CP2: ELECTROMAGNETISM

https://canvas.ox.ac.uk/courses/224992/pages/cp2-electromagnetism

Lecture 15:

Electromagnetic Induction

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$$\underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \underline{\nabla} \times \underline{\mathbf{B}} = \underline{\mathbf{J}} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1]With many thanks to Prof Neville Harnew and Prof Laura Herz

OUTLINE: 15. ELECTROMAGNETIC INDUCTION

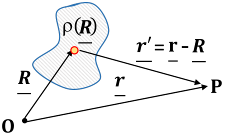
15.1 Summary : Electrostatics & Magnetostatics

15.2 Electromagnetic induction - outline

15.3 Faraday's and Lenz's Laws of Induction

15.4 Maxwell-Faraday equation in differential form

15.1.1 Summarising where we are : electrostatics



1. Coulomb's Law :

$$\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\underline{\mathbf{R}})}{(\underline{\mathbf{r}}-\underline{\mathbf{R}})^2}\,\widehat{(\underline{\mathbf{r}} - \underline{\mathbf{R}})}\, d\mathcal{V}$$

2. Gauss's Law :

$$\underbrace{\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = Q_{enc}/\epsilon_0}_{\text{integral form}} \quad \rightarrow \quad \underbrace{\underline{\nabla} \cdot \underline{\mathbf{E}} = \rho/\epsilon_0}_{\text{differential form}}$$

- Electric charges generate electric fields. Electric field lines begin and end on charge or at ∞.

3. Electric field is conservative :

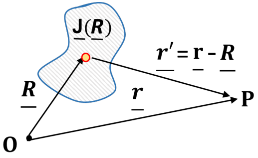
- $\oint \underline{\mathbf{E}} \cdot \underline{\mathbf{d}}\ell = 0$ (work done is independent of path)
- Stokes' Theorem : $\oint \underline{\mathbf{E}} \cdot \underline{\mathbf{d}}\ell = \int_S (\underline{\nabla} \times \underline{\mathbf{E}}) \cdot \underline{\mathbf{da}} \rightarrow \underline{\nabla} \times \underline{\mathbf{E}} = 0$
- $\rightarrow$ there is a well defined potential V such that $\underline{\mathbf{E}} = -\underline{\nabla}\, V$

  such a V always exists since : $\underline{\nabla} \times \underline{\mathbf{E}} = -\underline{\nabla} \times \underline{\nabla}\, V = 0$ (vector identity)

15.1.2 Summarising where we are : magnetostatics

1. Biot-Savart Law :

$$\underline{\mathbf{B}}(\underline{\mathbf{r}}) = \frac{\mu_0}{4\,\pi} \int_{\mathcal{V}} \frac{\underline{\mathbf{J}}(\underline{\mathbf{R}})}{(\underline{\mathbf{r}} - \underline{\mathbf{R}})^2} \times \widehat{(\underline{\mathbf{r}} - \underline{\mathbf{R}})}\, d\mathcal{V}$$



2. Gauss's Law for magnetism :

$$\underbrace{\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0}_{\text{integral form}} \;\rightarrow\; \underbrace{\underline{\nabla} \cdot \underline{\mathbf{B}} = 0}_{\text{differential form}}$$

   - No magnetic monopoles. Magnetic field lines form closed loops.

3. Ampere's Law :

   - Magnetic fields are generated by electric currents

$$\rightarrow \quad \oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0\, I_{enc} \quad \rightarrow \quad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0\, \underline{\mathbf{J}}$$

4. Continuity equation :

   - $\oint_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}} = -\frac{d}{dt} \int_{\mathcal{V}} \rho(\mathcal{V})\, d\mathcal{V} \;\rightarrow\; \underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial \rho}{\partial \mathbf{t}}$ (charge conserved)

Maxwell's equations in static limit

Electrostatics

Coulomb's law: $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^2} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r'$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \leftrightarrow \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q_V}{\varepsilon_0}$$

Maxwell 1: Gauss's law. Charge generates an electric field. Electric field lines begin and end on charge.

$$\nabla \times \mathbf{E} = 0 \leftrightarrow \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0$$

Maxwell 3: There is a well-defined electric scalar potential V, with: $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$

Magnetostatics

Biot-Savart law: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^2} \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r'$

$\nabla \cdot \mathbf{J} = 0$

$$\nabla \cdot \mathbf{B} = 0 \leftrightarrow \oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0$$

Maxwell 2: There are no magnetic monopoles. Magnetic field lines form closed loops.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \leftrightarrow \oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I$$

Maxwell 4: Electric currents generate magnetic fields.

