I ELECTROSTATICS

A MOSTLY REVISION

1. charge, Coulomb's law, superposition
2. electric field, field lines
3. Gauss' law
   (example)
4. curl $\underline{E} = 0$
5. potential

6. Poisson and Laplace equations
7. summary so far
8. electrostatic boundary conditions
9. conductors
   (example)

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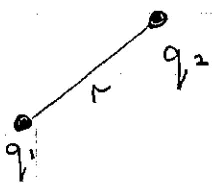
1. charge, Coulomb's law, superposition

Experimentally certain objects interact. Can describe the interaction by assigning a +ve or −ve charge. The force between objects with charges q_1, q_2 is



Coulomb's law

$$\underline{F} = \frac{q_1 q_2 \hat{\underline{r}}}{4\pi\epsilon_0 r^2}$$

unit vector along line between charges

N.B. (r^2)

ϵ_0: permittivity of free space $8.85\times10^{-12}\ C^2 N^{-1} m^{-2}$

- like charges repel, unlike charges attract
- charge due to unpaired electrons or protons, but Coulomb's law predated knowing this.

SUPERPOSITION Interaction between any two charges is not affected by the presence of other charges.

2. electric field, field lines

The electric field $\underline{E}(\underline{r})$ is the force that would be exerted on unit charge at position $\underline{r}$.

e.g. for a point charge q at the origin

$$\underline{E}(\underline{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\underline{r}}$$

superposition implies that the $\underline{E}$-field due to several charges is the vector sum of the fields due to each individual charge $q_i(\underline{r}_i')$

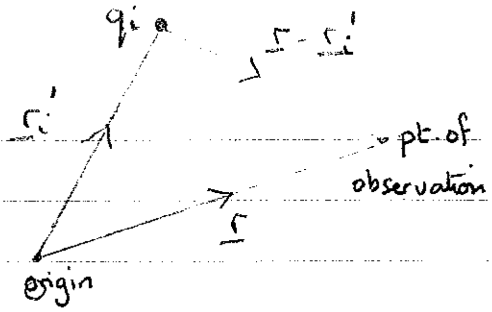
$$\underline{E}(\underline{r}) = \sum_i \frac{q_i\,\widehat{(\underline{r}-\underline{r}_i')}}{4\pi\varepsilon_0 |(\underline{r}-\underline{r}_i')|^2}$$

unit vector along $\underline{r}-\underline{r}_i'$

sum over charges labelled by i

position vector of charge i

position vector of point of observation



which generalises for a continuous charge distribution, to

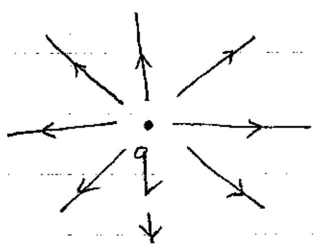
$$\underline{E}(\underline{r}) = \int_\tau \frac{\rho(\underline{r}')\,\widehat{(\underline{r}-\underline{r}')}}{4\pi\varepsilon_0 |\underline{r}-\underline{r}'|^2}\, d^3\underline{r}'$$

charge density

τ ← integrate over volume containing the charge

<u>field lines</u>: way of representing $\underline{E}$-field graphically
direction along $\underline{E}$
density $\propto |\underline{E}|$

check that this makes sense for a point charge:



remember that this is a 2-d cross section of a 3-d 'hedgehog'

density of lines at distance r from the charge is

$$\frac{k}{4\pi r^2}$$

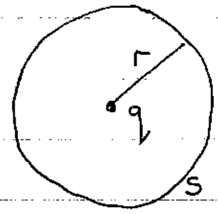
← no of lines
← surface area at distance r

$\propto E \Rightarrow$ expected $\frac{1}{r^2}$ dependence

3. <u>Gauss' law for electric fields</u>

follows from the inverse square law and superposition

- consider a charge q at the origin surrounded by a sphere of radius r



$$\int_S \underline{E} \cdot d\underline{S}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{q}{4\pi\epsilon_0 \cancel{r^2}} \overbrace{\hat{\underline{r}} \cdot \hat{\underline{r}}}^{1} \cancel{r^2} \sin\theta \, d\theta \, d\phi$$

$$= \frac{q}{4\pi\epsilon_0} \cdot 4\pi = \frac{q}{\epsilon_0}$$

(if surface is not a sphere this still holds as the flux through the surface will remain unchanged)

