ELECTROWEAK PHYSICS
Theory for the experimentalist

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CONTENTS

INTRODUCTION

The Standard Model of particle physics is naturally divided into the electroweak and strong interactions: each has a unique complication that affects both phenomenology and practical calculations. In the case of the strong interaction, it is the non-perturbative coupling at low momentum transfer that causes rich phenomena, which at present can only be described by heuristic models or intensive numerical calculations. In the case of the electroweak interaction, it is the breaking of the underlying symmetry that leads to a wide range of phenomena and complicates calculations. The electroweak interaction is the focus of this text.

The underlying symmetries of both sets of interactions arise from their descriptions as quantum gauge field theories. To gain a fundamental understanding of such theories, we proceed in the following steps: first, we discuss the geometrical construction of a classical gauge field theory (Chapter 1); second, we describe the quantization of the field and its physical interpretation (Chapter 2); third, we present the mechanism by which the gauge symmetry of the theory can be broken (Chapter 3); and finally, we provide the realization of these concepts in nature, the electroweak theory (Chapter 4).

After this introduction of the underlying principles of the theory, we turn to practical calculations. This begins with a discussion of path integrals and Feynman diagrams (Chapter 5), which are ubiquitous calculational tools that are indispensible in perturbative calculations. We follow with the application of these tools to calculations of scattering cross sections and particle widths (Chapter 6). At this point we can begin performing leading-order calculations of specific processes; however, the precision of electroweak measurements demands the inclusion of higher order calculations. Here we run into a fly in the ointment of quantum field theory: as soon as we go beyond leading order, we are faced with quantum contributions that produce unphysical results. This is resolved with a program of renormalization, which we describe in the context of both scalar and gauge field theories (Chapter 7).

The discussion of renormalization leads naturally into the specific measurements that fix the three parameters required to describe interactions of the electroweak gauge bosons with fermions (in the massless fermion approximation). The first parameter is taken to be the electromagnetic coupling, which describes the strength of the fermion-fermion-photon interaction and is measured using the magnetic moment of the electron (Chapter 8); this interaction is one of the elements of the renormalization program described in the prior chapter. The second parameter is the weak coupling, which describes the fermion-fermion-W boson interaction, and is indirectly determined using precise measurements of the muon lifetime (Chapter 9). The final parameter is the expectation value of the vacuum energy of the scalar field, which is indirectly determined using precise measurements of the Z boson mass (Chapter 10).

With the input gauge boson parameters in hand, we can now make a first-order prediction of the W boson mass, and demonstrate its sensitivity to other electroweak parameters through higher order corrections (Chapter 11). We then discuss the determination of the remaining parameters: the Higgs boson mass and flavour-diagonal Higgs-fermion-fermion Yukawa couplings (Chapter 12), measured through direct production of the Higgs boson; and off-diagonal Yukawa couplings manifested through the CKM matrix, measured through flavour-changing charged current processes such as B_s mixing (Chapter 13).

A completely open question is the nature of neutrinos and the corresponding parameters to describe their interactions. Their mixing can be parameterized with the PMNS matrix, akin to the CKM matrix, but we do not know the neutrino mass terms. These issues are touched on in Chapter 14.

CHAPTER 1

THE GEOMETRY OF FORCES

The development of the general theory of relativity was a conceptual breakthrough in understanding the force of gravity. Instead of the mysterious "action at a distance", gravity could be explained by the simple premise that all energy warps space and time. The curved trajectory of a particle in a gravitational field can be described as an unaltered path through a curved spacetime[1]; in mathematical terms, the path is a *geodesic*, the shortest distance from point a to point b.

It turns out that this concept can be extended to the strong and electroweak gauge forces, though here the curved space is internal. In this description, charged matter warps the internal space, causing the trajectories of other charged particles to curve in the projected spacetime. To describe this phenomenon, an additional layer of mathematics is required that combines spacetime with the internal space into a *fiber bundle*. This structure has more information than is physically accessible, however, and the internal space must be factored out in the end to account for the gauge independence of physical processes.

The detailed mathematical description of gauge forces is not generally provided in particle physics textbooks. It is included here for several reasons: first, it provides a valuable perspective in understanding the misnomer of "symmetry breaking"; second, it gives useful context for defining physically measurable quantities; and third, it describes the non-local

[1]One can visualize the effect as a bunching of space (or a stretching of time) near an object; space (time) is effectively more (less) dense in the vicinity of matter. A particle approaching matter will take less time to travel a given distance, accelerating towards the object. This effect is uniform for all matter, a crucial requirement for this intepretation.

aspects of the theory, which have applications in Dirac monopoles and the Aharanov-Bohm effect. A broader understanding of this geometrical construction could lead to more applications, in particular those that extend into the weak sector of the theory.