Magnetic vector and scalar potentials

Off syllabus, but worth a mention :

Magnetic vector potential A defined through: $\mathbf{B} = \nabla \times \mathbf{A}$

Such **A** always exists because: $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$

Inserting into Ampere's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A})$$
$$= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

There is a certain degree of freedom in which to choose **A** (i.e. can add any function with zero curl to it) – one convenient choice is to ensure $\nabla \cdot \mathbf{A} = 0$

Poisson equations for magnetostatics:
(one for each **J** & **A** coordinate)

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Magnetic scalar potential V_m: $\mathbf{B} = -\mu_0 \nabla V_m \longleftrightarrow V_m = -\frac{1}{\mu_0} \int_A^B \mathbf{B} \cdot d\mathbf{l}$

Caution: V_m is *pathway-dependent and not single-valued* because $\nabla \times \mathrm{B} \neq 0$.

But V_m can be used with care in simply-connected, current-free regions

Being a scalar, V_m is mathematically easier to use than the vector potential

15.2 Electromagnetic induction - outline

So far we have considered only stationary charges and steady currents :
$\rightarrow$ **electro**- and **magneto**-*statics* ($\frac{\partial \rho}{\partial t} = 0$ and $\frac{\partial J}{\partial t} = 0$)

Now consider what happens when either $\underline{\mathbf{E}}$ or $\underline{\mathbf{B}}$ varies with time :

1. Introduction: Electromagnetic Induction
2. Faraday's and Lenz's Laws of Induction
3. Self-Inductance and Mutual Inductance
4. The Transformer
5. Energy of the Magnetic Field

} Problem Set 4 (items 1–5)

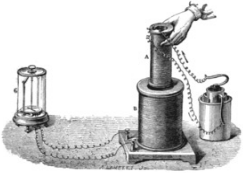
6. Charged Particles in E- and B-Fields

} Problem Set 5 (item 6)

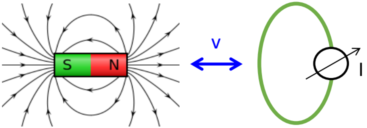
Origins of electromagnetic induction

1831: Michael Faraday carries out a series of experiments and observes:

Two coils are arranged in a way so that the magnetic flux density of coil A penetrates through coil B. He finds that if the B-field in one coil is changing, this induces an electrical current in coil B.



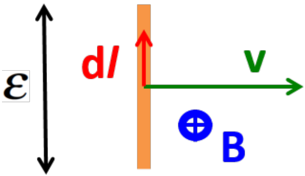
Moving a circuit (wire loop) through a magnetic field generates a current in the circuit. Moving instead the magnet w.r.t. the circuit, gives the same result. Varying the field strength, with circuit and magnet stationary, also generates a current.



→ A change with time in the magnetic flux density through a circuit causes an “electromotive force” that moves charges along the circuit.

15.3 Faraday's and Lenz's Laws of Induction

15.3.1 Motional electromotive force (EMF) from Lorentz force



- Consider a wire moving with velocity $\underline{\mathbf{v}}$ through a $\underline{\mathbf{B}}$-field
- Free charges in the wire experience a Lorentz force, perpendicular to $\underline{\mathbf{v}}$ & $\underline{\mathbf{B}}$:

  $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$

- This moves charge to one side/end of the wire, which will create an electric potential drop along the wire :

  $\mathcal{E} = \int_\ell \frac{dW}{q} = \int_\ell \frac{\underline{\mathbf{F}} \cdot \underline{\mathbf{d\ell}}}{q}$ (by definition, $V =$ work/unit charge)

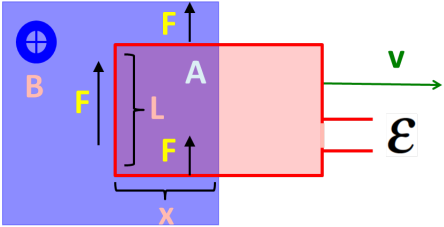
- Hence $\mathcal{E} = \int_\ell (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{d\ell}}$

  $\mathcal{E}$ is the *electromotive force* (or *electromotance*) (EMF)

- Note that $\mathcal{E}$ is *not* a force but a line integral over a force (i.e. a potential difference) !

