CP2: ELECTROMAGNETISM

https://canvas.ox.ac.uk/courses/224992/pages/cp2-electromagnetism

Lecture 19:

Motion in E & B Fields

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HT 2024

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

$$\frac{1}{\mu_0} \underline{\nabla} \times \underline{\mathbf{B}} = \underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

[1]With many thanks to Prof Neville Harnew and Prof Laura Herz

OUTLINE : 19. MOTION IN E & B FIELDS

19.1 Motion of charged particles in E and B fields

- Force on a charged particle in an $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ field :

$$\underline{\mathbf{F}} = q\left(\underbrace{\underline{\mathbf{E}}}_{\text{along } \underline{\mathbf{E}}} + \underbrace{\underline{\mathbf{v}} \times \underline{\mathbf{B}}}_{\perp \text{ to both } \underline{\mathbf{v}} \text{ and } \underline{\mathbf{B}}}\right)$$

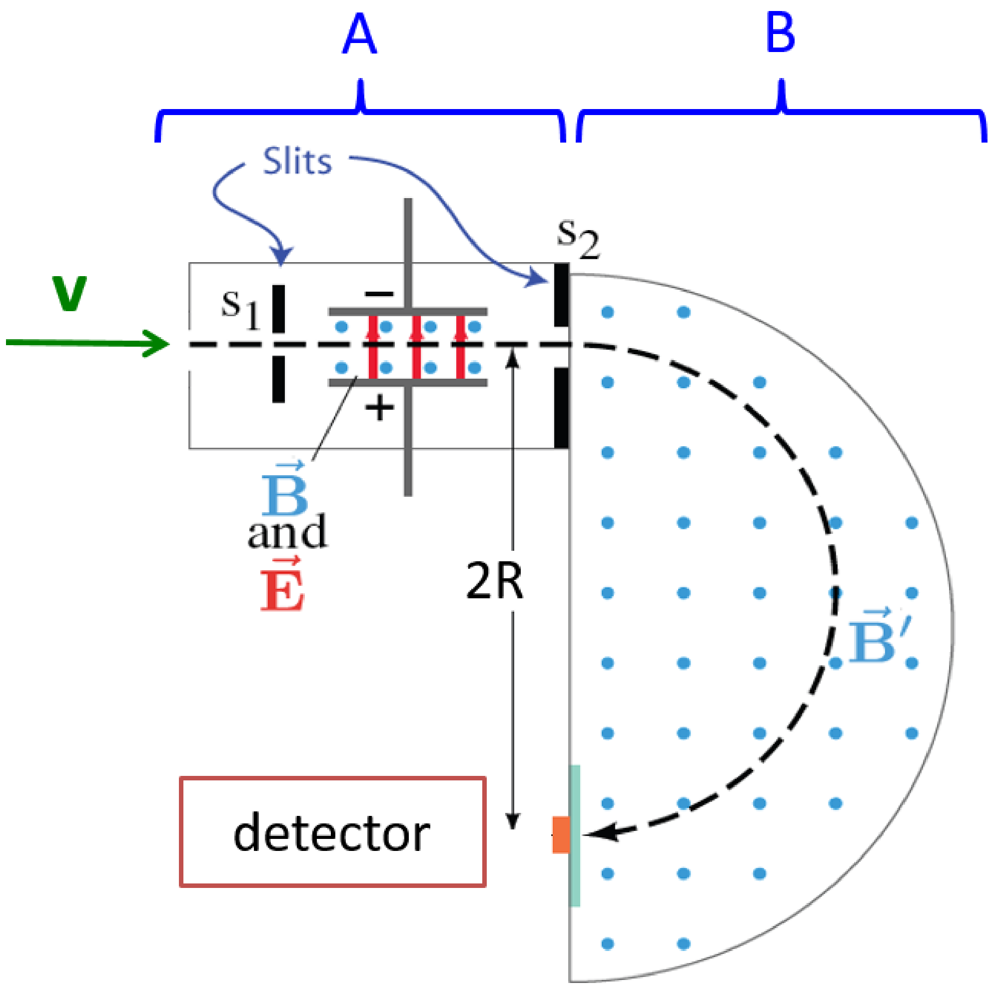
- Newton's Second Law gives equation of motion :

$$\underline{\mathbf{F}} = m\,\underline{\mathbf{a}} = m\,\underline{\ddot{\mathbf{r}}} = q\,(\,\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})$$

- Will demonstrate with 2 examples :
  1. Mass spectrometer
  2. Magnetic lens

19.2 Example : the mass spectrometer

Used for identifying small charged particles (molecules, ions) by their mass m



Stage A : The velocity filter

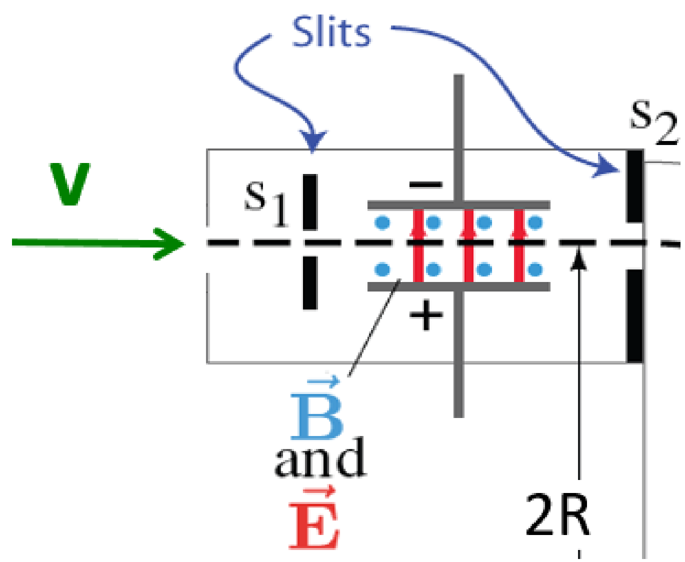
- The particle will pass through both slits if it experiences no net force inside the filter
- The region has both $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields

