CP2: ELECTROMAGNETISM

https://canvas.ox.ac.uk/courses/224992/pages/cp2-electromagnetism

Lecture 18:

Transformers & Magnetic Energy

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HT 2024

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0}\underline{\nabla} \times \underline{\mathbf{B}} = \underline{\mathbf{J}} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

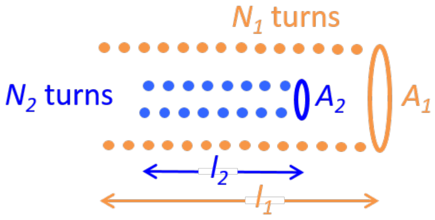
[1]With many thanks to Prof Neville Harnew and Prof Laura Herz

OUTLINE : 18. TRANSFORMERS, MAGNETIC ENERGY

18.1 Coaxial solenoids sharing the same area

From before, mutual inductance between coils :

$$M_{21} = M_{12} = \mu_0 \frac{N_1 N_2}{\ell_1} A_2 \; (= M)$$



- Self inductance of coils 1 & 2

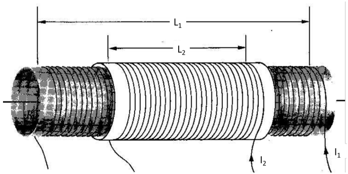
  $L_1 = \mu_0 \frac{N_1^2}{\ell_1} A_1$ and

  $L_2 = \mu_0 \frac{N_2^2}{\ell_2} A_2$

- If $A_1 = A_2$:

  $M = \left(\sqrt{\frac{\ell_2}{\ell_1}}\right) \sqrt{(L_1 L_2)}$

  If $\ell_1 = \ell_2$ then $M = \sqrt{(L_1 L_2)}$



- Hence the mutual inductance is proportional to the geometric mean of the self inductances

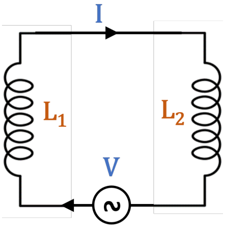
In general, circuits may not be tightly coupled $\rightarrow M = k\sqrt{(L_1 L_2)}$, where $k < 1$ (k is the *coefficient of coupling*)

18.2 Inductors in series and parallel

1. In series with no mutual inductance between coils:

$$V = L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt} = (L_1 + L_2)\frac{dI}{dt}$$

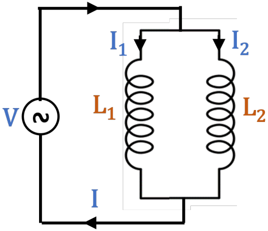
$$L = L_1 + L_2$$



2. In series with mutual inductance between coils:

$$V = (L_1 + M)\frac{dI}{dt} + (L_2 + M)\frac{dI}{dt}$$
$$= (L_1 + L_2 + 2M)\frac{dI}{dt}$$

$$L = L_1 + L_2 + 2M$$



3. In parallel with no mutual inductance:

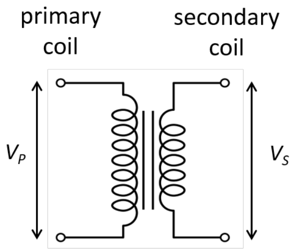
$$V = L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt} \quad \text{where } I = I_1 + I_2$$

Write: $V = L\frac{dI}{dt} \rightarrow V = L\left(\frac{dI_1}{dt} + \frac{dI_2}{dt}\right) = L\left(\frac{V}{L_1} + \frac{V}{L_2}\right)$

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

(**4.** with mutual inductance: $L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$)

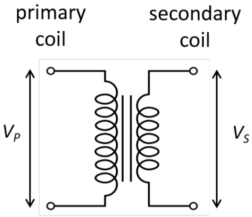
18.3 The transformer



- Transformer will step up or step down applied voltage V_P by a factor of the winding ratio
- There is no power consumed in the transformer (ideally) $\rightarrow$ input and output powers are equal :

$$V_S I_S = V_P I_P \;\rightarrow\; \frac{I_S}{I_P} = \frac{V_P}{V_S} = \frac{d\Phi_P}{d\Phi_S}\frac{N_P}{N_S} = \frac{1}{k}\frac{N_P}{N_S}$$

Transformer summary



Primary coil creates flux which permeates secondary coil, coupling their voltages:

Voltage Ratio:

$$\frac{V_S}{V_P} = \frac{d\Phi_S}{d\Phi_P}\frac{N_S}{N_P}$$

Current Ratio:

$$\frac{I_S}{I_P} = \frac{d\Phi_P}{d\Phi_S}\frac{N_P}{N_S}$$

18.4 Energy of the magnetic field

- Consider the energy stored in an inductor L:
- Change in current results in a back EMF $\mathcal{E} = -L\frac{dI}{dt}$

Energy of the magnetic field (cont.)

