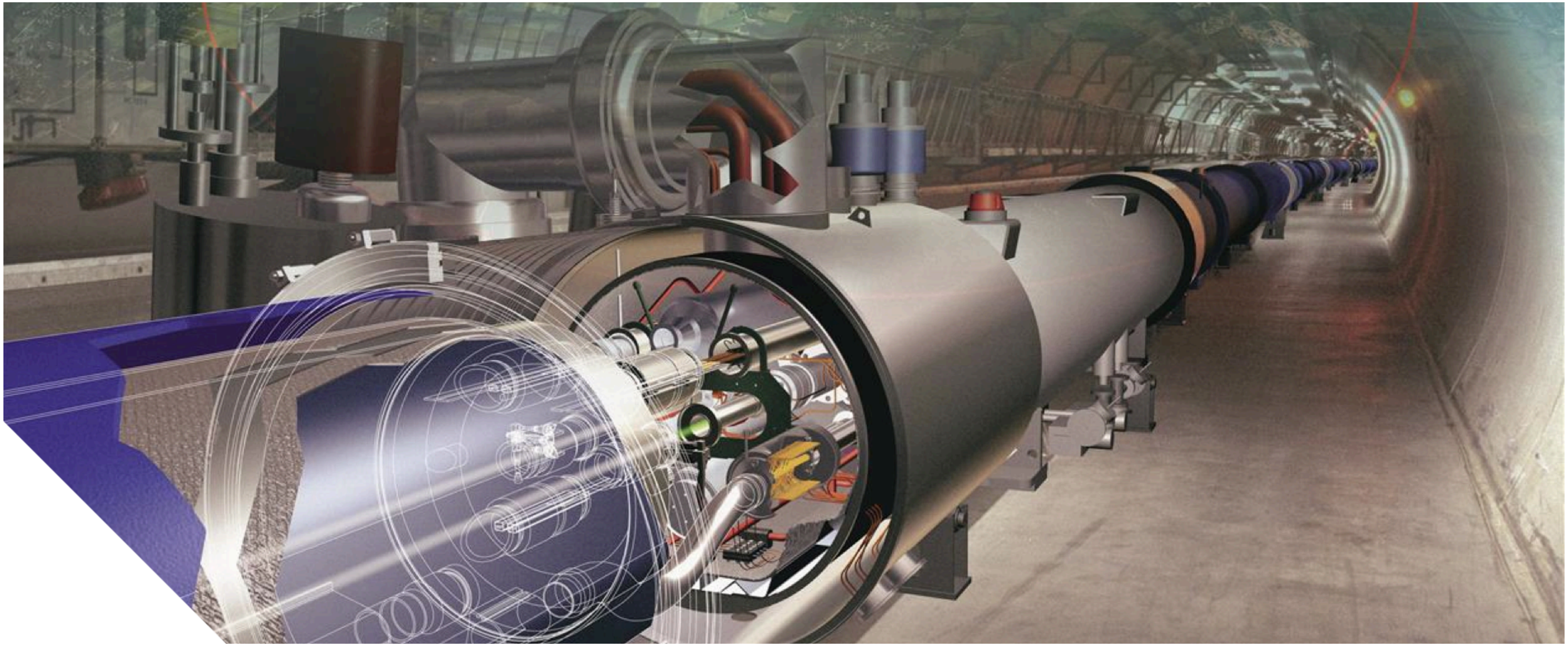


QCD – Lecture 3

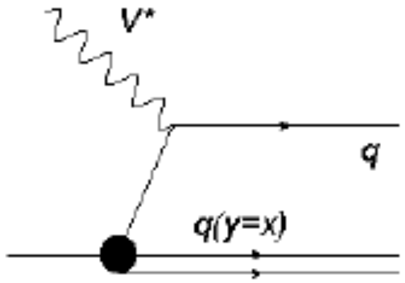
QCD-improved parton model and the DGLAP equations

Claire Gwenlan, Oxford, HT

QCD-improved parton model

Let's apply some of the ideas we have learnt to the Parton Model:

Formally the QPM calculate the cross section in terms of a convolution of the point-like V*q scattering and the parton distribution function.

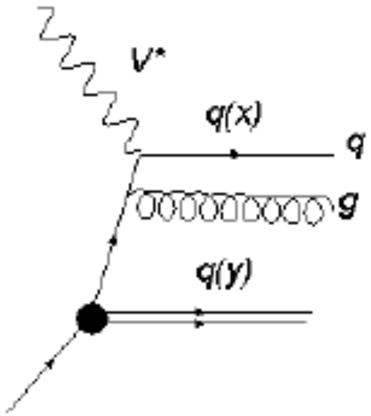


$$\frac{F_2(x)}{x} = \int d\,y\,d\,z\,\delta(x - zy)\,\sigma^{\text{point}}(z)\,q(y)$$

$$\sigma^{\text{point}} = q_i^2\,\delta(1 - z) \qquad z = x / y$$

$$\Rightarrow \frac{F_2(x)}{x} = q_i^2\,q(x) \Rightarrow y = x$$

QCD adds to this extra diagrams. E.g. a quark of momentum fraction y emits a gluon to become a quark of momentum fraction x, (y > x) before it interacts with V*.



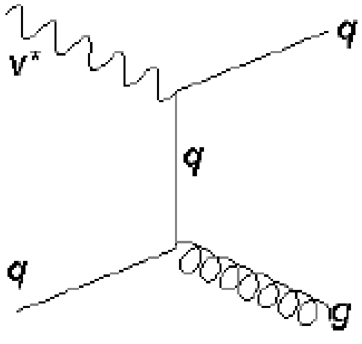
$$\frac{F_2(x)}{x} = \int d\,y\,d\,z\,\delta(x - zy)\,q(y)\left[\sigma^{\text{point}}(z) + \sigma^{V*q\to qg}\right]$$

A new term: $\sigma^{V*q\to qg}$

Is added to the point-like cross section.

see also, EG.
Halzen & Martin, Devenish & Cooper-Sarkar

$\sigma(V^* q - q g)$ is calculable from QCD Feynman rules.



$$V^* q q \text{ vertex } \Rightarrow e_i^2$$

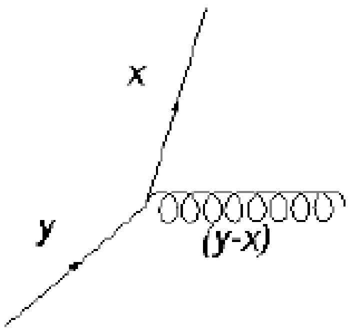
$$g q q \text{ vertex } \Rightarrow \alpha_s$$