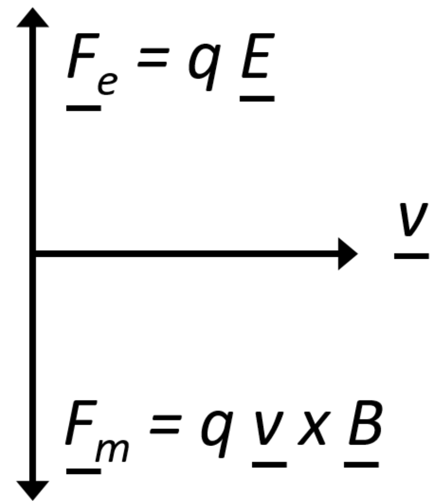
$$\underline{\mathbf{F}} = q\,(\,\underline{\mathbf{E}} \;+\; \underline{\mathbf{v}} \times \underline{\mathbf{B}}) = 0$$

$$\rightarrow \quad \text{need } \underline{\mathbf{E}} = -\underline{\mathbf{v}} \times \underline{\mathbf{B}} \rightarrow \; v = \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{B}}|}$$

$$(\underline{\mathbf{E}} \perp \underline{\mathbf{v}} \,\&\, \underline{\mathbf{B}})$$

- Will filter particles with $v = \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{B}}|}$ and the spread $\pm\Delta v$ is given by the slit width





Stage B : The mass filter

- This region has only a $\underline{\mathbf{B}}$ field

  $m\,\underline{\ddot{\mathbf{r}}} = q\,\underline{\dot{\mathbf{r}}} \times \underline{\mathbf{B}}$

  with $\underline{\mathbf{B}} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$ and $\underline{\dot{\mathbf{r}}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$

  $\rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \frac{q}{m} \begin{pmatrix} \dot{y}\,B \\ -\dot{x}\,B \\ 0 \end{pmatrix}$

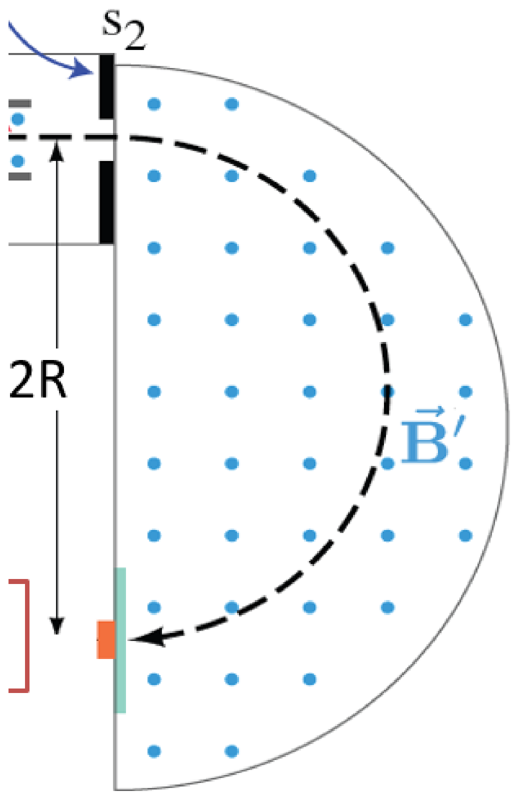
  $\rightarrow \ddot{z} = 0 \rightarrow v_z = \text{constant} \;\; (= 0)$

- $|\underline{\ddot{\mathbf{r}}}|^2 = \ddot{x}^2 + \ddot{y}^2 = \frac{q^2}{m^2} \underbrace{\left(\dot{x}^2 + \dot{y}^2\right)}_{v^2} B^2$

- Circular motion in $x - y$ plane with :

  $$\frac{v^2}{R} = \frac{q}{m} v\,B \rightarrow R = \frac{mv}{qB}$$

- Since q and v are constant, then $R \propto m$



Mass spectrometer summary

In the presence of both E- and B-fields, a charge experiences the force:

$$\mathbf{F}_{\mathrm{EM}} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Mass Spectrometer.