- **General Case:** $U = \frac{1}{2} L I^2 = \frac{1}{2} \Phi I$
- Consider magnetic flux through surface bound by loop :
  $\Phi = \int \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = \int (\underline{\nabla} \times \underline{\mathbf{A}}) \cdot \underline{\mathbf{da}} = \oint \underline{\mathbf{A}} \cdot \underline{\mathbf{d\ell}} \equiv LI$
- Energy : $U = \frac{1}{2} I \oint \underline{\mathbf{A}} \cdot \underline{\mathbf{d\ell}} = \frac{1}{2} \oint (\underline{\mathbf{A}} \cdot \underline{\mathbf{I}})\, d\ell$
  and generalised to volume currents : $U = \frac{1}{2} \int_{\mathcal{V}} (\underline{\mathbf{A}} \cdot \underline{\mathbf{J}})\, d\mathcal{V}$
- Ampere's Law : $\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \underline{\mathbf{J}} \;\rightarrow\; U = \frac{1}{2\,\mu_0} \int_{\mathcal{V}} \underline{\mathbf{A}} \cdot (\underline{\nabla} \times \underline{\mathbf{B}})\, d\mathcal{V}$
- Vector identity : $\underline{\nabla} \cdot (\underline{\mathbf{A}} \times \underline{\mathbf{B}}) = \underline{\mathbf{B}} \cdot (\underline{\nabla} \times \underline{\mathbf{A}}) - \underline{\mathbf{A}} \cdot (\underline{\nabla} \times \underline{\mathbf{B}})$
  $U = \frac{1}{2\,\mu_0} \left\{ \int_{\mathcal{V}} \underline{\mathbf{B}} \cdot (\underline{\nabla} \times \underline{\mathbf{A}})\, d\mathcal{V} \;-\; \int_{\mathcal{V}} \underline{\nabla} \cdot (\underline{\mathbf{A}} \times \underline{\mathbf{B}})\, d\mathcal{V} \right\}$
- Then, using $\underline{\mathbf{B}} = \underline{\nabla} \times \underline{\mathbf{A}}$ and the Divergence Theorem :
  $U = \frac{1}{2\,\mu_0} \left\{ \int_{\mathcal{V}} B^2\, d\mathcal{V} \;-\; \oint_S (\underline{\mathbf{A}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{da}} \right\}$
- This relationship holds true for any volume $\mathcal{V}$ that encloses all the current $\rightarrow$ *if* the integration volume is chosen as "all space", then the surface integral vanishes :

$$U = \frac{1}{2\,\mu_0} \int_{\mathcal{V}} B^2 d\mathcal{V} \quad \text{(where integration volume } \mathcal{V} = \text{all space)}$$

Summary of energy in E and B fields

Electric field energy

- In terms of circuits :

$$U_E = \frac{1}{2}\, C\, V^2$$
$$= \frac{1}{2}\, Q\, V$$

- In terms of fields :

$$U_E = \frac{\epsilon_0}{2} \int_{all\ space} E^2\, d\mathcal{V}$$

Magnetic field energy

- In terms of circuits :

$$U_M = \frac{1}{2}\, L\, I^2$$
$$= \frac{1}{2}\, \Phi\, I$$

- In terms of fields :

$$U_M = \frac{1}{2\,\mu_0} \int_{all\ space} B^2\, d\mathcal{V}$$

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Lecture 19:

Motion in E & B Fields

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OUTLINE : 19. MOTION IN E & B FIELDS

