CP2: ELECTROMAGNETISM

https://canvas.ox.ac.uk/courses/224992/pages/cp2-electromagnetism

Lecture 19:

Motion in E & B Fields

Claire Gwenlan[1]

University of Oxford

HT 2024

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

$$\frac{1}{\mu_0} \underline{\nabla} \times \underline{\mathbf{B}} = \underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

[1]With many thanks to Prof Neville Harnew and Prof Laura Herz

OUTLINE : 19. MOTION IN E & B FIELDS

19.1 Motion of charged particles in E and B fields

19.2 Example : the mass spectrometer

19.3 Example : magnetic lenses

19.1 Motion of charged particles in E and B fields

- Force on a charged particle in an $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ field :

$$\underline{\mathbf{F}} = q \left(\underbrace{\underline{\mathbf{E}}}_{\text{along } \underline{\mathbf{E}}} + \underbrace{\underline{\mathbf{v}} \times \underline{\mathbf{B}}}_{\perp \text{ to both } \underline{\mathbf{v}} \text{ and } \underline{\mathbf{B}}} \right)$$

- Newton's Second Law gives equation of motion :

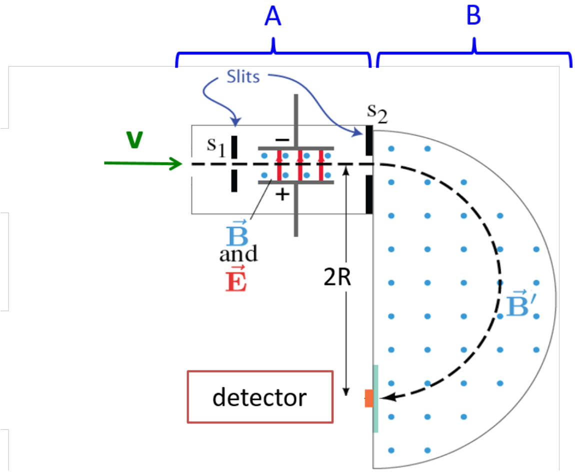
$$\underline{\mathbf{F}} = m\,\underline{\mathbf{a}} = m\,\underline{\ddot{\mathbf{r}}} = q\,(\,\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})$$

- Will demonstrate with 2 examples :

  1. Mass spectrometer
  2. Magnetic lens

19.2 Example : the mass spectrometer

Used for identifying small charged particles (molecules, ions) by their mass *m*



Stage A : The velocity filter

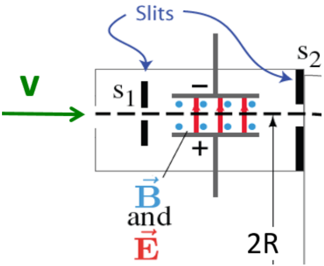
- The particle will pass through both slits if it experiences no net force inside the filter
- The region has both $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields

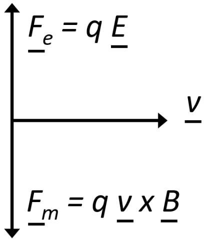
$$\underline{\mathbf{F}} = q\ (\ \underline{\mathbf{E}}\ +\ \underline{\mathbf{v}} \times \underline{\mathbf{B}}) = 0$$

$$\rightarrow \quad \text{need } \underline{\mathbf{E}} = -\underline{\mathbf{v}} \times \underline{\mathbf{B}} \rightarrow\ v = \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{B}}|}$$

$$(\underline{\mathbf{E}} \perp \underline{\mathbf{v}}\ \&\ \underline{\mathbf{B}})$$

- Will filter particles with $v = \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{B}}|}$ and the spread $\pm\Delta v$ is given by the slit width





Stage B : The mass filter

- This region has only a $\underline{\mathbf{B}}$ field

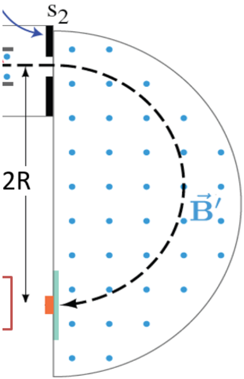
  $m\,\ddot{\underline{\mathbf{r}}} = q\,\dot{\underline{\mathbf{r}}} \times \underline{\mathbf{B}}$

  with $\underline{\mathbf{B}} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$ and $\dot{\underline{\mathbf{r}}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$