A. velocity filter:

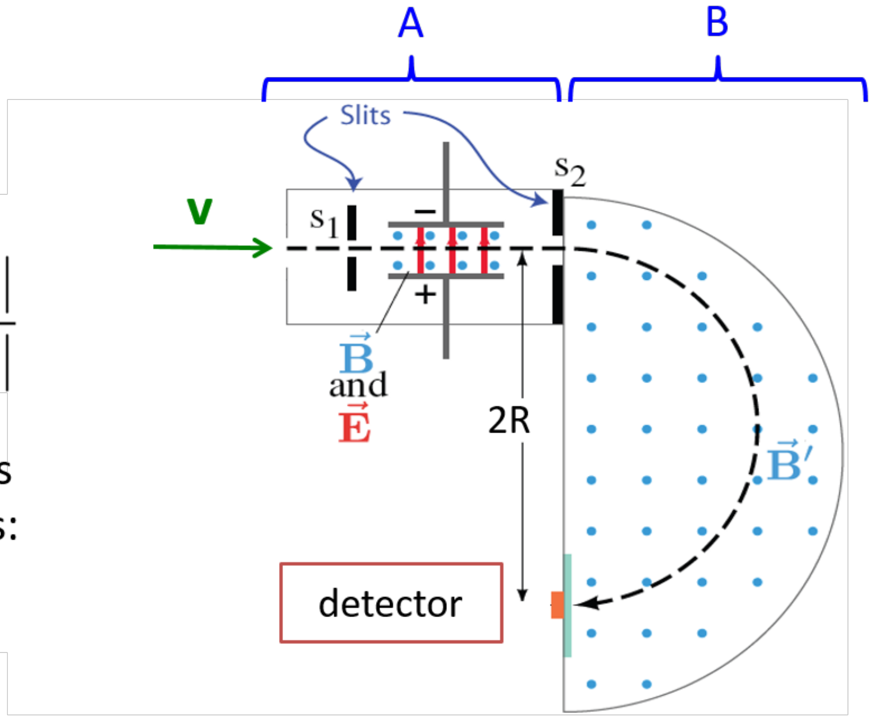
E&B-fields present. Charged particles pass through Stage A if their velocity equals the amplitude ratio:

$$v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$

B. Filter stage:

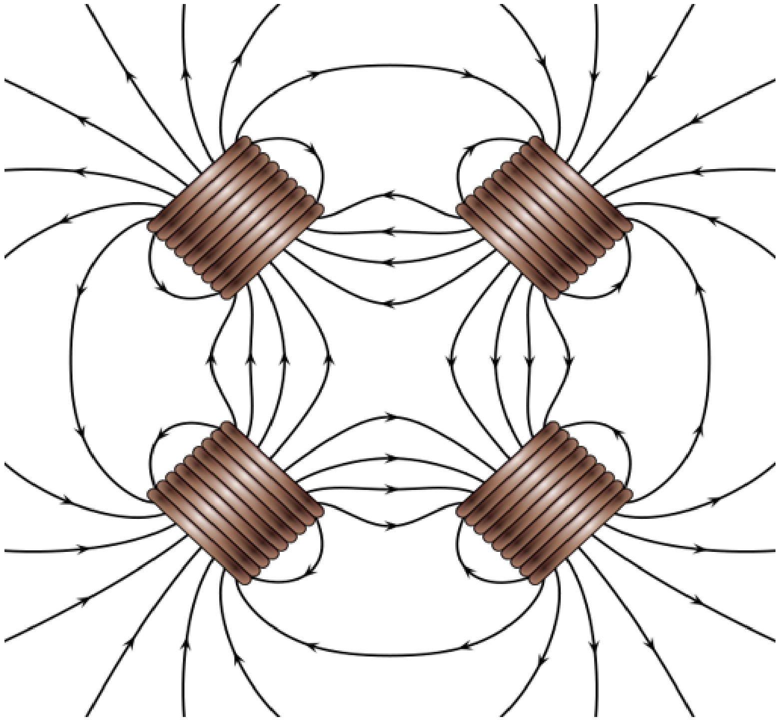
Only B-field present. Charged particles are forced on circular path with radius:

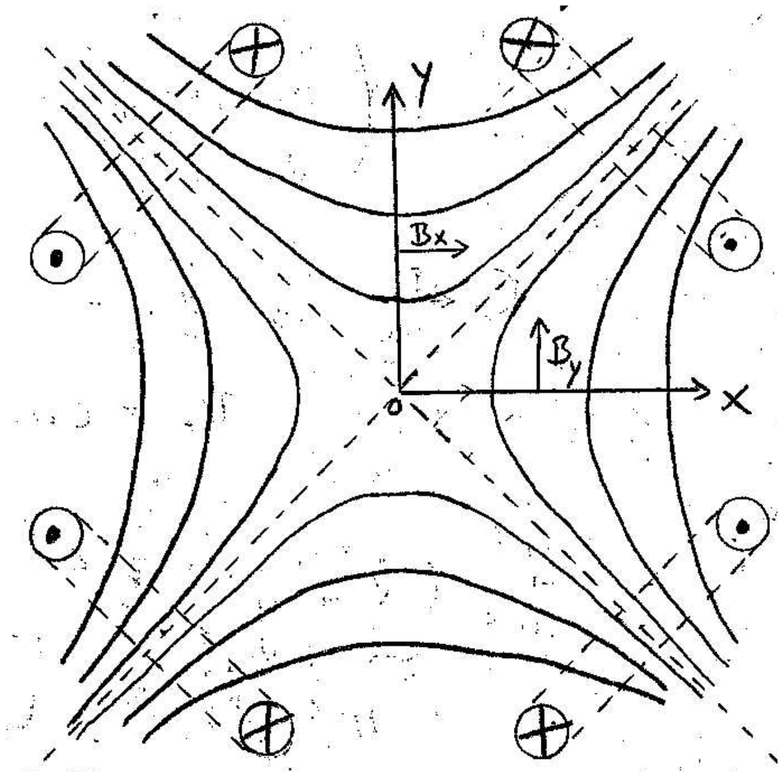
$$R = \frac{mv}{qB}$$



19.3 Example : magnetic lenses

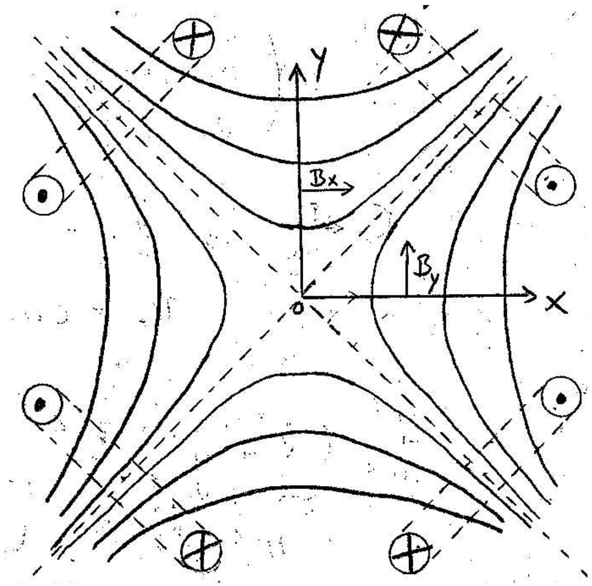
- Magnetic lenses are used for focusing and collimating charged particle beams (used in electron microscopes, particle accelerators etc.)
- Quadrupole lens : four identical coils in $x - y$ plane
- Sum of 4 dipole fields : for small values of x, y close to the axis of symmetry, $B_x \propto y$, $B_y \propto x$





Quadrupole lens

- Along x-axis : only B_y component
- Along y-axis : only B_x component
- No z-component (symmetry)



- Inside the lens, close to the z-axis

$$\underline{\mathbf{B}} = \begin{pmatrix} k\,y \\ k\,x \\ 0 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

- Equation of motion $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$

$$\mathrm{m} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ k\,y & k\,x & 0 \end{vmatrix} = q\,k \begin{pmatrix} -x\,\dot{z} \\ y\,\dot{z} \\ x\,\dot{x} - y\,\dot{y} \end{pmatrix}$$

- Assume particle travels at a small angle wrt the z-axis :

$$\rightarrow \;\; \dot{x}, \dot{y} \approx 0 \;\; \rightarrow \;\; \ddot{z} = 0 \;\; \rightarrow \;\; \dot{z} = v = \text{constant} \;\; \rightarrow \;\; z = v\,t$$

- Equations of motion in the $x - y$ plane :

$$\ddot{x} = -\frac{q}{m} k\,v\,x \quad \text{and} \quad \ddot{y} = \frac{q}{m} k\,v\,y$$

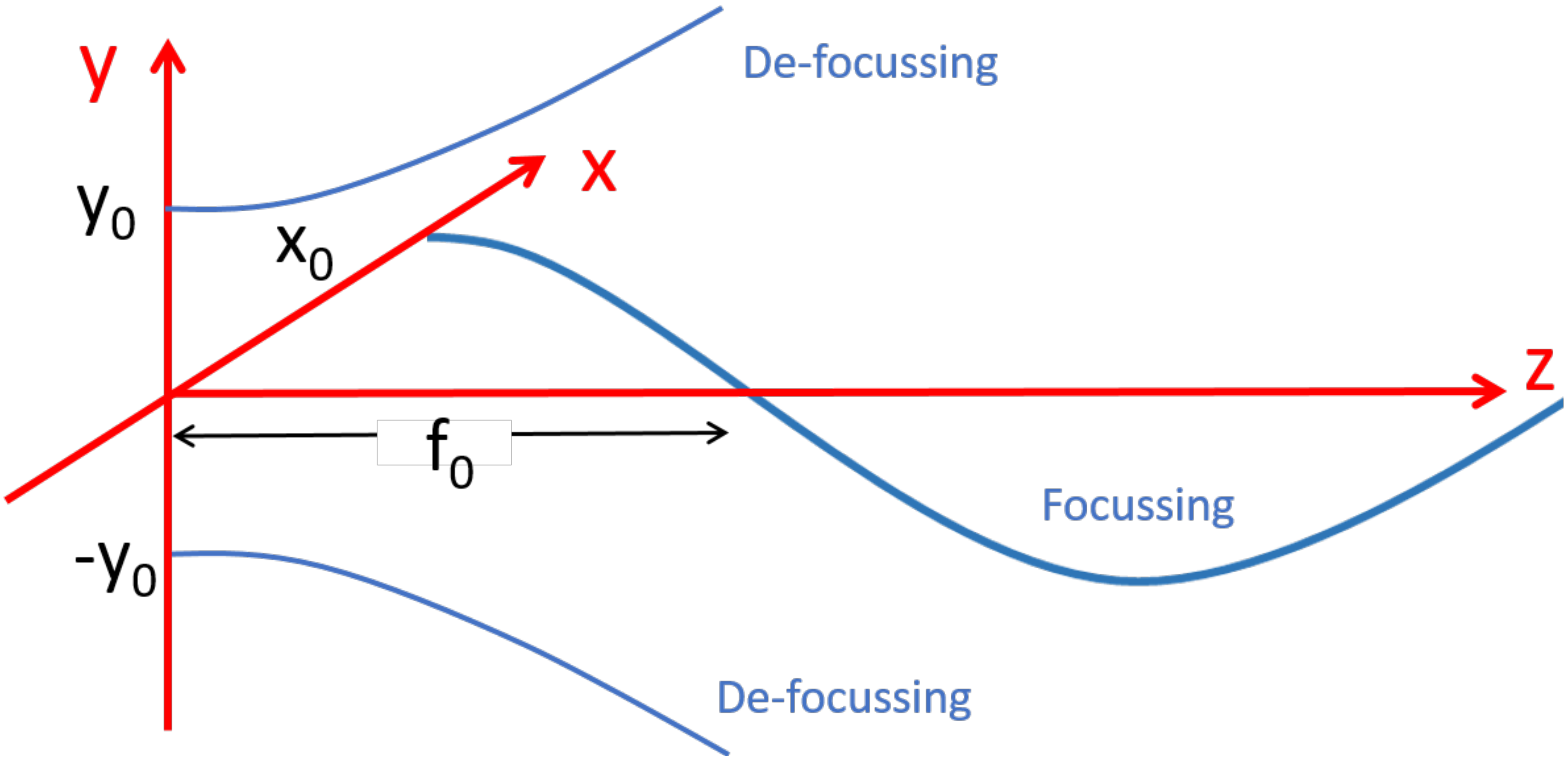
Quadrupole lens (cont.)