- if there are several charges inside S, superposition implies

$$\int_S \underline{E} \, d\underline{S} = \sum_i \frac{q_i}{\epsilon_0}$$

closed surface

sum over all charges within S

generalising for a continuous distribution of charge

$$\int_S \underline{E} \cdot d\underline{S} = \frac{1}{\epsilon_0}\int_\tau \rho \, d\tau$$

volume enclosed by S

integral form of Gauss' law

$\downarrow$ divergence thm.

$$\int_\tau \operatorname{div} \underline{E} \, d\tau = \frac{1}{\epsilon_0}\int_\tau \rho \, d\tau \qquad \forall \text{ volumes}$$

$$\therefore \operatorname{div} \underline{E} = \frac{\rho}{\epsilon_0}$$

differential form of Gauss' law

~~Problem~~ Example

Electric field is $\underline{E} = kr^2 \hat{\underline{r}}$

1. Find $\rho(\underline{r})$:

use $\quad \text{div } \underline{E} = \dfrac{\rho}{\epsilon_0}$

$$\rho = \epsilon_0 \text{div } \underline{E} = \frac{\epsilon_0}{r^2} \frac{\partial}{\partial r} \left(r^2 E_r \right) = \frac{\epsilon_0}{r^2} k 4 r^3 = 4 k r \epsilon_0$$

careful with div, grad, curl in non-Cartesian co-ord systems

2. Find charge in sphere of radius a centred on origin:

method 1: direct integration of $\rho(r)$

$$Q = \int_0^a 4 k r \epsilon_0 \, . \, 4\pi r^2 \, dr = 4\pi k \epsilon_0 a^4$$

method 2: Gauss thm

$$E(r=a) \, . \, 4\pi a^2 = \frac{Q}{\epsilon_0} \; ; \quad \therefore Q = 4\pi k \epsilon_0 a^4$$

4. to prove $\text{curl}\ \underline{E} = 0$

or, equivalently, $\oint \underline{E} \cdot \underline{dl} = 0$, $\underline{E}$ conservative

for a point charge q at the origin

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{\underline{r}}$$

in spherical polars $\underline{dl} = dr\,\hat{\underline{r}} + r\,d\theta\,\hat{\underline{\theta}} + r\sin\theta\,d\varphi\,\hat{\underline{\varphi}}$

$$\therefore \int_A^B \underline{E} \cdot \underline{dl} = \int_A^B \frac{q\,dr}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

depends on end-points, but not on path

if $A = B$ $\oint_C \underline{E} \cdot \underline{dl} = 0$

$\downarrow$ Stokes' thm

$$\int_S \text{curl}\,\underline{E} \cdot \underline{dS} = 0 \qquad \forall\ S \text{ bounded by } C$$

$$\therefore \text{curl}\,\underline{E} = 0$$

(for several charges superposition $\Rightarrow$ $\underline{E} = \underline{E}_1 + \underline{E}_2 + \dots$

$\therefore \text{curl}\,\underline{E} = \text{curl}\,\underline{E}_1 + \text{curl}\,\underline{E}_2 + \dots = 0$)

5. potential

$$\int_{\underline{R}}^{\underline{r}} \underline{E}(\underline{r}') \cdot \underline{dl}'$$

depends only on $\underline{r}$ and not on the path taken to get there

reference point

$\therefore$ sensible to define the electric potential

$$V(\underline{r}) = -\int_{\underline{R}}^{\underline{r}} \underline{E}(\underline{r}') \cdot \underline{dl}'$$

reminder of some maths:

$$\int_{\underline{r}_2}^{\underline{r}_1} \text{grad } V \cdot d\underline{l}'$$

$$= \int_{\underline{r}_2}^{\underline{r}_1} \left(\frac{\partial V}{\partial x'} dx' + \frac{\partial V}{\partial y'} dy' + \frac{\partial V}{\partial z'} dz' \right)$$