Quark "exchanged" propagator $\sim \dfrac{1}{{p_t}^2}$

Quark acquires $\boldsymbol{p_t}$ w.r.t proton because of the gluon emission

$$\int_{\min}^{\max} \frac{d\,{p_t}^2}{{p_t}^2} \Rightarrow \ln \frac{Q^2}{Q_0^2}$$

$$\sigma(V^* q \rightarrow q g) = e_i^2 \frac{\alpha_s}{2\pi} P_{qq}(z) \ln \frac{Q^2}{Q_0^2}$$



$$P_{qq}(z) = C_F \frac{1 + z^2}{1 - z}$$

Is the SPLITTING FUNCTION

Probability that quark of momentum ***yP*** splits to quark momentum ***xP*** and gluon of momentum ***(y-x)P***

z = x/y

see also, EG.
Halzen & Martin, Devenish & Cooper-Sarkar

scaling violation

$$\frac{F_2(x,Q^2)}{x} = \int_x^1 \frac{dy}{y} q(y)\left[e_i^2 \delta\left(1-\frac{x}{y}\right) + \sigma(V^* q - q g)\left(\frac{x}{y}, Q^2\right)\right]$$

$$= e_i^2 \int_x^1 \frac{dy}{y}\left[q(y)\,\delta\left(1-\frac{x}{y}\right)\right] + e_i^2\, \Delta \mathrm{q}\left(x, Q^2\right)$$

$$= e_i^2\left[q(x) + \Delta \mathrm{q}\left(x, Q^2\right)\right] = e_i^2\, q\left(x, Q^2\right)$$

where $\Delta \mathrm{q}\left(x, Q^2\right) = \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{Q_0^2} \int_x^1 \frac{dy}{y} q(y)\, P_{\mathrm{qq}}\left(\frac{x}{y}\right)$

The Q^2 dependence of $\sigma(V^* q - q g)$

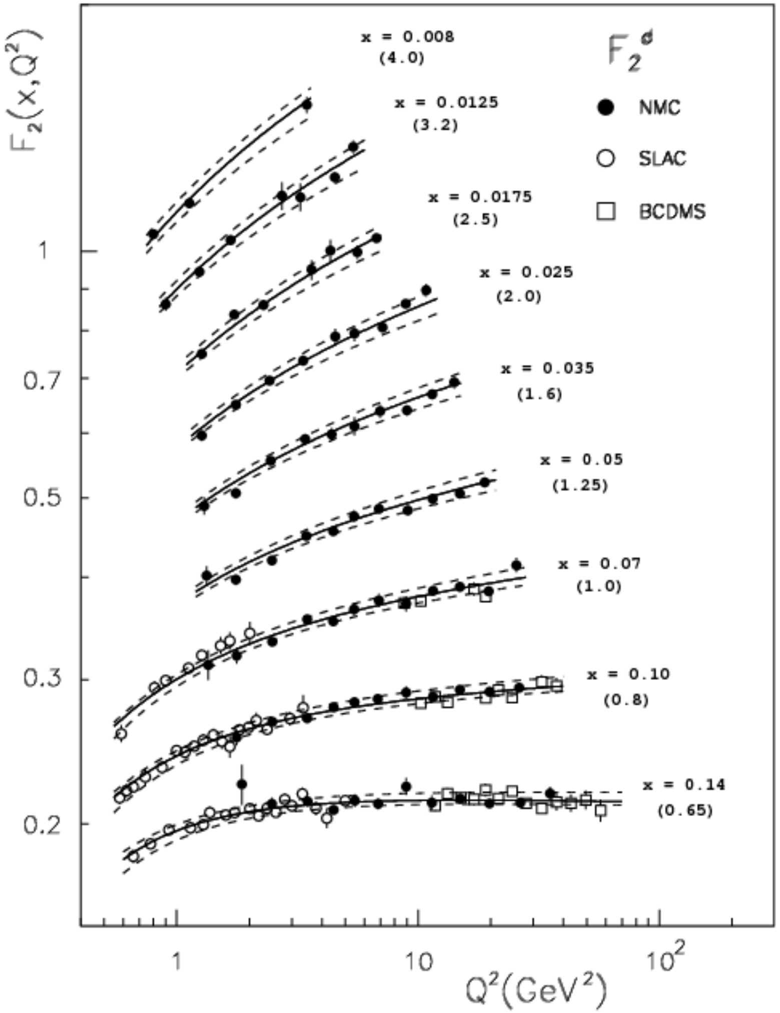
has been transferred into the parton distribution function $q(x) \to q\left(x, Q^2\right)$

$$F_2(x,Q^2) = \sum_i e_i^2\, x\left[q\left(x, Q^2\right) + \overline{q}\left(x, Q^2\right)\right]$$

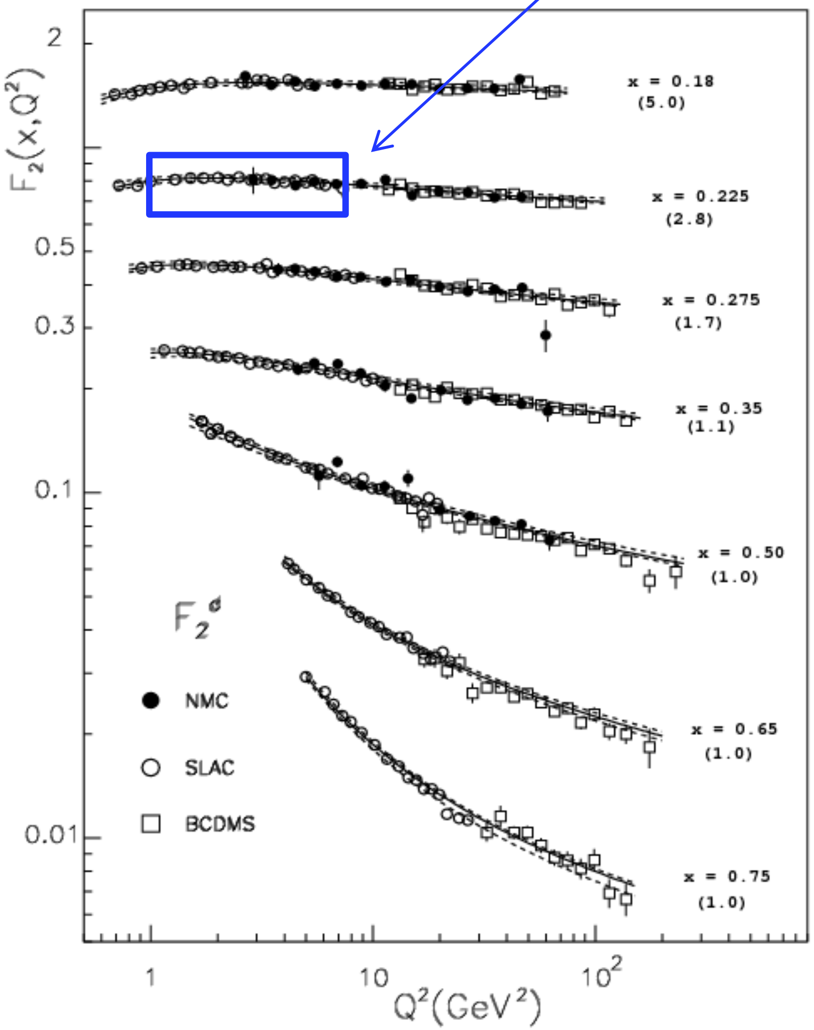
Bjorken scaling violation is predicted by QCD, but only ~$\ln Q^2$

observation of scaling violation

LO QCD: $F_2(x,Q^2) = \Sigma_i e_i^2(xq(x,Q^2) + xqbar(x,Q^2))$



Region of 1st SLAC measurement (1972)



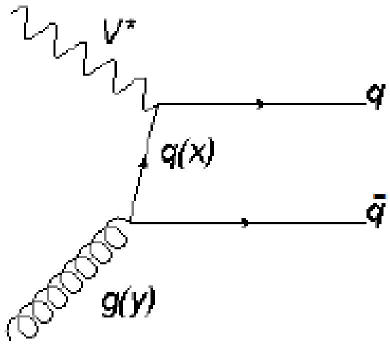
DGLAP evolution equations

$$\frac{\partial q(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 q(y,Q^2)\,P_{qq}\left(\frac{x}{y}\right)\frac{dy}{y}$$

Quark distribution evolve with Q^2

Shape of $q(x,Q^2)$ is **NOT** predicted, but its evolution **IS** (and is measurable).