19.1 Motion of charged particles in E and B fields

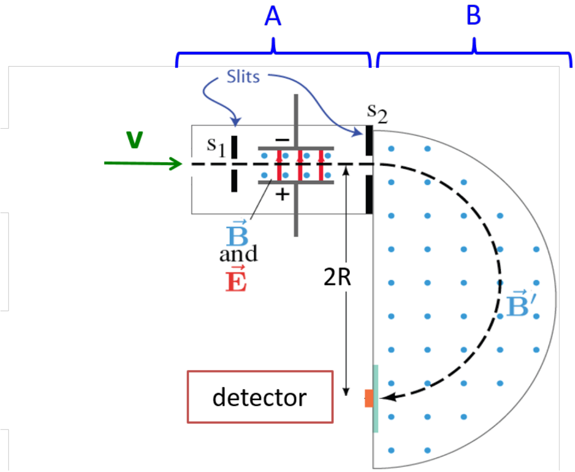
- Force on a charged particle in an $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ field :

- Newton's Second Law provides equation of motion :

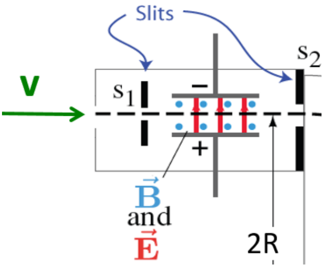
- Will demonstrate with 2 examples :
  1. Mass spectrometer
  2. Magnetic lens

19.2 Example : the mass spectrometer

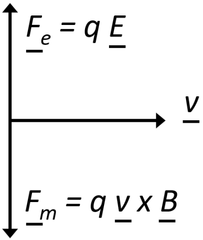
Used for identifying small charged particles (molecules, ions) by their mass m



Stage A : The velocity filter



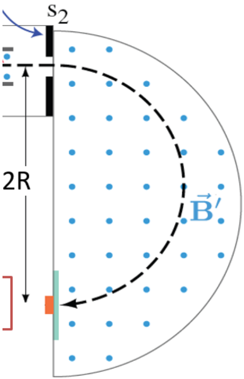
- The particle will pass through both slits if it experiences no net force inside the filter
- The region has both $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields



- Will filter particles with $v = \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{B}}|}$ and the spread $\pm\Delta v$ is given by the slit width

Stage B : The mass filter

- This region has only a **B** field



Mass spectrometer summary

In the presence of both E- and B-fields, a charge experiences the force:

$$\mathbf{F}_{\mathrm{EM}} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Mass Spectrometer.

A. velocity filter:

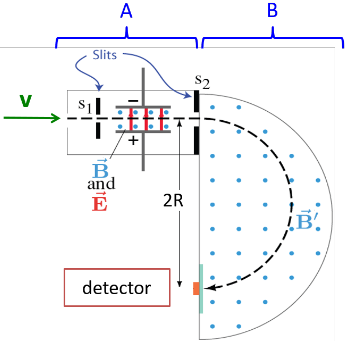
E&B-fields present. Charged particles pass through Stage A if their velocity equals the amplitude ratio:

$$v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$

B. Filter stage:

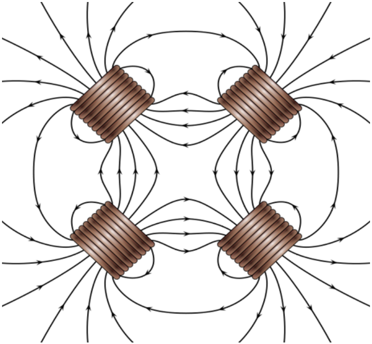
Only B-field present. Charged particles are forced on circular path with radius:

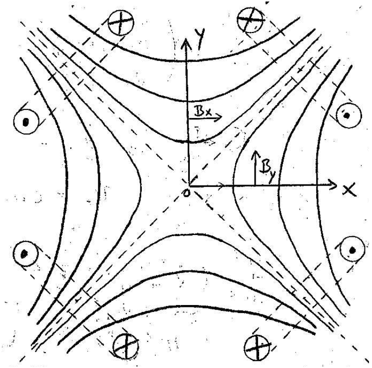
$$R = \frac{mv}{qB}$$



19.3 Example : magnetic lenses

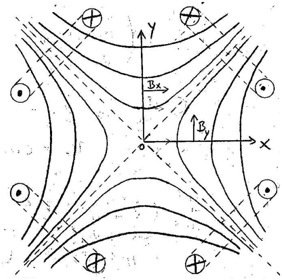
- Magnetic lenses are used for focusing and collimating charged particle beams (used in electron microscopes, particle accelerators etc.)
- Quadrupole lens : four identical coils in $x - y$ plane
- Sum of 4 dipole fields : for small values of x, y close to the axis of symmetry, $B_x \propto y$, $B_y \propto x$