This chapter provides a heuristic overview of the mathematics required to understand both the curvature of spacetime and of internal spaces. The spacetime *manifold* is introduced and combined with internal spaces into a fiber bundle. To describe particle trajectories, several additional structures are required: the spacetime *metric tensor*, the spacetime and fiber *connections*, and their resulting *curvature*.

Armed with this structure, a trajectory is then described as an integrated path length over a curved space; this is the familiar action of physics. Choosing the shortest path leads to the gravitational and gauge force field equations. This minimization is modified in the quantum theory: the minimum path length corresponds to the peak of a probability amplitude, rather than the exclusive path of a particle. This will be discussed in more detail in the next chapter.

The above description applies to a single particle travelling through a background space. In a multi-particle description, a field is used to provide initial and final configurations of particles. The trajectory of a single particle is replaced with the evolution of the field from initial to final states. This chapter ends by extending the action of a single-particle trajectory to the action of a field configuration over space.

1.1 The fiber bundle of the universe

In a geometric description, the topological structure of the universe directly determines the forces of nature. In the Standard Model, the topology is four large “external” dimensions (the *spacetime manifold*) and three internal spaces (the *gauge fibers*), which are the sets of transformations corresponding to the U(1), SU(2), and SU(3) groups. Elements of these groups can be characterized by points on a circle, a three-dimensional spherical surface, and an eight-dimensional spherical surface [2], respectively. Historical attempts to elevate these group spaces to additional spacetime dimensions [4] failed[2], and instead they are described by the mathematical construct of fibers [3].

A manifold is a space that can be locally mapped onto a Euclidean space, allowing a coordinate system to describe points in the space. More formally, a manifold is defined as a family of open sets U_i and mappings ϕ_i of points in each open set onto R^n. The family of all possible open sets and mappings is called an *atlas*. In general more than one mapping, or coordinate system, is needed to cover the manifold. For example, consider the circle, S^1. One coordinate system ϕ_i maps points in an open subset to $0 < x < \pi$ in R^1, but misses a point on S^1. A second coordinate system ϕ_j is needed, for example one that maps points in U_j to $-\pi < y < \pi$. A requirement of a differentiable manifold is that the transformation of a point contained in both U_i and U_j from one coordinate system to the other, $\phi_i \phi_j^{-1}$, is differentiable.

Remark: Every manifold that is locally R^n is a submanifold of R^{2n} and can therefore be embedded in R^{2n}. Thus, a curve can be embedded in a plane, but a plane needs a four-dimensional space for an embedding.

[2]The difficulties include the inability to reproduce the observed particle mass spectrum and to incorporate fermion chiral interactions [5].

A *vector* can be defined as the tangent to a one-parameter curve $\gamma(t)$ (a map from an interval in R to the manifold) at a point p. The vector $d\gamma_p/dt$ can be expressed in terms of a coordinate system in R^n as $V^\mu \partial/\partial x^\mu$, with $V^\mu = dx^\mu/dt$. Acting on a function gives the directional derivative of the function. Such vectors are known as *contravariant vectors*. The space of all possible vectors at point p (i.e., the tangents to all possible curves) is the *tangent space* T_pM. A set of vectors at all points in an open set U is a *vector field*.

The concepts of tangent spaces and vector fields can be combined to create the union of tangent spaces at all points in the manifold (TM). Together with a map π associating a vector in TM to the point p in the manifold where the vector's tangent space resides, a tangent bundle is defined. The tangent bundle can be locally mapped to R^{2n}, where n coordinates map the location on the manifold and n coordinates map the direction of the vector.

The tangent bundle is an example of a more general mathematical construct, the *fiber bundle*, in which the fibers are manifolds rather than tangent spaces (which are a particular type of manifold). In the case of the tangent bundle, the tangent space T_pM is the fiber over point p, and M is the base manifold. Since locally the space can be mapped into coordinates of the base and the fiber, it is locally a product bundle $M \times T_pM$. This is true of a general fiber bundle B, which is locally $M \times F$, where F is the fiber manifold. A point in B can be mapped into M ($\pi : B \to M$) and a point on M can be brought into the fiber "preimages" F_p with $\pi^1(p)$. A region in the manifold can be mapped to a region in the bundle using a section $s : M \to B$.

Examples: A fiber bundle which can be globally expressed as $M \times F$ is a *global product bundle* (examples are the five-dimensional space $M^4 \times S^1$ or the cylinder $C^2 = R^1 S^1$). An example of a fiber bundle that is only locally a product bundle is the Möbius strip, which is locally $R^1 \times S^1$.