Extend the formalism,



Maybe a gluon of momentum ***yP*** splits into a quark of momentum ***xP*** and an antiquark of momentum ***(y-x)P***

Splitting function $P_{qg}(z)$

$$\frac{\partial q(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y}\left[P_{qq}\left(\frac{x}{y}\right)q(y,Q^2) + P_{qg}\left(\frac{x}{y}\right)g(y,Q^2)\right]$$

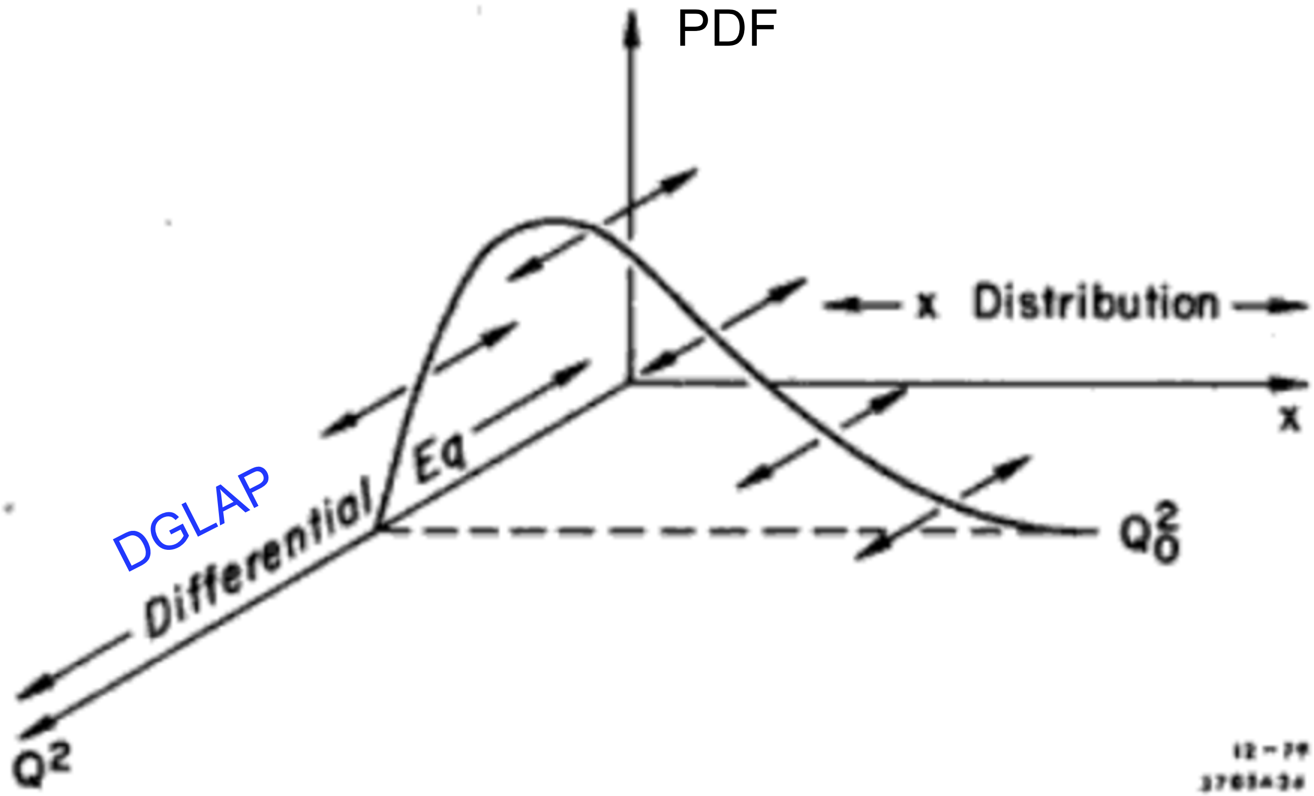
Quark evolution equation DGLAP

$$\frac{\partial g(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y}\left[P_{gq}\left(\frac{x}{y}\right)q(y,Q^2) + P_{gg}\left(\frac{x}{y}\right)g(y,Q^2)\right]$$

↑ Summed over all quark parents

gluon evolution equation

COUPLED differential equations



the theory predicts the rate at which the **parton distribution functions** (both quarks and gluons) evolve with energy scale of probe, Q^2

BUT it does not predict their x-dependence, ie. their starting shape

DGLAP evolution equations

DGLAP equations are a set of (2nf+1) coupled equations for the evolution of quarks and gluon

Dokshitzer
Gribov
Lipatov
Altarelli
Parisi

$$\frac{\partial}{\partial \ln Q^2} \begin{pmatrix} q(x,Q^2) \\ g(x,Q^2) \end{pmatrix} = \frac{\alpha_s}{2\pi} \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q \\ g \end{pmatrix}$$

where $P \otimes f \equiv \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}\right) f(y, Q^2)$

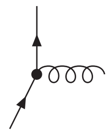
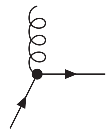
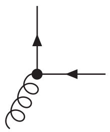
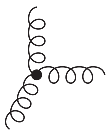
NB, previously we assumed that αs is constant;
taking the running of αs into account is somewhat subtle, but leads to the same evolution equation

with LO splitting functions:

$$P_{\bar{q}_i \bar{q}_j} = P_{q_i q_j} \equiv P_{qq}\, \delta_{ij},$$

$$P_{\bar{q}_i g} = P_{q_i g} \equiv P_{qg},$$

$$P_{g \bar{q}_i} = P_{g q_i} \equiv P_{gq}$$

Diagram	Splitting
	$P^{(0)}_{qq}(z) = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$
	$P^{(0)}_{gq}(z) = \frac{4}{3}\left[\frac{1+(1-z)^2}{z}\right]$
	$P^{(0)}_{qg}(z) = \frac{1}{2}\left[z^2+(1-z)^2\right]$
	$P^{(0)}_{gg}(z) = 6\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18}\right)\delta(1-z)\right]$

NB, **z=1 singularities cancelled by interference with virtual loop diagrams which give delta functions at z=1**
(use '+' prescription to regularize the 1/(1 – z))