- Equations of motion : $\ddot{x} = -\alpha^2 x$ & $\ddot{y} = \alpha^2 y$, where $\alpha = \sqrt{\frac{q\,k\,v}{m}}$

- Focal points in z direction $(x{=}0)$ at $f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$
- Use lens pair with 90° angle for collimating a charged beam

Quadrupole lens (cont.)

Lens pulls beam on-axis in x and removes particles deviating in y

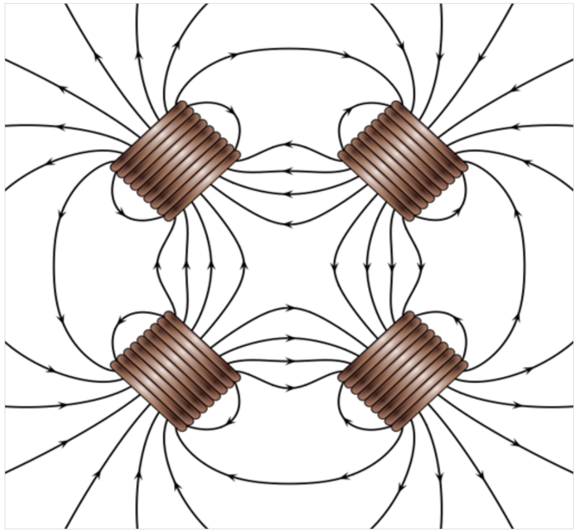


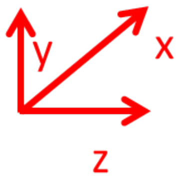
$$f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$$

Magnetic lens summary

Magnetic Lens.

$$\mathbf{B} = (\mathrm{k}\, y, \mathrm{k}\, x, 0)$$





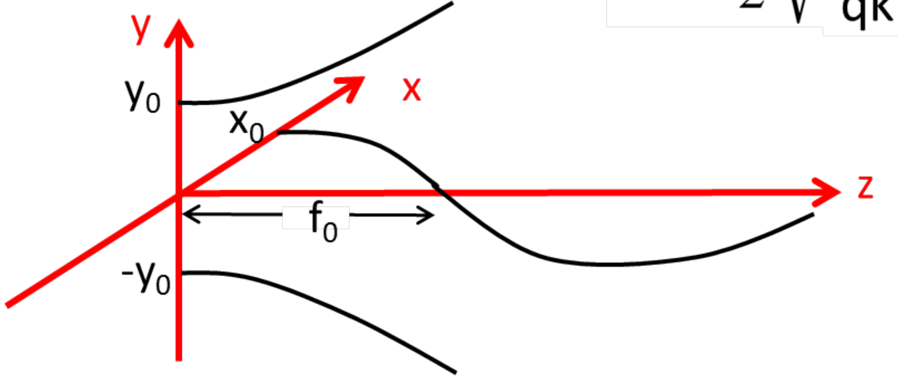
Equation of Motion: $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}$

Solutions:

$$y(z) = y_0 \cosh\sqrt{\frac{q\,\mathrm{k}}{vm}}z$$ de-focusing

$$x(z) = x_0 \cos\sqrt{\frac{q\,\mathrm{k}}{vm}}z$$ focusing with

$$f_0 = \frac{\pi}{2}\sqrt{\frac{vm}{\mathrm{qk}}}$$



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Lecture 20:

Displacement Current & Maxwell's Equations

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$$\underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

$$\frac{1}{\mu_0}\underline{\nabla} \times \underline{\mathbf{B}} = \underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

[1] With many thanks to Prof Neville Harnew and Prof Laura Herz

OUTLINE : 20. DISPLACEMENT CURRENT & MAXWELL'S EQUATIONS

20.1 Electrodynamics “before Maxwell”

Time-varying B-fields generate E-fields. *However*, time-varying E-fields do not seem to create B-fields in this version.

Is there something wrong?

20.2 Revisit Ampere's Law

- Therefore Ampere's Law in its current form violates the continuity equation (and hence charge conservation) !
- But this is not surprising since Ampere's Law is derived from the Biot-Savart Law assuming that $\frac{\partial}{\partial t}(\rho) = 0$

  $\rightarrow$ we have to "fix" Ampere's Law !

20.3 Fixing Ampere's Law : displacement current

- Add a term to Ampere's Law to make it compatible with the continuity equation : $\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\rho)$

  The term $\left(\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$ is called the *displacement current* $\underline{\mathbf{J}}_D$ (note that it is actually a time-varying electric field)
- Time-varying $\underline{\mathbf{E}}$ fields now generate $\underline{\mathbf{B}}$ fields and vice versa. Also satisfies charge conservation.

