

4. GUIDED WAVES

A. Transmission lines

1. Introduction
2. Telegraph equations v, z
3. boundary between transmission lines
4. termination by a load
5. coaxial line
6. e.m. waves on a coaxial line

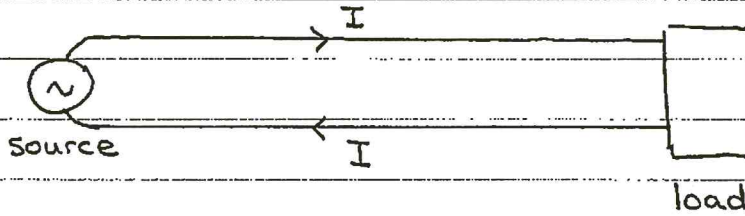
B. Waveguides

GUIDED WAVES

A. Transmission Lines

1. Introduction

Considers the circuit



unless length of line $\ll \lambda$ of radiation flowing along it
 or, equivalently, line signal takes to $\ll \frac{L}{\omega}$
 travel along wire ω \uparrow freq. of source

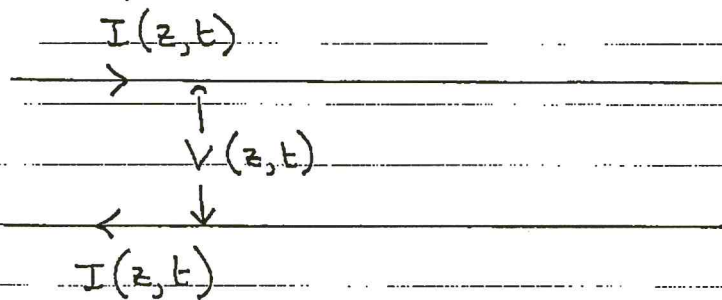
need to consider wires as an explicit element in the circuit called a transmission line

(we shall assume that the distance between the wires remains constant)

the necessity to connect source and load by a transmission line causes

delays (e.g. $\sim 5 \text{ ns m}^{-1}$ in typical dielectric cable connecting 2 computers, say)
 reflections if line not matched to load \uparrow i.e. same impedance
 dispersion

2. Telegraph equations



L, C inductance, capacitance per unit length of line

$$V(z+dz, t) = V(z, t) - Ldz \frac{\partial I(z, t)}{\partial t}$$

↓ Taylor expansion

$$= V(z, t) + \frac{\partial V(z, t)}{\partial z} dz$$

$$\therefore \frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} \quad (1)$$

Let Q be the charge between z and z + dz

$$C dz V = Q$$

$$C dz \frac{\partial V}{\partial t} = \frac{\partial Q}{\partial t}$$

but $\frac{\partial Q}{\partial t} = I(z, t) - I(z+dz, t) = -\frac{\partial I}{\partial z} dz$

$$\therefore \frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t} \quad (2)$$

comparing $\frac{\partial}{\partial z}$ (1) and $\frac{\partial}{\partial t}$ (2) \Rightarrow

$$\frac{\partial^2 V}{\partial z^2} = LC \frac{\partial^2 V}{\partial t^2}$$

wave equation with velocity

$$v = \frac{1}{\sqrt{LC}}$$

$$V(z, t) = f(z-vt) + g(z+vt)$$

using (2)

$$\frac{\partial I}{\partial z} = Cv f' - Cv g'$$

$$\therefore I = Cv f - Cv g = \sqrt{\frac{C}{L}} f - \sqrt{\frac{C}{L}} g$$

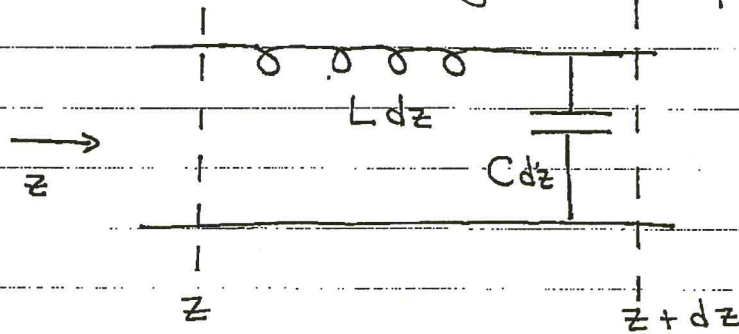
NB $f'(u) \equiv \frac{df}{du}$
ie differential of f with respect to its argument