Devenish & Cooper-Sarkar, p77

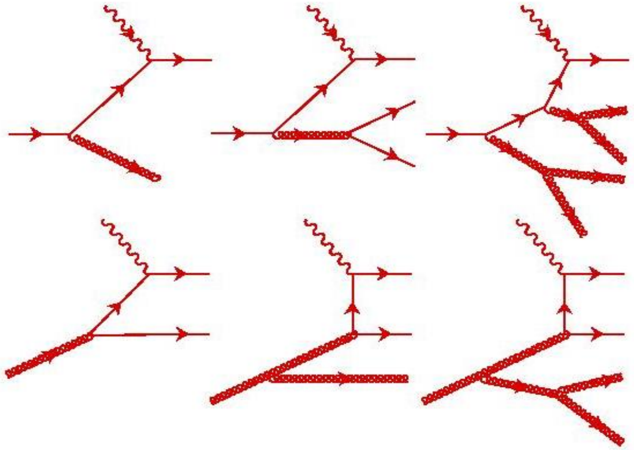
calculations at higher orders

going beyond LO:

splitting functions are given in QCD by perturbative expansion in **α_s**

$$P_{ij} = P_{ij}^{(0)} + (\alpha_s/2\pi) P_{ij}^{(1)} + (\alpha_s/2\pi)^2 P_{ij}^{(2)} + \cdots$$

where, at LO, the splitting functions are as defined previously



ALSO, F2 no longer so neatly expressed in terms of **parton distributions**, and FL is no longer zero

EG. at NLO:

$$\frac{F_2(x,Q^2)}{x} = \int_x^1 \frac{d\xi}{\xi}\left[\sum_i e_i^2 q_i(\xi,Q^2) C_q\left(\frac{x}{\xi},\alpha_s\right) + \sum_i e_i^2\, g(\xi,Q^2) C_g\left(\frac{x}{\xi},\alpha_s\right)\right]$$

$$C_q(z,\alpha_s) = \delta(1-z) + \frac{\alpha_s}{2\pi} C_q^1(z) \qquad C_g(z,\alpha_s) = \frac{\alpha_s}{2\pi} C_g^1(z)$$

$$F_L(x,Q^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[\frac{8}{3}\left(\frac{x}{\xi}\right)^2 F_2(\xi,Q^2) + \sum_i e_i^2\, 4\left(\frac{x}{\xi}\right)^2\left(1-\frac{x}{\xi}\right)\xi g(\xi,Q^2)\right]$$

BUT everything is still **perturbatively calculable**, apart from the starting **parton distributions**

NLO splitting functions

$$P^{(1)}_{\rm ps}(x) = 4\,C_F n_f\left(\frac{20}{9}\frac{1}{x} - 2 + 6x - 4\mathrm{H}_0 + x^2\left[\frac{8}{3}\mathrm{H}_0 - \frac{56}{9}\right] + (1+x)\left[5\mathrm{H}_0 - 2\mathrm{H}_{0,0}\right]\right)$$

$$P^{(1)}_{\rm qg}(x) = 4\,C_A n_f\left(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{\rm qg}(-x)\mathrm{H}_{-1,0} - 2p_{\rm qg}(x)\mathrm{H}_{1,1} + x^2\left[\frac{44}{3}\mathrm{H}_0 - \frac{218}{9}\right]\right.$$
$$+4(1-x)\left[\mathrm{H}_{0,0} - 2\mathrm{H}_0 + x\mathrm{H}_1\right] - 4\zeta_2 x - 6\mathrm{H}_{0,0} + 9\mathrm{H}_0\Big) + 4\,C_F n_f\Big(2p_{\rm qg}(x)\Big[\mathrm{H}_{1,0} + \mathrm{H}_{1,1} + \mathrm{H}_2$$
$$-\zeta_2\Big] + 4x^2\left[\mathrm{H}_0 + \mathrm{H}_{0,0} + \frac{5}{2}\right] + 2(1-x)\left[\mathrm{H}_0 + \mathrm{H}_{0,0} - 2x\mathrm{H}_1 + \frac{29}{4}\right] - \frac{15}{2} - \mathrm{H}_{0,0} - \frac{1}{2}\mathrm{H}_0\Big)$$

$$P^{(1)}_{\rm gq}(x) = 4\,C_A C_F\left(\frac{1}{x} + 2p_{\rm gq}(x)\left[\mathrm{H}_{1,0} + \mathrm{H}_{1,1} + \mathrm{H}_2 - \frac{11}{6}\mathrm{H}_1\right] - x^2\left[\frac{8}{3}\mathrm{H}_0 - \frac{44}{9}\right] + 4\zeta_2 - 2\right.$$
$$-7\mathrm{H}_0 + 2\mathrm{H}_{0,0} - 2\mathrm{H}_1 x + (1+x)\left[2\mathrm{H}_{0,0} - 5\mathrm{H}_0 + \frac{37}{9}\right] - 2p_{\rm gq}(-x)\mathrm{H}_{-1,0}\Big) - 4\,C_F n_f\Big(\frac{2}{3}x$$
$$-p_{\rm gq}(x)\left[\frac{2}{3}\mathrm{H}_1 - \frac{10}{9}\right]\Big) + 4\,C_F{}^2\Big(p_{\rm gq}(x)\left[3\mathrm{H}_1 - 2\mathrm{H}_{1,1}\right] + (1+x)\left[\mathrm{H}_{0,0} - \frac{7}{2} + \frac{7}{2}\mathrm{H}_0\right] - 3\mathrm{H}_{0,0}$$
$$+1 - \frac{3}{2}\mathrm{H}_0 + 2\mathrm{H}_1 x\Big)$$

$$P^{(1)}_{\rm gg}(x) = 4\,C_A n_f\left(1 - x - \frac{10}{9}p_{\rm gg}(x) - \frac{13}{9}\left(\frac{1}{x} - x^2\right) - \frac{2}{3}(1+x)\mathrm{H}_0 - \frac{2}{3}\delta(1-x)\right) + 4\,C_A{}^2\Big(27$$
$$+(1+x)\left[\frac{11}{3}\mathrm{H}_0 + 8\mathrm{H}_{0,0} - \frac{27}{2}\right] + 2p_{\rm gg}(-x)\left[\mathrm{H}_{0,0} - 2\mathrm{H}_{-1,0} - \zeta_2\right] - \frac{67}{9}\left(\frac{1}{x} - x^2\right) - 12\mathrm{H}_0$$
$$-\frac{44}{3}x^2\mathrm{H}_0 + 2p_{\rm gg}(x)\left[\frac{67}{18} - \zeta_2 + \mathrm{H}_{0,0} + 2\mathrm{H}_{1,0} + 2\mathrm{H}_2\right] + \delta(1-x)\left[\frac{8}{3} + 3\zeta_3\right]\Big) + 4\,C_F n_f\Big(2\mathrm{H}_0$$
$$+\frac{2}{3}\frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x)\left[4 - 5\mathrm{H}_0 - 2\mathrm{H}_{0,0}\right] - \frac{1}{2}\delta(1-x)\Big)\ .$$

$$P_{ab} = \frac{\alpha_s}{2\pi}P^{(0)} + \frac{\alpha_s^2}{16\pi^2}P^{(1)}$$

Curci, Furmanski & Petronzio '80

from Gavin Salam

NNLO splitting functions

NNLO, $P_{ab}^{(2)}$: Moch, Vermaseren & Vogt '04

from Gavin Salam

constraints on splitting functions

- the renormalised parton densities obey sum rules to conserve fermion number and flavour

$$\int_0^1 dx[q_i(x,Q^2) - \bar{q}_i(x,Q^2)] = v_i, \quad \text{where } v_i = 2,1,0,\ldots \text{ for the u, d, s,}\ldots$$