Summary : Ampere's Law with Maxwell's correction

Ampere's Law does not comply with the Equation of Continuity:

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply divergence: $\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{=0 \text{ always}} = \mu_0 \underbrace{\nabla \cdot \mathbf{J}}_{= -\frac{\partial \rho}{\partial t}}$

$= 0$ only for statics!

This lack of charge conservation is unphysical! As a solution, add a so-called "displacement current" to **J**, which will ensure compliance with the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot (\underbrace{\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}_{\text{displacement current } \mathbf{J}_D})$$

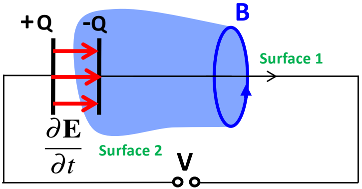
Obtain **Ampere's Law with "displacement current"**: $\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$

Using Stoke's theorem : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot d\underline{\mathbf{a}}$

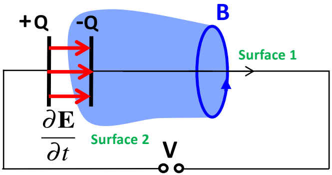
gives the integral form : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \mu_0 \underbrace{\int_S \underline{\mathbf{J}} \cdot d\underline{\mathbf{a}}}_{I_{enc}} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$

20.4 Example : Ampere's Law and a charging capacitor

- This is the first example, showing why Ampere's Law fails without adding the displacement current : a straight wire, and add a capacitor into the circuit
- Previously we used Ampere's Law to calculate the magnetic field along Amperian loop $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I_{enc}$



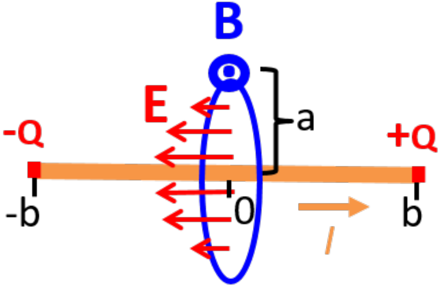
Ampere's Law and a charging capacitor (cont.)



- In differential form : $\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \left(\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$

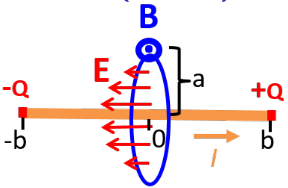
20.5 Example : B-field of a short current-carrying wire

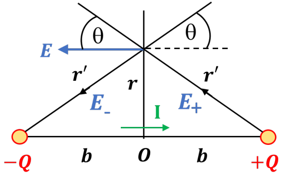
- Recall B-field from Biot-Savart Law at a distance a from centre of a wire of length $2b \rightarrow \; B = \frac{\mu_0 I}{2\pi a} \frac{b}{\sqrt{b^2+a^2}}$
- Again, Ampere's Law fails. Need to use displacement current.
- $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$
- Wire is short, so charge builds up at the ends giving time-varying $\underline{\mathbf{E}}$-field



B-field of a short current-carrying wire (cont.)

- Form Amperian loop of radius a, and integrate $\frac{\partial E}{\partial t}$ over enclosed area
- Calculate **E**-field due to two point charges at wire ends, $\pm b$





20.6 Maxwell's equations

$$\oint_S \mathbf{E} \,.\, \mathbf{da} = \frac{Q}{\varepsilon_0} \leftrightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss's Law: Charge generates an electric field. Electric field lines begin and end on charge.

$$\oint_C \mathbf{E} \,.\, \mathbf{d}\boldsymbol{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \,.\, \mathbf{da}$$

$$\leftrightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Maxwell-Faraday's Law: time-varying magnetic fields create electric fields (induction)

$$\oint_S \boldsymbol{B} \,.\, \boldsymbol{da} = 0 \leftrightarrow \nabla \cdot \mathbf{B} = 0$$

There are **no magnetic monopoles.** Magnetic field lines form closed loops.

$$\oint_C \mathbf{B} \,.\, \mathbf{d}\boldsymbol{l} = \mu_0 I + \mu_0 \, \varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \,.\, \mathbf{da}$$

$$\leftrightarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Ampere's Law with Maxwell's correction: electric currents and time-varying electric fields generate magnetic fields

Maxwell's equations, together with the Lorentz force : $\underline{\mathbf{F}} = q\,(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})$ summarise the entire theoretical content of classical electrodynamics

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Lecture 21:

Electromagnetic Waves & Energy Flow

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$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

$$\frac{1}{\mu_0}\underline{\nabla} \times \underline{\mathbf{B}} = \underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

[1]With many thanks to Prof Neville Harnew and Prof Laura Herz

OUTLINE : 21. ELECTROMAGNETIC WAVES & ENERGY FLOW

21.1 Electromagnetic waves in vacuum

- In the absence of electric charge or current
  $\rightarrow \;\; \rho = 0$ and $\underline{\mathbf{J}} = 0$:
- Maxwell's Equations become :

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 0 \qquad\qquad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad\qquad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \epsilon_0 \, \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

(note the symmetry between the $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields)

Electromagnetic waves in vacuum (cont.)