An important additional construct on the fiber bundle relates to coordinate transformations on the fiber. Two mappings in an overlap region can lead to different "coordinates"; the mapping from one to the other ($\phi_i \phi_j^1$) is an element of the *structure group* G, and is known as a transition function (g_{ij}). A fiber bundle is normally labelled by (B, M, F, G, π). In a *principle fiber bundle* a point in the fiber can be mapped locally to the structure group G.

Example: In the case of a tangent bundle the general linear structure group $GL(n, R)$ maps R^n onto itself.

The topology of the universe is a principal fiber bundle U containing the spacetime manifold M and fiber manifolds corresponding to the groups SU(3) $\times$ SU(2) $\times$ U(1) in a local region of spacetime. Since physical results do not depend on positions in the fiber space, but only differences between points, the physically measurable construction has a reduced structure. It is the product of two principle fiber bundles, divided by the structure group $[(P \times P)/G]$.

1.2 Spacetime metric

To describe dynamical processes we need the basic concept of distances and angles. This requires a mapping of vectors to real numbers, which then allows a metric that sets vector lengths and angles.

A mapping ω of a vector to R^1 is called a *covariant vector* (or a *one-form*). In a coordinate system defined on a patch of a manifold, a basis for covariant vectors dx^μ can be defined so that the covariant vector takes the form $\omega = \omega_\mu dx^\mu$. The space of all maps of vectors in a tangent space is called the cotangent space, $T_p * M$. The concept of vectors can be extended to tensors: a tensor of type (q, r) has q contravariant vector components and r covariant vector components. The vector V is a tensor of type $(1, 0)$, and ω is a tensor of type $(0, 1)$.

The infinitesimal distance along a curve $d\gamma/dt$ is to first order the length of the vector V at the point p multiplied by $dt = \epsilon$ along the curve. The notion of the length of a vector is thought of in terms of its magnitude; formally we create a $(2, 0)$ tensor and use a $(0, 2)$ tensor to map it to a real number (i.e., an inner product). The square root of this number is the vector's length. In the case of a distance along a curve, the $(0, 2)$ tensor is the metric g and the integral of $\sqrt{g}$ over a curve gives the length of the curve. Frequently the distance is written as ds and, in terms of coordinate components,

$$ds^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu. \tag{1.1}$$

More rigorously, the metric at a given point p on a manifold can be expressed in a coordinate system as

$$g(p) = \frac{1}{2} g_{\mu\nu}(p)[dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu]. \tag{1.2}$$

The inverse $(2, 0)$ tensor g^{-1} can be similarly defined in terms of a coordinate system with a contravariant basis. The components of a vector have the units of the coordinate system, so the spacetime metric components have no units. The metric of a flat spacetime is the familiar diagonal Minkowski metric; in general it is possible to diagonalize the metric. Because the metric is not positive-definite, the spacetime manifold is *pseudo-riemannian*.

Historical interlude: In the early twentieth century, Kaluza and Klein discovered that a metric can be written in the five-dimensional spacetime $M^4 \times S^1$ with a component that can be interpreted as the electromagnetic field; the equations of general relatively then contain the electromagnetic field equations.

In the five-dimensional spacetime the ground state corresponds to the flat diagonal metric $g^0_{AB} = (\eta_{\mu\nu}, -\Phi_0)$, where $\eta_{\mu\nu} = (1, -1, -1, -1)$ and $\Phi_0 = R^2$, with R the radius of the circle S^1. Expansions about this ground state produce a metric

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} - B_\mu B_\nu \Phi & B_\mu \Phi \\ B_\nu \Phi & -\Phi \end{pmatrix}, \tag{1.3}$$

where all terms are generally functions of x^μ and $x^5 \equiv \theta$. The metric is not diagonal and is thus referred to as a *non-coordinate* metric. It is straightforward to calculate the inverse of the metric,

$$g^{AB} = \begin{pmatrix} g^{\mu\nu} & B_\nu g^{\mu\nu} \\ B_\mu g^{\mu\nu} & -\Phi^{-1} + g^{\mu\nu} B_\mu B_\nu \end{pmatrix}. \tag{1.4}$$