15.3.2 Relation to magnetic flux

- Now consider a wire circuit loop being pulled with velocity $\underline{\mathbf{v}}$ out of a region containing a $\underline{\mathbf{B}}$-field



- EMF on vertical side:

$$\mathcal{E} = \int_\ell (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{d\ell}} = v\,B\,L$$

- No contribution to EMF from horizontal sides (since $\underline{\mathbf{v}} \times \underline{\mathbf{B}} \perp \underline{\mathbf{d\ell}}$)
- Define *magnetic flux*: $\Phi = \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$
- Rate of change of flux: $\frac{d\Phi}{dt} = \frac{d}{dt} \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = \frac{d}{dt} \int_S B\,da$ (since $\underline{\mathbf{B}} \parallel \underline{\mathbf{da}}$)
- $\frac{d\Phi}{dt} = \frac{d}{dt}(B\,L\,x) = B\frac{dx}{dt}\,L = -v\,B\,L = -\mathcal{E}$

  (negative since x *decreases* with positive v)
- In general, $\mathcal{E}$ from magnetic flux: $\frac{d\Phi}{dt} = \frac{d}{dt} \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = -\mathcal{E}$

15.3.3 EMF due to time-varying B-fields

- So far we have considered EMFs induced due to the motion of charges in the presence of magnetic fields, i.e. due to the Lorentz force (these are so-called "motional EMFs")
- Faraday also observed that EMFs are induced if magnetic fields vary in time, even if no charges are moving

  $\rightarrow$ concluded that time-varying $\underline{\mathbf{B}}$-fields give rise to electric fields, which result in an electric potential $\mathcal{E}$

$$\mathcal{E} = \oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell} = -\int_S \frac{\partial \underline{\mathbf{B}}}{\partial t} \cdot \underline{\mathbf{da}} \quad \text{(Maxwell-Faraday equation)}$$

  (sometimes called "Faraday's Law" in the literature, but strictly describes only EMFs due to time-varying magnetic fields)

  $\rightarrow$ used *together* with the Lorentz force ($\underline{\mathbf{F}}/q = \underline{\mathbf{v}} \times \underline{\mathbf{B}}$), Faraday's Law (sometimes called the Universal Flux Rule), which was first developed empirically, can be derived

Faraday's and Lenz's Laws of Induction

- **Faraday's Law (or the Universal Flux Rule) :**

  The induced electromotive force (EMF) $\mathcal{E}$ in any closed circuit is equal to (the negative of) the time rate of change of the magnetic flux Φ through the circuit :

  $$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$$

  This states that *any* change in magnetic flux through a loop, no matter the reason, i.e. whether it be from a time-dependent $\underline{\mathbf{B}}$-field or to a changing circuit, induces an EMF $\mathcal{E}$ in the loop

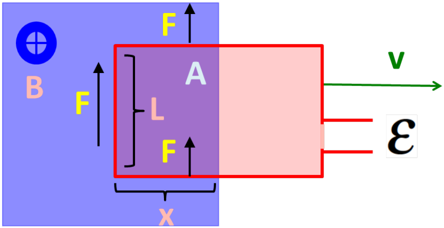
- **Lenz's Law :**

  The induced EMF gives rise to a current whose magnetic field opposes the original change in magnetic flux that caused it

  (Lenz's Law *is* the minus sign in Faraday's Law)

Example: consequence of Lenz's Law

- Revisit circuit loop being pulled with velocity $\underline{\mathbf{v}}$ out of a region containing a $\underline{\mathbf{B}}$-field
- $\mathcal{E}$ from magnetic flux:



$$\mathcal{E} = -\frac{d\Phi}{dt}$$

What consequence does the minus sign have?