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 0 \qquad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0\, \epsilon_0\, \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

This gives us a *wave equation* in $\underline{\mathbf{B}}$:

$$\underline{\nabla}^2\, \underline{\mathbf{B}} - \epsilon_0\, \mu_0\, \underline{\ddot{\mathbf{B}}} = 0$$

together with :

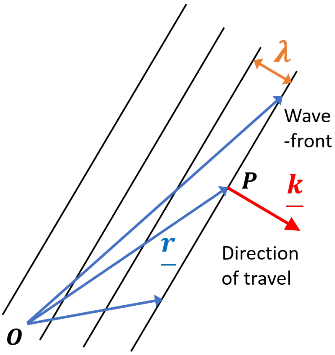
$$\underline{\nabla}^2\, \underline{\mathbf{E}} - \epsilon_0\, \mu_0\, \underline{\ddot{\mathbf{E}}} = 0$$

- These equations have general solutions (in 1D) of the form :
- $E\,(x, t) = F\,(x - ct) + G\,(x + ct)$ and
  $B\,(x, t) = F'\,(x - ct) + G'\,(x + ct)$

  where F, G, F', G' are *any* functions of $(x - ct), (x + ct)$

21.2 Electromagnetic waves : 3D plane wave solutions

Plane waves



- $\underline{\mathbf{k}}$ is in the direction normal to the wave-fronts
- All points P form a wave-front with the same phase
- Maxima are separated by the wavelength λ where $\lambda = 2\pi/k$
- Phase velocity (or propagation velocity) of wave-fronts given by $c = \omega/k$

21.3 Divergence, time derivative, curl of $\underline{E}$ and $\underline{B}$

- The divergence of $\underline{\mathbf{E}}$:

- The time derivative of $\underline{\mathbf{E}}$:

- The curl of $\underline{\mathbf{E}}$:

21.4 Electromagnetic waves : speed of propagation

- To get speed of propagation, substitute $\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$ into the wave equation $\underline{\nabla}^2\,\underline{\mathbf{E}} = \epsilon_0\,\mu_0\,\frac{\partial^2}{\partial\,t^2}\underline{\mathbf{E}}$

21.5 Relationship between $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$

- Electric and magnetic fields in vacuum are *perpendicular* to direction of propogation $\rightarrow$ *EM waves are transverse*

- $\underline{\mathbf{E}}$, $\underline{\mathbf{B}}$ & $\underline{\mathbf{k}}$ are mutually orthogonal (NB. $\underline{\mathbf{k}} \times \underline{\mathbf{B}} = kB \sin \frac{\pi}{2} \underline{\hat{\mathbf{E}}}$)
- $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ are in phase and lie in the plane of the wavefront
- Field magnitude ratio : $|\underline{\mathbf{E}}|/|\underline{\mathbf{B}}| = \frac{c^2}{\omega} k = c = \frac{1}{\sqrt{\mu_0 \, \epsilon_0}}$

21.6 Electromagnetic wave travelling along the z direction

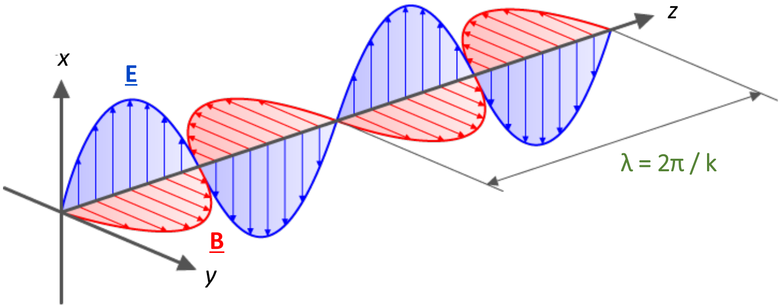


Figure adapted from Physics Stack Exchange

$$\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \sin\left(\omega(t - z/c)\right) \underline{\hat{\mathbf{x}}}$$

$$\underline{\mathbf{B}} = \underline{\mathbf{B}}_0 \sin\left(\omega(t - z/c)\right) \underline{\hat{\mathbf{y}}}$$

21.7 Characteristic impedance of free space

Take the ratio $Z = \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{H}}|}$ where $|\underline{\mathbf{H}}| = \frac{1}{\mu_0}\,|\underline{\mathbf{B}}|$

Z has units $[V\,m^{-1}]\,/\,[\,A\,m^{-1}\,] =$ [Ohms]

Z is called the *characteristic impedance of free space*

$$Z = \mu_0\,\frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{B}}|} = \mu_0\,c = \frac{\mu_0}{\sqrt{\mu_0\,\epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7\,\Omega$$

Electromagnetic waves : summary

In vacuum, free of charge or currents (ρ, **J** = 0):

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

$\rightarrow$

$$\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \ddot{\mathbf{E}}$$

$$\nabla^2 \mathbf{B} = \varepsilon_0 \mu_0 \ddot{\mathbf{B}}$$

Wave equations in **E, B**!

Electromagnetic waves propagate in free space:

Plane EM wave fronts: $\mathbf{E} = \mathbf{E_0} \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\}$ with wavelength $\lambda = \frac{2\pi}{k}$

Propagation velocity of wave fronts: $c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \, \mathrm{m\,s^{-1}}$

Relationship between E and B:
(in phase and mutually orthogonal
with wave vector **k**)

$$\mathbf{B} = \frac{\mathbf{k}}{\omega} \times \mathbf{E} \qquad \mathbf{E} = -c^2 \frac{\mathbf{k}}{\omega} \times \mathbf{B} \qquad \frac{|\mathbf{E}|}{|\mathbf{B}|} = c$$

Impedance of free space: $Z = \frac{|\mathbf{E}|}{|\mathbf{B}|/\mu_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\,\Omega$

21.8 Energy flow and the Poynting vector

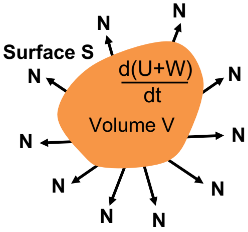
- Consider how the energy in an EM wave can leave a volume
- Recall : energy density of electric field : $u_e = \frac{1}{2}\,\epsilon_0\,\underline{\mathbf{E}}^2$

  energy density of magnetic field : $u_m = \frac{1}{2\,\mu_0}\,\underline{\mathbf{B}}^2$

Energy flow and the Poynting vector (cont.)