Quadrupole lens

- Along x-axis : only B_y component
- Along y-axis : only B_x component
- No z-component (symmetry)



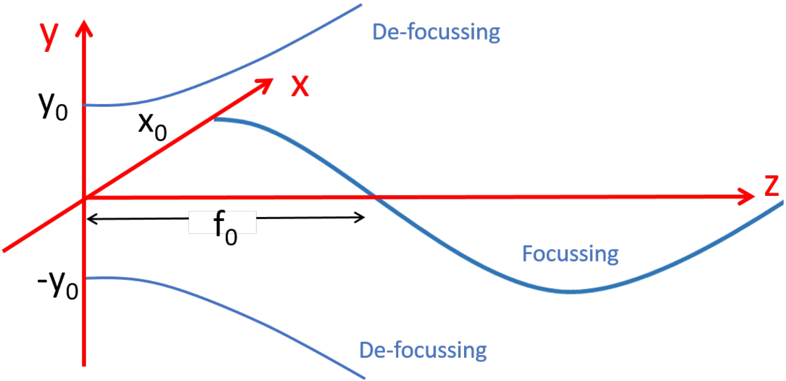
Quadrupole lens (cont.)

- Equations of motion : $\ddot{x} = -\alpha^2 x$ & $\ddot{y} = \alpha^2 y$, where $\alpha = \sqrt{\frac{q\,k\,v}{m}}$

- Focal points in z direction (x=0) at $f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$
- Use lens pair with 90° angle for collimating a charged beam

Quadrupole lens (cont.)

Lens pulls beam on-axis in x and removes particles deviating in y

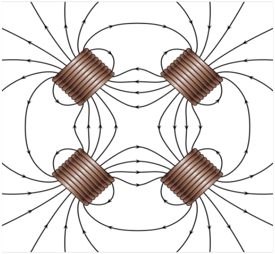


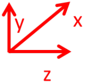
$$f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$$

Magnetic lens summary

Magnetic Lens.

$$\mathbf{B} = (\mathrm{k}\, y, \mathrm{k}\, x, 0)$$





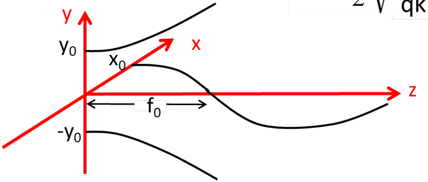
Equation of Motion: $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}$

Solutions:

$$y(z) = y_0 \cosh \sqrt{\frac{q\,\mathrm{k}}{vm}}\, z$$ de-focusing

$$x(z) = x_0 \cos \sqrt{\frac{q\,\mathrm{k}}{vm}}\, z$$ focusing with

$$f_0 = \frac{\pi}{2}\sqrt{\frac{vm}{\mathrm{qk}}}$$



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Lecture 20:

Displacement Current & Maxwell's Equations

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$$\underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

$$\frac{1}{\mu_0} \underline{\nabla} \times \underline{\mathbf{B}} = \underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

[1]With many thanks to Prof Neville Harnew and Prof Laura Herz

OUTLINE : 20. DISPLACEMENT CURRENT & MAXWELL'S EQUATIONS

20.1 Electrodynamics “before Maxwell”

Time-varying B-fields generate E-fields. *However*, time-varying E-fields do not seem to create B-fields in this version.
Is there something wrong ?

20.2 Revisit Ampere's Law

- Therefore Ampere's Law in its current form violates the continuity equation (and hence charge conservation) !
- But this is not surprising since Ampere's Law is derived from the Biot-Savart Law assuming that $\frac{\partial}{\partial t}(\rho) = 0$

  $\rightarrow$ we have to "fix" Ampere's Law !

20.3 Fixing Ampere's Law : displacement current

- Add a term to Ampere's Law to make it compatible with the continuity equation : $\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\rho)$

  The term $\left(\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$ is called the *displacement current* $\underline{\mathbf{J}}_D$ (note that it is actually a time-varying electric field)
- Time-varying $\underline{\mathbf{E}}$ fields now generate $\underline{\mathbf{B}}$ fields and vice versa. Also satisfies charge conservation.

Summary : Ampere's Law with Maxwell's correction

Ampere's Law does not comply with the Equation of Continuity:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \text{apply divergence:} \quad \underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{\substack{=0 \\ \text{always}}} = \mu_0 \underbrace{\nabla \cdot \mathbf{J}}_{= -\frac{\partial \rho}{\partial t}}$$

$= 0$ only for statics!