- Current will flow in a circuit in a direction soas to oppose the change in flux that caused it
- In this example, the flux cutting area A is *decreasing* with time

$\rightarrow$ current must flow in a direction to *increase* the flux

$\rightarrow$ current will flow *clockwise*, reinforcing existing $\underline{\mathbf{B}}$-field

(Alternatively, consider the Lorentz force produced by the current: $\underline{\mathbf{F}}' = I\,\underline{\mathbf{d}\ell} \times \underline{\mathbf{B}}$; a clockwise current flow provides a force in a direction to *oppose* the motion)

15.4 Maxwell-Faraday equation in differential form

- Maxwell-Faraday equation in integral form :

$$\oint_{\ell} \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell} = -\int_S \frac{\partial \underline{\mathbf{B}}}{\partial t} \cdot \underline{\mathbf{da}}$$

Apply Stokes' theorem to LHS :

$$\int_S (\underline{\nabla} \times \underline{\mathbf{E}}) \cdot \underline{\mathbf{da}} = -\int_S \frac{\partial \underline{\mathbf{B}}}{\partial t} \cdot \underline{\mathbf{da}}$$

- Gives the Maxwell-Faraday equation in differential form :

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

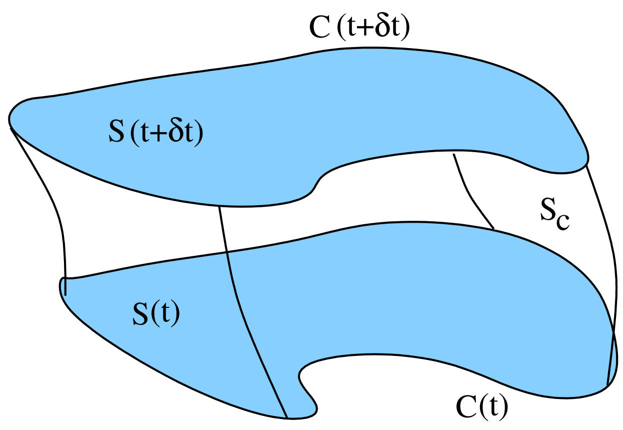
- Any time-varying magnetic field generates an electric field which results in an electric potential $\mathcal{E}$

(c.f. $\underline{\nabla} \times \underline{\mathbf{E}} = 0$ for electro-/magneto-statics)

Faraday's Law (or the Universal Flux Rule) :

The induced electromotive force (EMF) $\mathcal{E}$ in any closed circuit is equal to (the negative of) the time rate of change of the magnetic flux Φ through the circuit :

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot d\mathbf{S}$$



General Case: moving circuit with time varying B-field

Why? *(lecture note extracts taken from David Tong, Cambridge)*

- Consider moving loop C(t) – change in flux through surface S has two terms: one because B may change and one because C is changing
- In a small time δt:

$$\delta\Phi = \Phi(t+\delta t) - \Phi(t) = \int_{S(t+\delta t)} \mathbf{B}(t+\delta t)\cdot d\mathbf{S} - \int_{S(t)} \mathbf{B}(t)\cdot d\mathbf{S}$$
$$= \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t}\,\delta t \cdot d\mathbf{S} + \left[\int_{S(t+\delta t)} - \int_{S(t)}\right] \mathbf{B}(t)\cdot d\mathbf{S} + \mathcal{O}(\delta t^2)$$

$$d\mathbf{S} = (\underline{\mathbf{d}\ell} \times \mathbf{v})\delta t$$

where $d\boldsymbol{l}$ is a line element along C(t) and $\mathbf{v}$ is the velocity of point on C

- Consider closed surface created by S(t), S(t+δt) and the cylindrical region swept out by C(t), called S_C – since $\nabla \cdot \underline{\mathbf{B}} = 0$ then:

$$\left[\int_{S(t+\delta t)} - \int_{S(t)}\right] \mathbf{B}(t)\cdot d\mathbf{S} = -\int_{S_c} \mathbf{B}(t)\cdot d\mathbf{S}$$

- And so: $$\frac{d\Phi}{dt} = \lim_{\delta t\to 0}\frac{\delta\Phi}{\delta t} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} - \int_{C(t)} (\mathbf{v}\times\mathbf{B})\cdot \underline{\mathbf{d}\ell}$$

Having also used: $(\underline{\mathbf{d}\ell} \times \mathbf{v})\cdot\mathbf{B} = \underline{\mathbf{d}\ell}\cdot(\mathbf{v}\times\mathbf{B})$

- Finally, using Maxwell: $\oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}$

$$\frac{d\Phi}{dt} = -\int_C (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot \underline{\mathbf{d}\ell} = -\mathcal{E}$$