- $\frac{dU}{dt} = - \int_{\mathcal{V}} \underline{\mathbf{J}} \cdot \underline{\mathbf{E}}\, d\mathcal{V} - \frac{1}{\mu_0} \int_{\mathcal{V}} \underline{\nabla} \cdot (\underline{\mathbf{E}} \times \underline{\mathbf{B}})\, d\mathcal{V}$
- Apply divergence theorem and write as :

$$\left(\frac{dU}{dt} + \frac{dW}{dt}\right) = - \oint_S \left(\frac{1}{\mu_0} \underline{\mathbf{E}} \times \underline{\mathbf{B}}\right) \cdot \underline{\mathbf{da}}$$



- Poynting's theorem (energy conservation for EM fields) :

$$-\left(\frac{dU}{dt} + \frac{dW}{dt}\right) = \oint_S \underline{\mathbf{N}} \cdot \underline{\mathbf{da}}$$

where :

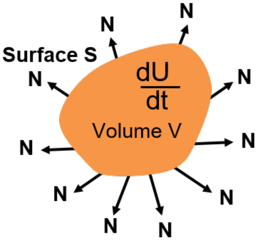
$$\underline{\mathbf{N}} = \frac{1}{\mu_0} \underline{\mathbf{E}} \times \underline{\mathbf{B}}$$

[Rate of energy loss of fields and particles in $\mathcal{V}$] = [Rate at which energy escapes through surface S bounding $\mathcal{V}$]

[Poynting vector]

Energy flow and the Poynting vector (cont.)

- Poynting's theorem (integral form) :
  $\int_{\mathcal{V}} \underline{\nabla} \cdot \underline{\mathbf{N}}\, d\mathcal{V} = -\int_{\mathcal{V}} \frac{\partial u}{\partial t}\, d\mathcal{V} - \int_{\mathcal{V}} \underline{\mathbf{J}} \cdot \underline{\mathbf{E}}\, d\mathcal{V}$
  where the energy density $u = \frac{1}{2}\epsilon_0\, E^2 + \frac{1}{2\mu_0}\, B^2$



- In differential form : $\underline{\nabla} \cdot \underline{\mathbf{N}} = -\frac{\partial u}{\partial t} - \underline{\mathbf{J}} \cdot \underline{\mathbf{E}}$
- Note also that in free space $(\rho = 0\,, \underline{\mathbf{J}} = \underline{\mathbf{0}})$:

  $\underline{\nabla} \cdot \underline{\mathbf{N}} = -\frac{\partial u}{\partial t}$ (c.f. $\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial \rho}{\partial t}$)

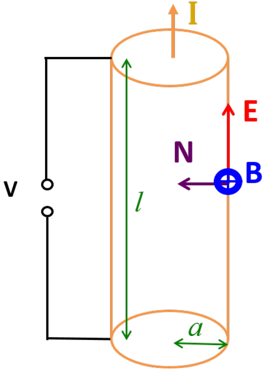
  $\rightarrow$ the Poynting vector is to energy what $\underline{\mathbf{J}}$ is to charge

  Poynting vector $\underline{\mathbf{N}}$ is the power per unit area flowing through the surface bounded by volume $\mathcal{V}$ (it also gives direction of flow).
  Units of $\underline{\mathbf{N}}$: $[W\, m^{-2}]$

- For EM waves, the intensity is the time-average of $|\underline{\mathbf{N}}|$:
  $\Im = \langle|\underline{\mathbf{N}}|\rangle = \frac{1}{\mu_0} E_0\, B_0 \underbrace{\left\langle \cos^2(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right\rangle}_{1/2} = \frac{1}{2\,\mu_0\, c}\, E_0^2$

Example : Poynting vector for a long resistive cylinder

- Calculate Poynting vector at the surface of a wire with applied potential difference V and current I : $\underline{\mathbf{N}} = \frac{1}{\mu_0}\underline{\mathbf{E}} \times \underline{\mathbf{B}}$

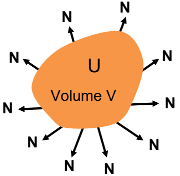


- Total power dissipated in wire : $P = -\int_S \underline{\mathbf{N}} \cdot \underline{\mathbf{da}} = V\,I$ as expected from circuit theory

Poynting Vector : summary

Total electromagnetic energy U contained in volume V:

$$U = \int_V \underbrace{\frac{1}{2}\left(\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right)}_{\text{energy density, } u = \frac{dU}{dV}} \mathrm{d}V$$



In free space:

$$-\frac{dU}{dt} = \oint_S \mathbf{N}\,.\,\mathbf{da} \quad \text{with} \quad \mathbf{N} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

Poynting vector

Energy flow rate out of volume V

Power per unit area through area bounding V

$[N]=W/m^2$

The intensity I of an EM wave is given by the time-average over the magnitude of the Poynting vector:

$$I = \langle|\mathbf{N}|\rangle = \frac{1}{2c\mu_0}{E_0}^2$$