This lack of charge conservation is unphysical! As a solution, add a so-called "displacement current" to **J**, which will ensure compliance with the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot (\underbrace{\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}_{\text{displacement current } \mathbf{J}_D})$$

Obtain **Ampere's Law with "displacement current"**:

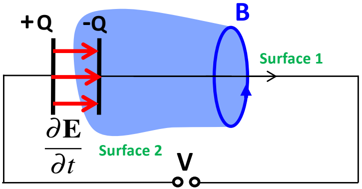
$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

Using Stoke's theorem : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot d\underline{\mathbf{a}}$

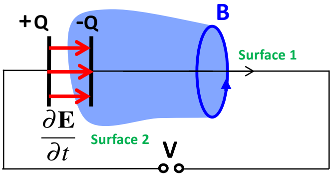
gives the integral form : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \mu_0 \underbrace{\int_S \underline{\mathbf{J}} \cdot d\underline{\mathbf{a}}}_{I_{enc}} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$

20.4 Example : Ampere's Law and a charging capacitor

- This is the first example, showing why Ampere's Law fails without adding the displacement current : a straight wire, and add a capacitor into the circuit
- Previously we used Ampere's Law to calculate the magnetic field along Amperian loop $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 \, I_{enc}$



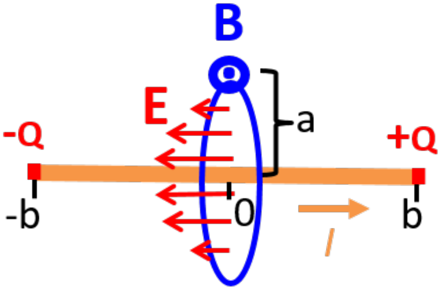
Ampere's Law and a charging capacitor (cont.)



- In differential form : $$\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \left(\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

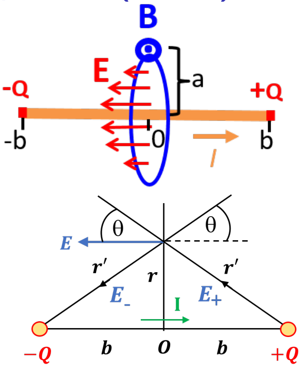
20.5 Example : B-field of a short current-carrying wire

- Recall B-field from Biot-Savart Law at a distance a from centre of a wire of length $2b \rightarrow \; B = \frac{\mu_0 I}{2\pi a} \frac{b}{\sqrt{b^2+a^2}}$
- Again, Ampere's Law fails depending on which path we use. Need to use displacement current.
- $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$
- Wire is short, so charge builds up at the ends giving time-varying $\underline{\mathbf{E}}$-field



B-field of a short current-carrying wire (cont.)

- Form Amperian loop of radius a, and integrate $\frac{\partial E}{\partial t}$ over enclosed area
- Calculate **E**-field due to two point charges at wire ends, $\pm b$



20.6 Maxwell's equations

$$\oint_S \mathbf{E}\,.\,\mathbf{da} = \frac{Q}{\varepsilon_0} \leftrightarrow \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss's Law: Charge generates an electric field. Electric field lines begin and end on charge.

$$\oint_C \mathbf{E}\,.\,\mathbf{d}\boldsymbol{l} = -\int_S \frac{\partial\mathbf{B}}{\partial t}\,.\,\mathbf{da} \leftrightarrow \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

Maxwell-Faraday's Law: time-varying magnetic fields create electric fields (induction)

$$\oint_S \boldsymbol{B}\,.\,\boldsymbol{da} = 0 \leftrightarrow \nabla\cdot\mathbf{B} = 0$$

There are **no magnetic monopoles.** Magnetic field lines form closed loops.

$$\oint_C \mathbf{B}\,.\,\mathbf{d}\boldsymbol{l} = \mu_0 I + \mu_0\,\varepsilon_0 \int_S \frac{\partial\mathbf{E}}{\partial t}\,.\,\mathbf{da} \leftrightarrow \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Ampere's Law with Maxwell's correction: electric currents and time-varying electric fields generate magnetic fields

Maxwell's equations, together with the Lorentz force : $\underline{\mathbf{F}} = q\,(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})$ summarise the entire theoretical content of classical electrodynamics