$$= \int_{\underline{r}_2}^{\underline{r}_1} \delta V = V(\underline{r}_1) - V(\underline{r}_2)$$

to show $\underline{E} = -\text{grad } V$:

$$V(\underline{r}_1) - V(\underline{r}_2) \equiv \int_{\underline{r}_2}^{\underline{r}_1} \text{grad } V \cdot d\underline{l}'$$

$$= -\int_{\underline{R}}^{\underline{r}_1} \underline{E} \cdot d\underline{l}' + \int_{\underline{R}}^{\underline{r}_2} \underline{E} \cdot d\underline{l}' = -\int_{\underline{r}_2}^{\underline{r}_1} \underline{E} \cdot d\underline{l}'$$

$$\therefore \underline{E} = -\text{grad } V$$

$$\left(\text{check} \quad \text{curl } \underline{E} = -\text{curl grad } V \equiv 0 \quad \checkmark \right)$$

<u>comments</u>

(i) for a point charge at the origin

$$V(\underline{r}) = -\int_{\infty}^{\underline{r}} \frac{q}{4\pi\epsilon_0 r'^2} \underbrace{\hat{\underline{r}}' \cdot d\underline{l}'}_{dr'} = \frac{q}{4\pi\epsilon_0 r}$$

sensible and usual reference point

(ii) potential obeys superposition principle – usually easier to add contributions from different charges to V than to $\underline{E}$ because V is a scalar.

(iii) for a charge distribution (superposition again)

$$V(\underline{r}) = \int_\tau \frac{\rho(\underline{r}')}{4\pi\epsilon_0 |\underline{r}-\underline{r}'|} \, d^3\underline{r}'$$

(iv) work needed to move a charge from $\underline{r}_1$ to $\underline{r}_2$ is

$$W = - \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{l} = - q \int_{\underline{r}_1}^{\underline{r}_2} \underline{E} \cdot d\underline{l}$$

force on charge
$q\underline{E}$

$$\equiv q \{ V(\underline{r}_2) - V(\underline{r}_1) \}$$

<u>potential difference</u> between $\underline{r}_1$ and $\underline{r}_2$ is the <u>work per unit charge</u> needed to move a body from $\underline{r}_1$ to $\underline{r}_2$.

6. <u>Poisson and Laplace equations</u>

$$\text{div } \underline{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss' law)}$$

and $\underline{E} = -\text{grad } V$

$$\therefore \text{ div grad } V \equiv \underline{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \qquad \text{Poisson's equation}$$