  $\rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \frac{q}{m} \begin{pmatrix} \dot{y}\,B \\ -\dot{x}\,B \\ 0 \end{pmatrix}$

  $\rightarrow \ddot{z} = 0 \rightarrow v_z = \text{constant} \;\; (= 0)$

- $|\ddot{\underline{\mathbf{r}}}|^2 = \ddot{x}^2 + \ddot{y}^2 = \frac{q^2}{m^2} \underbrace{\left(\dot{x}^2 + \dot{y}^2\right)}_{v^2} B^2$



- Circular motion in $x - y$ plane with :

  $$\frac{v^2}{R} = \frac{q}{m} v\,B \rightarrow R = \frac{mv}{qB}$$

- Since q and v are constant, then $R \propto m$

Mass spectrometer summary

In the presence of both E- and B-fields, a charge experiences the force:

$$\mathbf{F}_{\text{EM}} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Mass Spectrometer.

A. velocity filter:

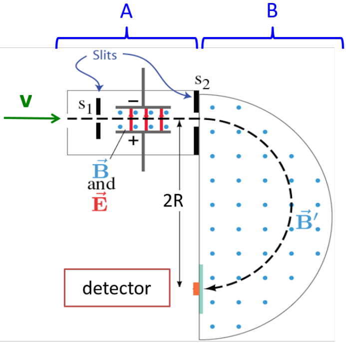
E&B-fields present. Charged particles pass through Stage A if their velocity equals the amplitude ratio:

$$v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$

B. Filter stage:

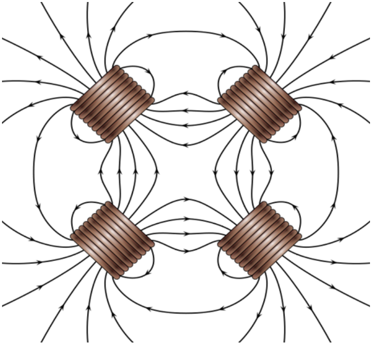
Only B-field present. Charged particles are forced on circular path with radius:

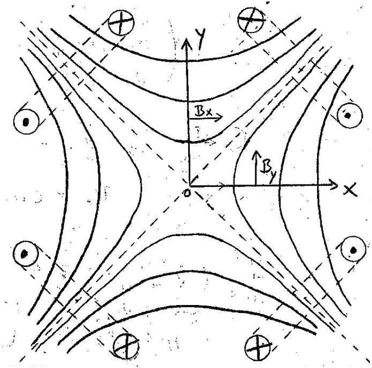
$$R = \frac{mv}{qB}$$



19.3 Example : magnetic lenses

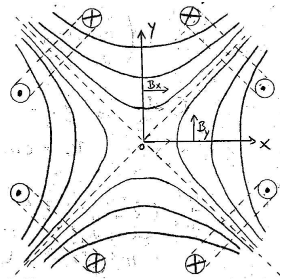
- Magnetic lenses are used for focusing and collimating charged particle beams (used in electron microscopes, particle accelerators etc.)
- Quadrupole lens : four identical coils in $x - y$ plane
- Sum of 4 dipole fields : for small values of x, y close to the axis of symmetry, $B_x \propto y$, $B_y \propto x$





Quadrupole lens

- Along x-axis : only B_y component
- Along y-axis : only B_x component
- No z-component (symmetry)
- Inside the lens, close to the z-axis



$$\underline{\mathbf{B}} = \begin{pmatrix} k\,y \\ k\,x \\ 0 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

- Equation of motion $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$

$$\mathrm{m} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ k\,y & k\,x & 0 \end{vmatrix} = q\,k \begin{pmatrix} -x\,\dot{z} \\ y\,\dot{z} \\ x\,\dot{x} - y\,\dot{y} \end{pmatrix}$$

- Assume particle travels at a small angle wrt the z-axis :

$$\rightarrow \;\; \dot{x}, \dot{y} \approx 0 \;\; \rightarrow \;\; \ddot{z} = 0 \;\; \rightarrow \;\; \dot{z} = v = \text{constant} \;\; \rightarrow \;\; z = v\,t$$