$$\therefore Z = \frac{V}{I} = \pm \sqrt{\frac{L}{C}}$$

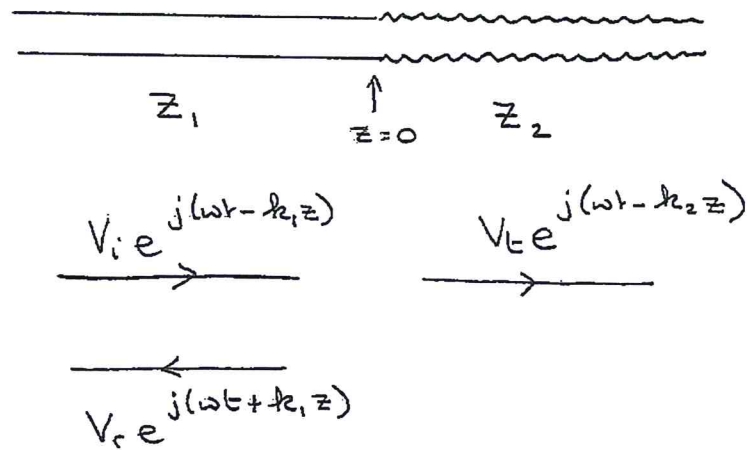
+ve for waves travelling to $+\frac{\hat{x}}$
 -ve " " " " " " " " $-\frac{\hat{x}}$

↑
 characteristic impedance of
 the transmission line

N.B. equivalent circuit for a length dz of loss free line:



3. boundary between transmission lines



$$\text{At } z=0 \quad V_i = V_2$$

$$\therefore V_i + V_r = V_t$$

$$I_1 = I_2$$

$$\therefore \frac{V_i}{Z_1} - \frac{V_r}{Z_1} = \frac{V_t}{Z_2}$$

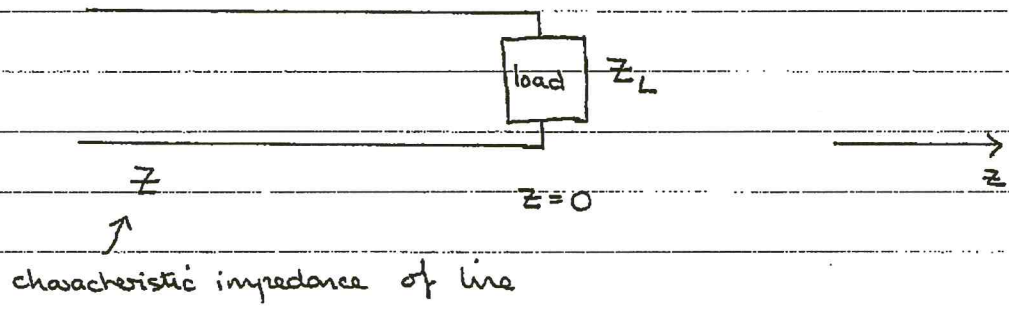
$$\therefore \frac{V_r}{V_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\frac{V_t}{V_i} = \frac{2Z_2}{Z_2 + Z_1}$$

$$\text{Power} = IV = \frac{V^2}{Z}$$

$$\therefore \text{expect } \frac{V_i^2}{Z_1} = \frac{V_r^2}{Z_1} + \frac{V_t^2}{Z_2} \quad \checkmark$$