if there is no charge

$$\underline{\nabla^2 V = 0} \qquad \text{Laplace equation}$$

7. <u>summary so far</u>

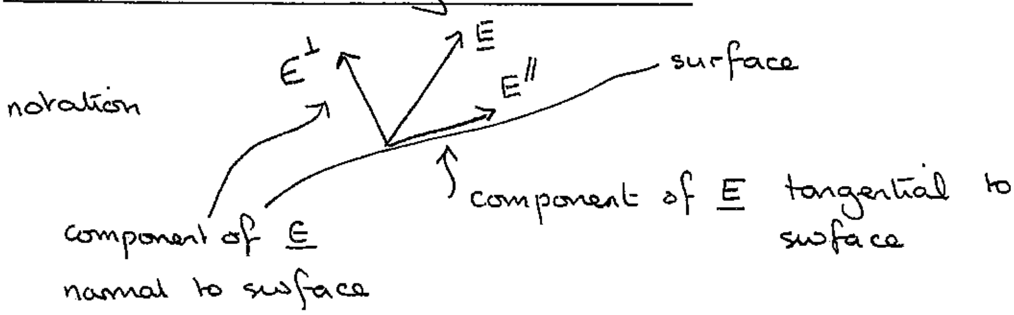
Coulomb's law + superposition

$$\Downarrow$$

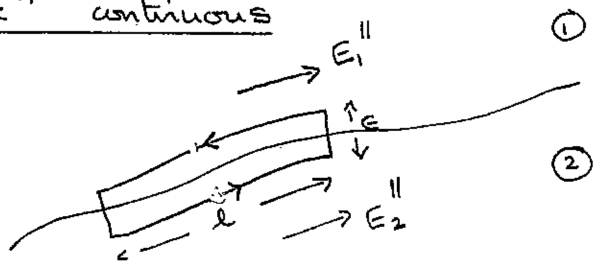
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & \rho \arrow[dl, shift right, "V(\underline{r}) = \int_\tau \frac{\rho(\underline{r}')\, d^3\underline{r}'}{4\pi\varepsilon_0 |\underline{r}-\underline{r}'|}"'] \arrow[dr, shift left, "\underline{E}(\underline{r}) = \int_\tau \frac{\rho(\underline{r}')\, (\widehat{\underline{r}-\underline{r}'})}{4\pi\varepsilon_0 |\underline{r}-\underline{r}'|^2} d^3\underline{r}'"] & \\
V \arrow[ur, shift right, "\nabla^2 V = -\frac{\rho}{\varepsilon_0}"'] \arrow[rr, shift left, "\underline{E} = -\operatorname{grad} V"] & & \underline{E} \arrow[ul, shift left, "\operatorname{div}\underline{E} = \frac{\rho}{\varepsilon_0},\ \operatorname{curl}\underline{E} = 0"] \arrow[ll, shift left, "V = -\int \underline{E}\cdot d\underline{l}"]
\end{tikzcd}
$$

(integral form of Gauss' law for situations of high symmetry)

8. Electrostatic boundary conditions



(i) $E^{\parallel}$ continuous



$$\oint \underline{E} \cdot \underline{dl} = 0$$

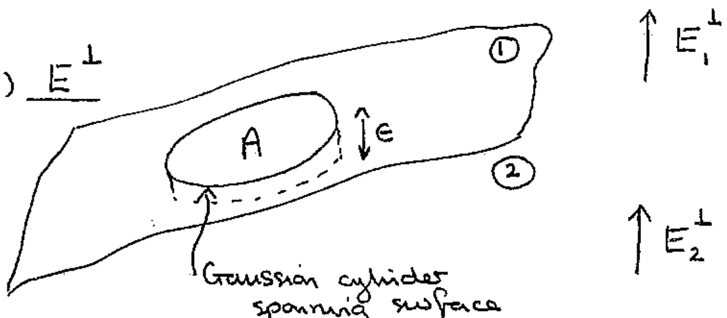
take $\epsilon \to 0$ so can ignore ends of loop ⊥ to surface
l small enough that $\underline{E}$ does not vary

$$\therefore \left(E_2^{\parallel} - E_1^{\parallel}\right) l = 0$$

$$\underline{E_1^{\parallel} = E_2^{\parallel}}$$

<u>tangential component of $\underline{E}$ is continuous</u>

(ii) $E^\perp$



charge density σ on surface

take $\epsilon \to 0$ so can ignore sides of cylinder
take A small enough that can concider $\underline{E}$ constant

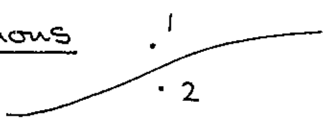
$$\int \underline{E} \cdot \underline{dS} = E_1^\perp A - E_2^\perp A = \frac{\sigma A}{\epsilon_0}$$

surface charge density

$$E_1^\perp - E_2^\perp = \frac{\sigma}{\epsilon_0}$$

(N.B. useful way of calculating σ given $E_1^\perp$, $E_2^\perp$)

(iii) V continuous



$$\Delta V = -\int_2^1 \underline{E} \cdot \underline{dl}$$

change in V across surface

as 1, 2 approach surface $d\underline{l} \to 0$ $\therefore$ $\Delta V \to 0$ (unless $\underline{E} \to \infty$ i.e. infinite force)

9. Conductors

material in which charge is free to move

(i) $\underline{E} = 0$ inside a conductor
(because if $\underline{E} \neq 0$, charge will move around until $\underline{E} = 0$)

(ii) $\rho = 0$ inside a conductor

$$\text{div}\,\underline{E} = \frac{\rho}{\epsilon_0} \qquad \text{and} \quad \underline{E} = 0$$