- Equations of motion in the $x - y$ plane :

$$\ddot{x} = -\frac{q}{m} k\,v\,x \quad \text{and} \quad \ddot{y} = \frac{q}{m} k\,v\,y$$

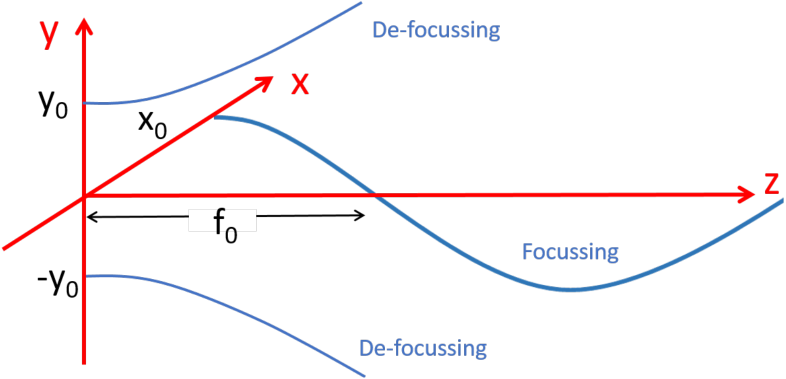
Quadrupole lens (cont.)

- Equations of motion : $\ddot{x} = -\alpha^2 x$ & $\ddot{y} = \alpha^2 y$, where $\alpha = \sqrt{\frac{q\,k\,v}{m}}$

- Focal points in z direction (x=0) at $f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$
- Use lens pair with 90° angle for collimating a charged beam

Quadrupole lens (cont.)

Lens pulls beam on-axis in x and removes particles deviating in y

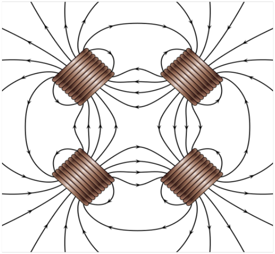


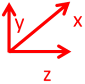
$$f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$$

Magnetic lens summary

Magnetic Lens.

$$\mathbf{B} = (\mathrm{k}\,y, \mathrm{k}\,x, 0)$$





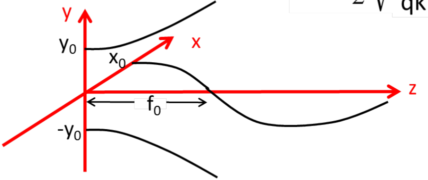
Equation of Motion: $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}$

Solutions:

$$y(z) = y_0 \cosh \sqrt{\frac{q\,\mathrm{k}}{vm}} z$$ de-focusing

$$x(z) = x_0 \cos \sqrt{\frac{q\,\mathrm{k}}{vm}} z$$ focusing with

$$f_0 = \frac{\pi}{2}\sqrt{\frac{vm}{\mathrm{qk}}}$$



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Lecture 20:

Displacement Current & Maxwell's Equations

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$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

$$\frac{1}{\mu_0} \underline{\nabla} \times \underline{\mathbf{B}} = \underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

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OUTLINE : 20. DISPLACEMENT CURRENT & MAXWELL'S EQUATIONS

20.1 Electrodynamics “before Maxwell”

Time-varying B-fields generate E-fields. *However*, time-varying E-fields do not seem to create B-fields in this version.
Is there something wrong?

20.2 Revisit Ampere's Law

- Therefore Ampere's Law in its current form violates the continuity equation (and hence charge conservation) !
- But this is not surprising since Ampere's Law is derived from the Biot-Savart Law assuming that $\frac{\partial}{\partial t}(\rho) = 0$

  $\rightarrow$ we have to "fix" Ampere's Law !

20.3 Fixing Ampere's Law : displacement current

- Add a term to Ampere's Law to make it compatible with the continuity equation : $\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\rho)$

  The term $\left(\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$ is called the *displacement current* $\underline{\mathbf{J}}_D$ (note that it is actually a time-varying electric field)
- Time-varying $\underline{\mathbf{E}}$ fields now generate $\underline{\mathbf{B}}$ fields and vice versa. Also satisfies charge conservation.