- and overall momentum conservation requires

where Σ is a sum over all quark flavours

$$\int_0^1 dx\, x[\Sigma(x,Q^2) + g(x,Q^2)] = 1, \qquad \Sigma = \sum_i (q_i + \bar{q}_i)$$

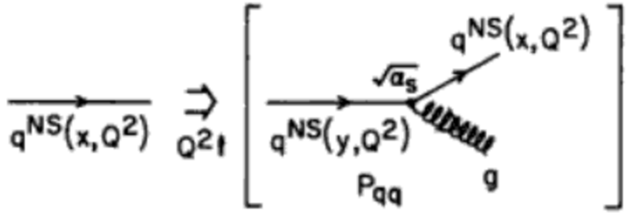
- this imposes the following **constraints on the splitting functions**:

$$\begin{aligned}
\int_0^1 dz\, P_{qq}(z) &= 0 \\
\int_0^1 dz\, z[P_{qq}(z) + P_{gq}(z)] &= 0 \\
\int_0^1 dz\, z[2n_f P_{qg}(z) + P_{gg}(z)] &= 0
\end{aligned}$$

singlet and non-singlet combinations

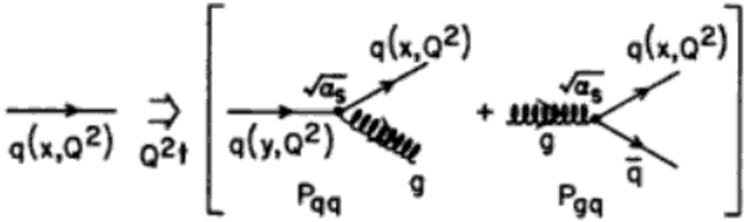
- useful to define **singlet distribution, Σ**

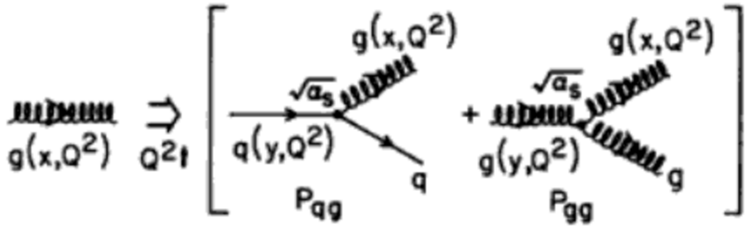
$$\Sigma(x,Q^2) = \sum_i \left(q_i(x,Q^2) + \bar{q}_i(x,Q^2)\right)$$



- and **non-singlet** (or flavourful) **distributions, q^{NS}**, of which the most obvious are the **valence combinations**

$$q_i^{NS}(x,Q^2) \quad = \quad q_i(x,Q^2) - \bar{q}_i(x,Q^2)$$





- it can be shown that:

$$\frac{\partial}{\partial \ln Q^2}\begin{pmatrix}\Sigma \\ g\end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix}P_{qq} & 2n_f P_{qg} \\ P_{gq} & P_{gg}\end{pmatrix} \otimes \begin{pmatrix}\Sigma \\ g\end{pmatrix}$$

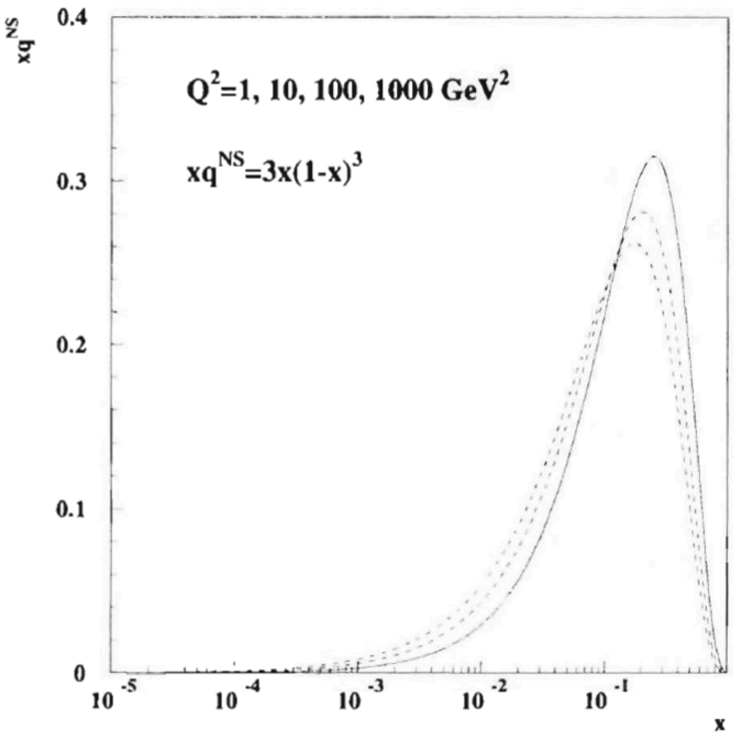
singlet and gluon coupled

$$\frac{\partial q^{\rm NS}}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} P_{qq} \otimes q^{\rm NS}$$

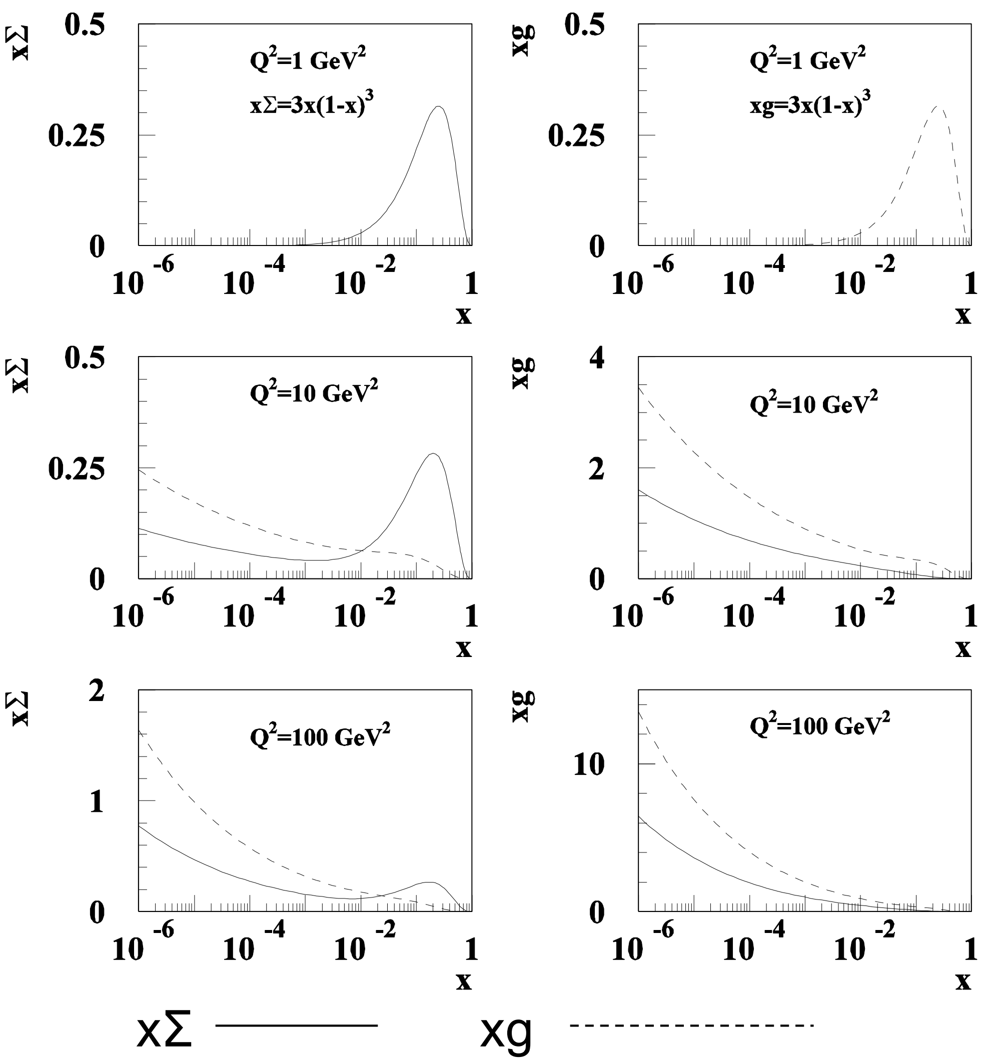
non-singlet evolves independently of gluon

