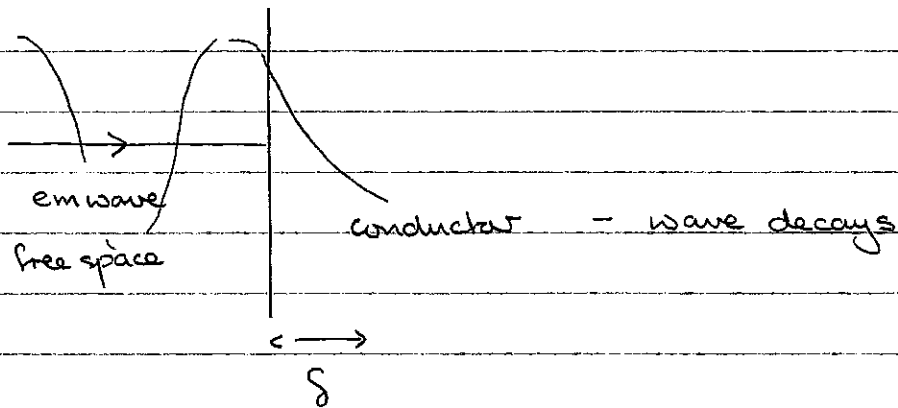
$$\therefore k^2 - \kappa^2 = 0 \quad (\text{cf } (7))$$

$$2k\kappa = \mu_0 \sigma \omega \quad (\text{cf } (8))$$

$$\therefore k = \kappa = \sqrt{\frac{\mu_0 \sigma \omega}{2}} \quad (\text{as before})$$

NB2 skin depth

$$\delta = \frac{1}{k} = \sqrt{\frac{2}{\mu\omega\sigma}}$$



typical metal, visible,  $\delta \sim$  few nm  
 microwave  $\sim \mu\text{m}$   
 radio  $\sim \text{mm}$

(ii) 'poor' conductor

$$\frac{\sigma}{\epsilon\epsilon_0\omega} \ll 1$$

$$K = \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0\omega^2}{2} \left( 1 + \frac{\sigma^2}{2(\epsilon\epsilon_0\omega)^2} - 1 \right)^{1/2}}$$

$$= \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0}{2 \cdot 2}} \cdot \omega \cdot \frac{\sigma}{\epsilon\epsilon_0\omega} = \frac{\sigma}{2} \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$$

skin depth  
of a poor  
conductor is  
independent of  
 $\omega$

$$k = \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0}{2}} \cdot \omega \sqrt{2} \left( 1 + \frac{\sigma^2}{4(\epsilon\epsilon_0\omega)^2} \right)^{1/2}$$

$$= \omega \sqrt{\mu\mu_0\epsilon\epsilon_0} \left( 1 + \frac{\sigma^2}{8(\epsilon\epsilon_0\omega)^2} \right)$$

↑  
can ignore of

(iii) dielectric  $\sigma = 0$   $k = 0$  ✓

$$k = \omega \cdot \frac{1}{v} \quad \checkmark$$