Summary : Ampere's Law with Maxwell's correction

Ampere's Law does not comply with the Equation of Continuity:

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply divergence: $\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{=0 \text{ always}} = \mu_0 \underbrace{\nabla \cdot \mathbf{J}}_{= -\frac{\partial \rho}{\partial t}}$

$= 0$ only for statics!

This lack of charge conservation is unphysical! As a solution, add a so-called "displacement current" to **J**, which will ensure compliance with the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot (\underbrace{\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}_{\text{displacement current } \mathbf{J_D}})$$

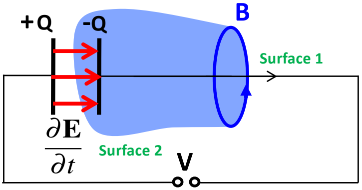
Obtain **Ampere's Law with "displacement current"**: $\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$

Using Stoke's theorem : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot d\underline{\mathbf{a}}$

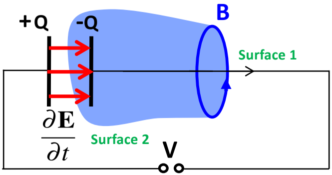
gives the integral form : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \mu_0 \underbrace{\int_S \underline{\mathbf{J}} \cdot d\underline{\mathbf{a}}}_{I_{enc}} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$

20.4 Example : Ampere's Law and a charging capacitor

- This is the first example, showing why Ampere's Law fails without adding the displacement current : a straight wire, and add a capacitor into the circuit
- Previously we used Ampere's Law to calculate the magnetic field along Amperian loop $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 \, I_{enc}$



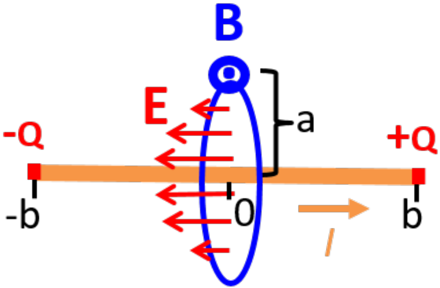
Ampere's Law and a charging capacitor (cont.)



- In differential form : $$\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \left(\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

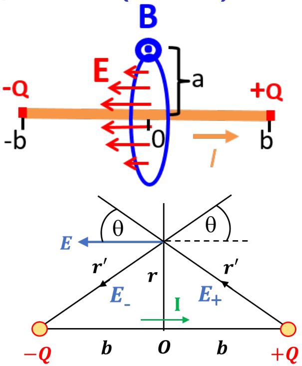
20.5 Example : B-field of a short current-carrying wire

- Recall B-field from Biot-Savart Law at a distance a from centre of a wire of length $2b$ → $B = \frac{\mu_0 I}{2\pi a} \frac{b}{\sqrt{b^2+a^2}}$
- Again, Ampere's Law fails depending on which path we use. Need to use displacement current.
- $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$
- Wire is short, so charge builds up at the ends giving time-varying $\underline{\mathbf{E}}$-field



B-field of a short current-carrying wire (cont.)

- Form Amperian loop of radius a, and integrate $\frac{\partial E}{\partial t}$ over enclosed area
- Calculate **E**-field due to two point charges at wire ends, $\pm b$



20.6 Maxwell's equations

$$\oint_S \mathbf{E} \,.\, \mathbf{da} = \frac{Q}{\varepsilon_0} \leftrightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss's Law: Charge generates an electric field. Electric field lines begin and end on charge.

$$\oint_C \mathbf{E} \,.\, \boldsymbol{dl} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \,.\, \mathbf{da} \leftrightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Maxwell-Faraday's Law: time-varying magnetic fields create electric fields (induction)

$$\oint_S \boldsymbol{B} \,.\, \boldsymbol{da} = 0 \leftrightarrow \nabla \cdot \mathbf{B} = 0$$

There are **no magnetic monopoles.** Magnetic field lines form closed loops.

$$\oint_C \mathbf{B} \,.\, \boldsymbol{dl} = \mu_0 I + \mu_0 \,\varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \,.\, \mathbf{da} \leftrightarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Ampere's Law with Maxwell's correction: electric currents and time-varying electric fields generate magnetic fields

Maxwell's equations, together with the Lorentz force : $\underline{\mathbf{F}} = q\,(\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})$ summarise the entire theoretical content of classical electrodynamics