4. Termination by a load



$V_i e^{j(\omega t - kz)}$ (rightward wave)
 $V_r e^{j(\omega t + kz)}$ (leftward wave)

at $z=0$ " $V = I Z_L$ "

$$\begin{aligned}
 V &= V_i e^{j(\omega t - kz)} + V_r e^{j(\omega t + kz)} \\
 I &= \frac{V_i}{Z} e^{j(\omega t - kz)} - \frac{V_r}{Z} e^{j(\omega t + kz)}
 \end{aligned}$$

$$\therefore V_i + V_r = \left(\frac{V_i}{Z} - \frac{V_r}{Z} \right) Z_L$$

$$\therefore \frac{V_r}{V_i} = \frac{Z_L - Z}{Z_L + Z}$$

①

impedance matched: $Z_L = Z \quad \therefore V_r = 0$
 max. power transfer to load

open circuit $Z_L = \infty \quad I = 0$ at $z=0 \quad \therefore V_r = V_i$



short closed circuit $Z_L = 0 \quad V = 0$ at $z=0 \quad \therefore V_r = -V_i$
 (phase change of π)

input impedance

if line has length l

$$Z_{in} \equiv \frac{V(-l)}{I(-l)} = \left\{ \frac{V_i e^{jkl} + V_r e^{-jkl}}{V_i e^{jkl} - V_r e^{-jkl}} \right\} Z$$

$$= \text{(using ①)} \left\{ \frac{(Z_L + Z) e^{jkl} + (Z_L - Z) e^{-jkl}}{(Z_L + Z) e^{jkl} - (Z_L - Z) e^{-jkl}} \right\} Z$$

for a $\lambda/2$ line $e^{\pm jkl} = e^{\pm j\pi} = -1$

$\therefore Z_{in} = Z_L$

for a $\lambda/4$ line $e^{\pm jkl} = e^{\pm j\pi/2} = \pm i$

$\therefore Z_{in} = \frac{Z^2}{Z_L}$

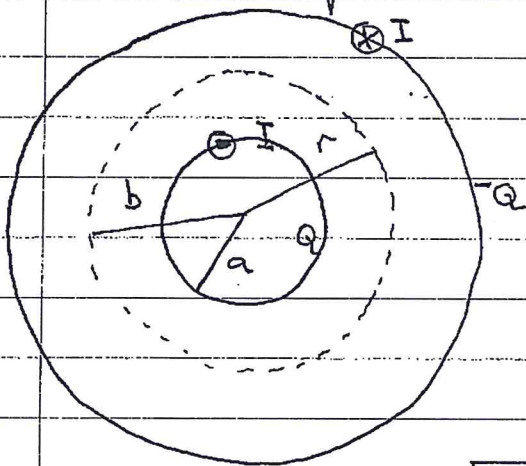
(can be placed between a long line & a load to aid impedance matching)

N.B. in general Z_L and Z_{in} are complex
 Z is real for a loss-free line

5. Coaxial transmission line (to calculate v, Z)

(i) capacitance

(all extensive quantities defined per unit length)



Gauss:

$$E \cdot 2\pi r = \frac{Q}{\epsilon \epsilon_0}$$

$$\therefore V = - \int_b^a \frac{Q}{2\pi \epsilon \epsilon_0 r} dr = \frac{Q}{2\pi \epsilon \epsilon_0} \ln \frac{b}{a}$$

$$\therefore C = \frac{Q}{V} = \frac{2\pi \epsilon \epsilon_0}{\ln \frac{b}{a}}$$

(ii) inductance

Ampère:

$$B \cdot 2\pi r = \mu_0 I$$

$$\therefore \phi = \int_a^b \frac{\mu_0 I}{2\pi r} dr = \frac{\mu_0 I}{2\pi} \ln \frac{b}{a}$$

$$\therefore L = \frac{\phi}{I} = \frac{\mu_0 \ln \frac{b}{a}}{2\pi}$$

$$\therefore v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \text{ as expected}$$

the waves are e.m. waves - can either view them in terms of V, I in wires or B, E in the space around the wires