g → qqbar does not feed back to flavour

evolution of parton distributions



- evolution of **non-singlet** combination does not involve gluon and hence valence distributions evolve slowly…
- whereas evolution of **singlet** combination and **gluon** are coupled



sea = $x\Sigma$ valence-like and gluon zero at Q^2=1

sea = $x\Sigma$ zero and gluon valence-like at Q^2=1

theoretical guidance for the PDF shapes?

Do we have theoretical expectations for the shapes of parton distributions?

Non perturbative approaches

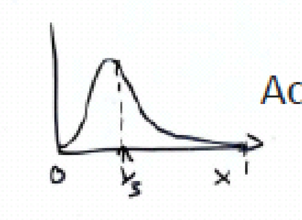
- quantum statistics
- bag models
- lattice gauge theory

Not yet good enough

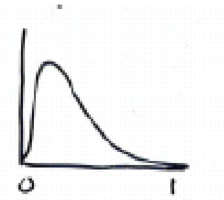
Constraints at high x and low x

$$\sim x^{\alpha}(1-x)^{\beta}$$

3 valence quarks of ≈ $^{1}/_{3}$ momentum each,
but smeared because confined in proton $\Delta x \Delta p \sim \hbar$



Actually observe

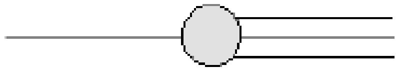


Valence momentum has decreased, $q \to q + g$

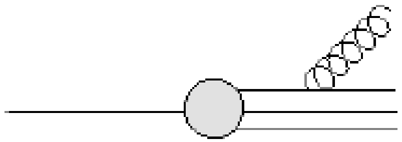
counting rules and high x

Try and explain the high *x* behaviour as well. $(1-x) \to 0$ i.e. no momentum left for the other partons – they are just, $q(x) \xrightarrow{x\to 1} (1-x)^{2n_s-1}$

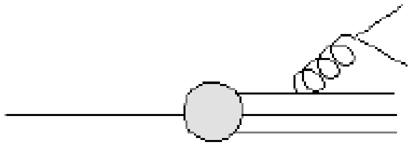
where n is the minimum number of spectators



Valence quarks $n_s = 2 \to (1-x)^3$



Gluons $n_s = 3 \to (1-x)^5$



Sea quarks $n_s = 4 \to (1-x)^7$

Why $(2n_s - 1)$? (Brodsky & Farrar PRL31, p1183)

It relates to the dimensions of the operators in the Lagrangian... anyway – the values for the powers above had only moderate success. E.g.,

$$u_v \sim (1-x)^3, \quad \mathrm{dv} \sim (1-x)^4$$

Diquark explanations...?

gluon $\sim (1-x)^5$ not too bad, but only at low Q^2

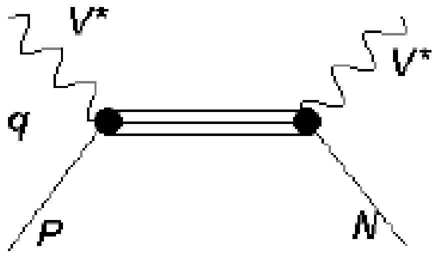
Sea, anything from $(1-x)^4 \to (1-x)^{10}$ has been fitted! $\sim (1-x)^7$ at $Q^2 \approx 4\ \mathrm{GeV}^2$

Regge theory and low x

In the low *x* region $x = \dfrac{Q^2}{2\, p \cdot q}$

we have large $p \cdot q \Rightarrow$ large W^2

Large (centre of mass energy)^2 for the V^*- nucleon sub-process.



$W^2 = s(V^* p) = 2\, p \cdot q$

$$\sigma(V^* p \to X) = \frac{1}{S(V^* p)} \operatorname{Im}(V^* p \to V^* p)$$ Optical theorem

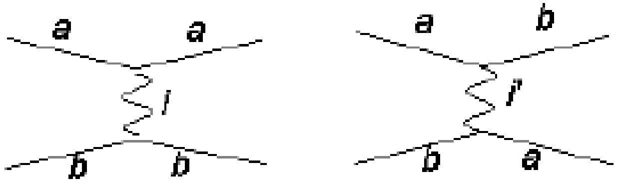
Could virtual boson-hadron scattering have anything in common with real boson-hadron scattering? E.g. photon-hadron *γP* and hadron-hadron *PP, πP, kP* etc. scattering.

If so, then the predictions of Regge theory for high energy scattering may be relevant.

The prediction is that, $\mathrm{Im}(\mathrm{ab} \to \mathrm{ab}) = \sum_i \gamma^{\mathrm{ab}} S_{\mathrm{ab}}^{\alpha_i}$

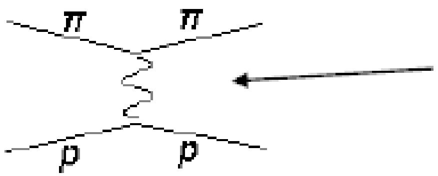
Hence, from the optical theorem, $\sigma(\mathrm{ab} \to X) = \sum_i \gamma^{\mathrm{ab}} S_{\mathrm{ab}}^{\alpha_i - 1}$

The sum is over Regge exchanges,



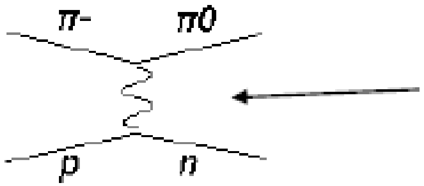
Which could mediate the elastic scattering

E.g.