(iii) $\therefore$ any net charge resides on the surface

(iv) a conductor is an equipotential (including the surface)
for two points of the conductor a, b

$$V_{\underline{a}} - V_{\underline{b}} = -\int_{\underline{b}}^{\underline{a}} \underline{E} \cdot d\underline{l} = 0 \qquad \therefore V_{\underline{a}} = V_{\underline{b}}$$

(v) just outside the surface $E^{\parallel}_{out} = 0$ $\quad E^{\perp}_{out} = \dfrac{\sigma}{\epsilon_0}$

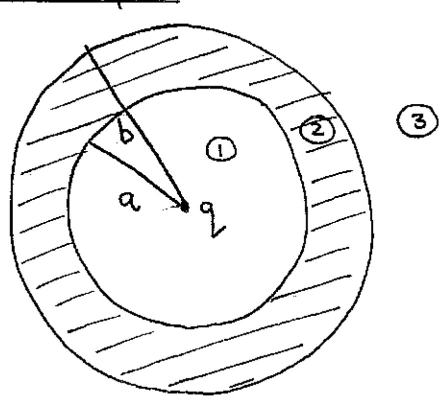
follows from boundary conditions

$$E^{\parallel}_{in} = E^{\parallel}_{out}$$

$$E^{\perp}_{out} - E^{\perp}_{in} = \frac{\sigma}{\epsilon_0}$$

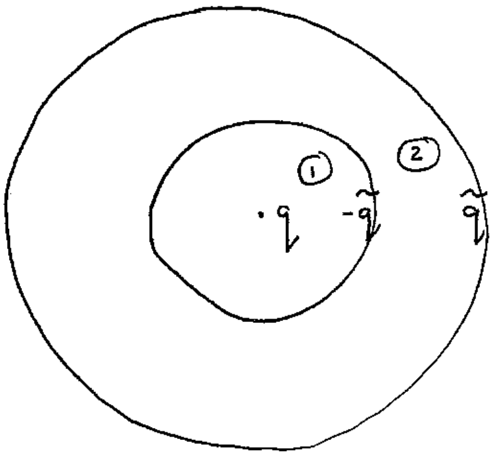
and $E^{\parallel}_{in} = E^{\perp}_{in} = 0$.

example



charge q at origin surrounded by uncharged spherical conducting shell

find $\underline{E}_1, \underline{E}_2, \underline{E}_3$
V_1, V_2, V_3



induced charge on surfaces (see (iii))

Gauss' law

$$\underline{E}_1 = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\underline{r}} \quad ; \quad \underline{E}_2 = \frac{(q-\tilde{q})}{4\pi\varepsilon_0 r^2}\hat{\underline{r}} \quad ; \quad \underline{E}_3 = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\underline{r}}$$

$$= 0 \quad \text{(from (i))}$$

$$\therefore q = \tilde{q}$$

$$V_3 = -\int_{\infty}^{r} \underline{E}_3 \cdot d\underline{r} = \frac{q}{4\pi\varepsilon_0 r}$$

$$V_2 = -\int_{\infty}^{b} \underline{E}_3 \cdot d\underline{r} - \int_{b}^{r} \underline{E}_2 \cdot d\underline{r} = \frac{q}{4\pi\varepsilon_0 b}$$ constant, good (see (iv))

$$V_1 = -\int_{\infty}^{b} \underline{E}_3 \cdot d\underline{r} - \int_{b}^{a} \underline{E}_2 \cdot d\underline{r} - \int_{a}^{r} \underline{E}_1 \cdot d\underline{r} = \frac{q}{4\pi\varepsilon_0 b} - \frac{q}{4\pi\varepsilon_0 a} + \frac{q}{4\pi\varepsilon_0 r}$$

check (v) :

just outside outer surface

$$E^{\perp}(r=b) \equiv E_3(r=b) = \frac{q}{4\pi\varepsilon_0 b^2} = \frac{\sigma^{outer}}{\varepsilon_0}$$

just outside inner surface

$$E^{\perp}(r=a) = -E_1(r=a) = \frac{-q}{4\pi\varepsilon_0 a^2} = \frac{\sigma^{inner}}{\varepsilon_0}$$

$\underline{E}_1$ is along $+\hat{\underline{r}}$
unit normal to surface is along $-\hat{\underline{r}}$