6. em waves travelling along a coaxial line

need a solution which obeys Maxwell's equations and the boundary conditions on the surfaces of the conductors:

$$B_{\perp} = 0 \quad E_{\parallel} = 0$$

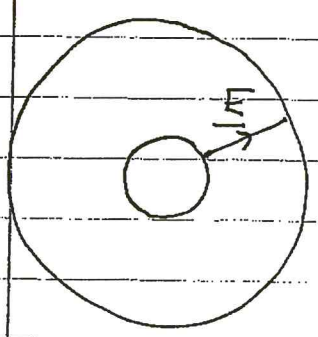
(from $\text{div } \underline{B} = 0$) ($\underline{E} = 0$ inside conductors and E_{\parallel} continuous)

check that a solution that works is

$$\underline{E} = \frac{A}{r} e^{j(\omega t - kz)} \hat{r} ; \quad \underline{B} = \frac{A \sqrt{\epsilon_0 \mu_0}}{r} e^{j(\omega t - kz)} \hat{\theta}$$

amplitude

can fix A by looking at the line voltage



$$V = - \int_b^a \frac{A}{r} e^{j(\omega t - kz)} dr$$

$$= A \ln \frac{b}{a} e^{j(\omega t - kz)}$$

$$\equiv V_0 e^{j(\omega t - kz)}$$

$$\therefore \underline{E} = \frac{V_0}{\ln b/a} \cdot \frac{1}{r} e^{j(\omega t - kz)} \hat{r}$$

$$\underline{B} = \frac{V_0}{\ln b/a} \sqrt{\mu_0 \epsilon_0} \frac{1}{r} e^{j(\omega t - kz)} \hat{\theta}$$

we have two other boundary conditions:

on E^\perp (ie. Gauss' law) can be used to give the charge on the plates

on B^\parallel (ie. Ampere's law)

assuming B^\parallel is small inside the conductor and can be ignored this can be used to give the current

$$2\pi a B(r=a) = \mu_0 I$$

$$\therefore I_0 = \frac{2\pi}{\mu_0} \frac{V_0}{\ln b/a} \sqrt{\epsilon_0 \mu_0}$$

$$\therefore Z \equiv \frac{V_0}{I_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\ln(b/a)}{2\pi} \equiv \sqrt{\frac{L}{C}} \quad \text{as before}$$

2. Waveguides

- hollow metal conductor with cross section of constant shape
- em waves reflect off walls to set up an e.m. wave which transports energy along guide.
- if walls perfectly conducting, there are no losses
- most efficient method of power transport for frequencies $\gtrsim 10^3$ MHz

Wave equation for lossless waveguides

assume:

1. wave propagates in a linear dielectric (metallic walls are o.k.: they just provide boundary conditions)
2. zero attenuation
3. propagation is in a straight line along \hat{z}

(N.B. differs from Sec. 3B (plane waves in dielectric) because waves are not plane $\therefore \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \neq 0$)

Maxwell's equations \Rightarrow

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{v}\right)^2 - k^2 \right\} E_z = 0 \quad (1)$$

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{v}\right)^2 - k^2 \right\} B_z = 0 \quad (2)$$

T.E. modes $E_z = 0$

solve (2) using separation of variables, and boundary condition that $B_{\perp} = 0$ to give

$$B_z = \sum_{m,n} B_{m,n} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{j(\omega t - kz)}$$

and a dispersion relation

$$k^2 = \left(\frac{\omega}{v}\right)^2 - \frac{m^2 \pi^2}{a^2} - \frac{n^2 \pi^2}{b^2}$$

for a given mode to propagate

$$\omega > \omega_{m,n} = v \pi \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{1/2}$$