Quantum number of the exchange!
Could be like **ρ⁰**
Could be flavourless "Pomeron"

$$\mathrm{Im}(\pi\mathrm{p} \to \pi\mathrm{p}) = A\, s^{1.08} + B\, s^{0.5}$$



Can only be like **ρ⁻** $\Rightarrow C\, s^{0.5}$

Total cross sections of the form,

$$\sim \sum_i s^{a_i - 1} \Rightarrow A\, s^{0.08} + B\, s^{-0.5} \Rightarrow$$ Evidence

Much of HEP in 60's and 70's measured such reactions.

total cross section measurements

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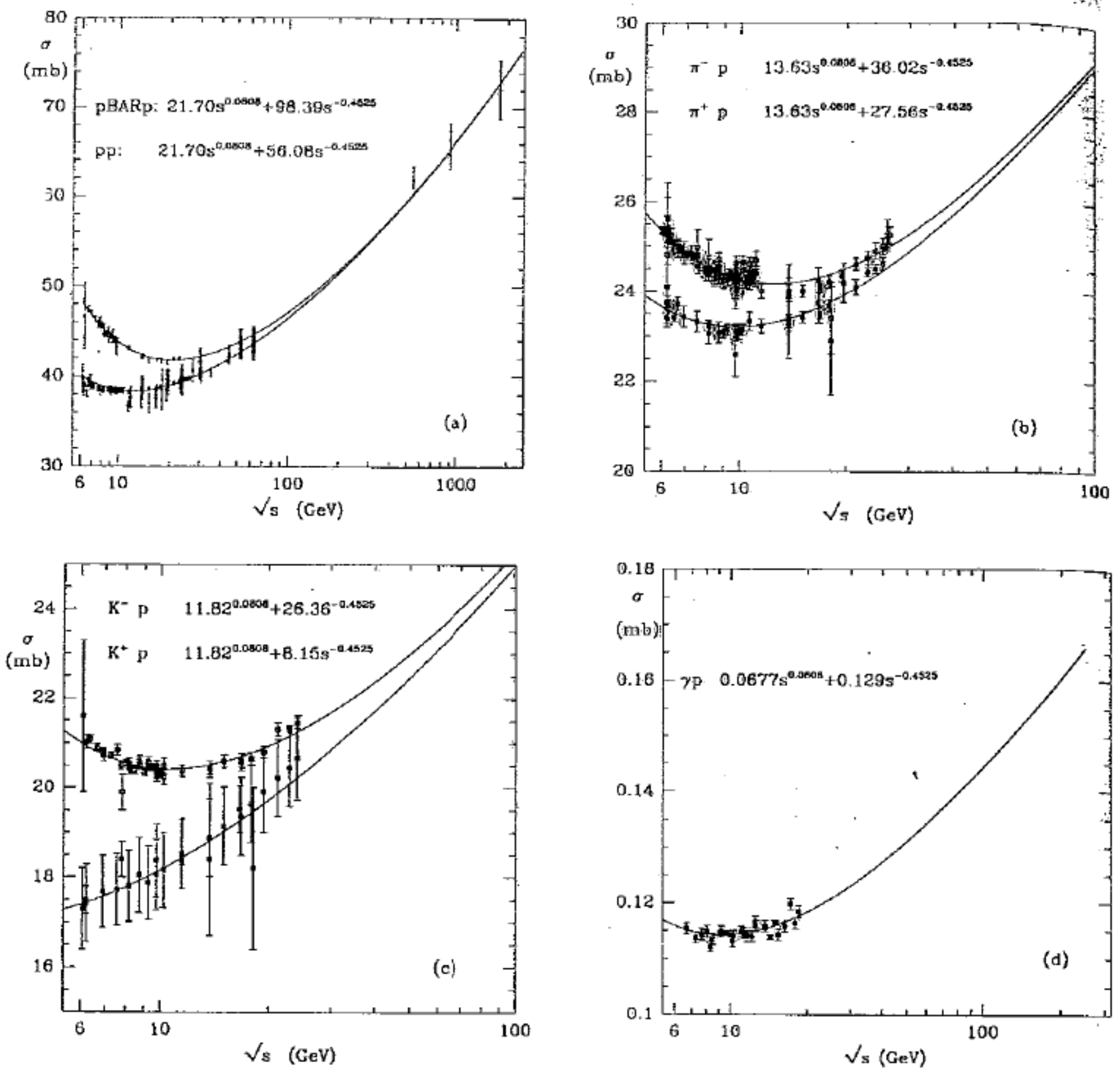


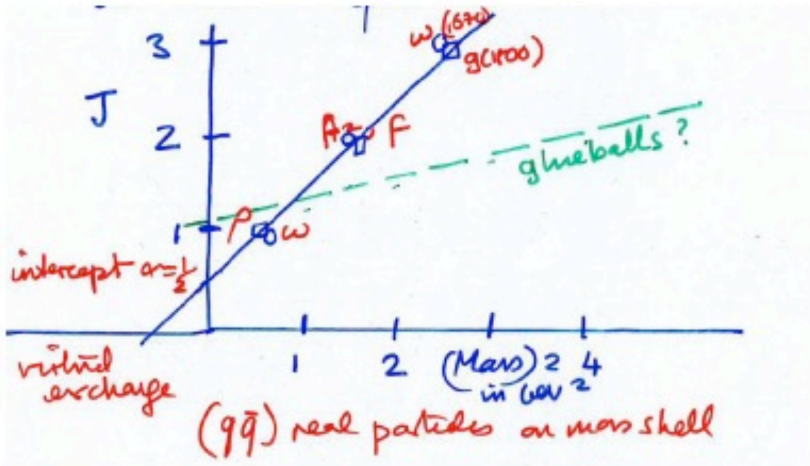
Fig. 1. Data for total cross sections with fits of type (2).

Regge and Pomeron trajectories

What is the Regge exchange?

In such reactions the exchange cannot be on mass shell – no real particle is exchanged. But, the virtual exchange may have the same quantum numbers as a series of particles.

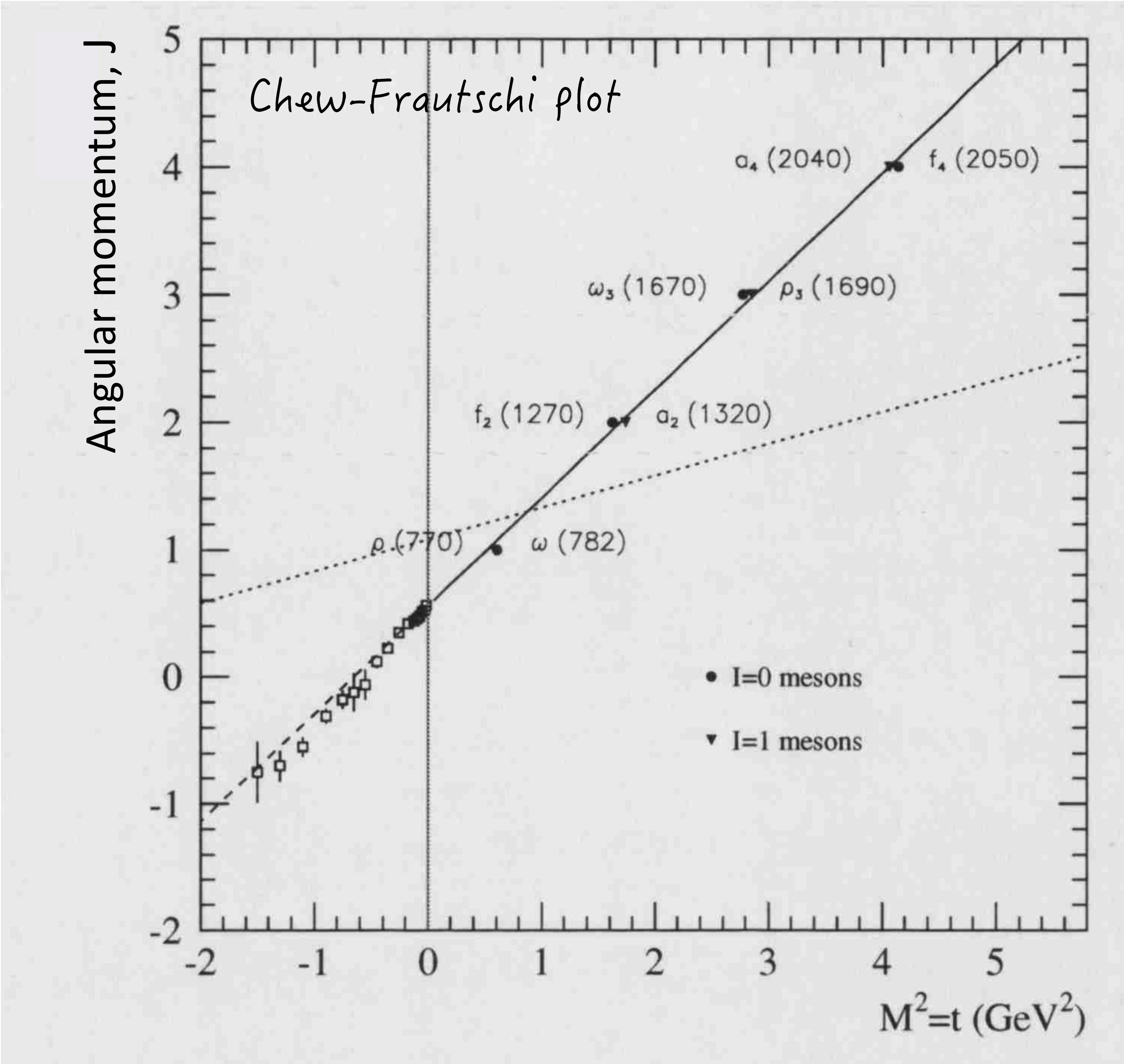
Plot J vs (mass)^2 for series of particles with the same quantum no's.



This is the Regge trajectory (in fact two degenerate Regge trajectories). It's intercept $\alpha \approx {}^1/_2$ is the parameter which controls the energy dependence, $s^{\alpha-1}$

The Pomeron Regge trajectory has $\alpha \approx 1.08$ hence $s^{0.08}$ slowly rising cross sections. May be associated with glueballs.

measurements



relation to low x PDFs

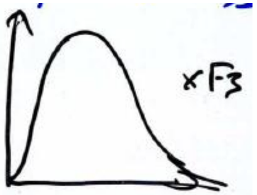
What has this to do with low x?

$$\sigma(V^* p \to X) \sim \sum_i s(V^* p)^{\alpha_i - 1} \propto x^{1-\alpha} \qquad x = \frac{Q^2}{2\,p\cdot q} = \frac{Q^2}{s(V^* p)}$$

What are the appropriate exchanges?

For the parts of the cross section related to $\boldsymbol{xF_3}$ it is the ρ/A_2 trajectory of intercept $\alpha = {}^1/_2$ this has $|u\,\bar{u} - d\,\bar{d}\rangle$ quark **flavour** and goes together with $\rho^{\pm}$, $A_2{}^{\pm}$, so it is 'non singlet' and hence associated to the valence part of the cross section.

$\Rightarrow\; x\,F_3 \sim x^{1-\alpha} \sim x^{1/2}$ as observed at moderate Q^2:



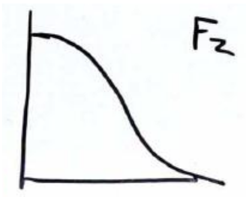
$$\Rightarrow x\,u_v,\; x\,d_v \;\sim\; x^{a_u},\; x^{b_u}$$

$$a_u \simeq b_u \simeq 0.5$$

For the parts of the cross section related to F_2 the Pomeron trajectory is more important than the ω/f because $\alpha_P > \alpha_{\omega,f}$

These trajectories have **no flavour** → **"singlet"**

$\Rightarrow\; F_2 \sim x^{1-\alpha} \sim x^{-0.08}$ Flattish as observed at moderate Q^2



⇒ low *x* behaviour of { gluons / singlet quarks } predicted as:

$$x\,S \sim x^{-\lambda_s} \quad \text{where } \lambda_s \simeq \lambda_g \simeq 0.08$$

$$x\,g \sim x^{-\lambda_g}$$

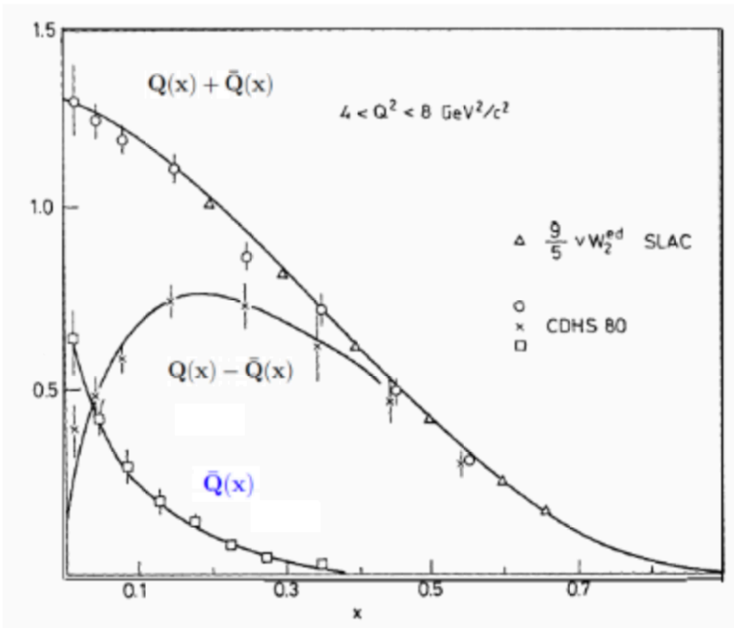
So we have some idea how to start parameterising PDFs

$$x\, q(x) \sim x^{\alpha} (1 - x)^{\beta}$$

And we know the powers will change with Q^2 but the change is perturbatively calculable – QCD

So, parameterise at Q_0^2 and use DGLAP equations to evolve to other Q^2 and then fit to data.

Be sure Q_0^2 is large enough that perturbation theory is valid.



J. Steinberger, Nobel Lecture 1988.

extras