A general coordinate transformation along the fifth dimension leads to the usual gauge transformation. To see this, write $B_\mu = \xi A_\mu$ and fix the S^1 metric to its ground state $\Phi = \Phi_0$. Consider the transformation of the off-diagonal term $g_{\mu\theta}$ under a translation $\theta = \theta' - \xi\epsilon(x)$. Then $\partial\theta/\partial x'^{\mu'} = -\xi\delta^\mu_{\mu'}\partial_\mu\epsilon(x)$ (since $x'^{\mu'} = x^\mu$), and

$$g'_{\mu'5} = g_{\mu 5} \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial\theta}{\partial\theta'} + g_{5\mu} \frac{\partial\theta}{\partial x'^{\mu'}} \frac{\partial x^\mu}{\partial\theta'} + g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial\theta'} + g_{55} \frac{\partial\theta}{\partial x'^{\mu'}} \frac{\partial\theta}{\partial\theta'}, \tag{1.5}$$

giving

$$B'_{\mu'}\Phi_0 = \delta^{\mu}_{\mu'} B_\mu \Phi_0 + \delta^{\mu}_{\mu'} \Phi_0 \xi \partial_\mu \epsilon(x). \tag{1.6}$$

Writing $B_\mu = \xi A_\mu$, we get $A'_\mu = A_\mu + \partial_\mu \epsilon(x)$. Thus, a coordinate transformation in the fifth dimension corresponds to a gauge transformation of the "field" that is the deviation from the ground state in this dimension. This observation was made in an attempt to unify gravity and electromagnetism, but problems arose in the predictions of the masses and charges of the particles. This has led to the description of the universe not as a five-dimensional spacetime, but rather a fiber bundle consisting of a four-dimensional manifold and group fibers.

1.3 Connections

While the metric definition provides a method for determining the inner product of vectors, it does not directly provide a comparison of vectors at different points in the manifold. This is because the metric can change from point to point. To compare vectors, one needs to transport them to a common point with a common coordinate system. This is done by *parallel transport*. Parallel transport keeps a vector at a constant angle with respect to the tangent of the trajectory. The difference between vectors $V'_\mu(x')$ and $V_\mu(x)$ at a common point x will have two components, one from the transport $[V'_\mu(x') - V'_\mu(x)]$ and the other from the intrinsic difference $[V'_\mu(x) - V_\mu(x)]$. In flat space there will be no difference arising from the transport. In curvilinear space there will be a difference

$$V'_\mu(x') - V'_\mu(x) = \Gamma^\nu_{\mu\lambda} V_\nu dx^\lambda. \tag{1.7}$$

Combining the two pieces, the total difference in the vectors is

$$DV_\mu = (\partial_\lambda V_\mu - \Gamma^\nu_{\mu\lambda} V_\nu) dx^\lambda. \tag{1.8}$$

The right-hand side of the equation makes it is clear that the covariant derivative is a $(0,1)$ tensor (i.e. a covariant vector) with a chosen coordinate basis dx^λ. The covariant derivative is along a particular direction. The parallel transport of a vector corresponds to $DV^\mu = 0$; when the direction of the covariant derivative coincides with the direction of the vector, the vector is transported along a geodesic curve. The linear coefficient $\Gamma^\nu_{\mu\lambda}$ is known as the connection and is clearly dependent on the covariant coordinate basis (it is therefore not a tensor). The connection can be written in terms of the metric as

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2} g^{\nu\kappa} [\partial_\mu g_{\kappa\lambda} + \partial_\lambda g_{\kappa\mu} - \partial_\kappa g_{\mu\lambda}]. \tag{1.9}$$

Extending the concept of parallel transport to a fiber bundle proceeds as follows. Consider a curve γ in M that is "lifted" with a section σ to $\tilde{\gamma}$ in P such that its tangent vector is purely along M; this is known as the "horizontal lift" of γ. Moving along this vector moves from one fiber to another, with no change in the group position if the fiber space is "flat". One can separate a general vector in the fiber space in terms of a horizontal vector in the direction of another fiber, and a vertical vector in the direction along a fiber. In a general fiber space, parallel translating an object in group space along the horizontal lift can change its group position:

$$\frac{dg(t)}{dt} = \omega(X) g(t), \tag{1.10}$$

where X is the tangent vector to the curve in M and g is determined by a particular section mapping points in M to points in P. Different horizontal lifts are possible, bringing the curve to different points on the fiber. The connection on the fiber is ω analogous to the connection in the spacetime manifold. The connection has a direction in spacetime along a curve, as in the case of the connection in spacetime, and is not a tensor. It is also specific to a particular point in the group space (i.e. specific to the lift). To go to a different point in group space, one can either use a transition function on the fiber, or change to a different horizontal lift, or section. In this case the connection transforms as

$$\omega' = g^{-1}\omega g + g^{-1}dg; \tag{1.11}$$

In both cases the connection is a horizontal lift, so its propagation along fibers does not change. In the geometrical description of the gauge forces, ω is related to the gauge potential A; for the electromagnetic force the relation is $\omega = ieA$ (for charge $q = e$). A particular choice of lift corresponds to a choice of gauge. Consequently, a change in the lift corresponds to a "gauge transformation." The presence of a gauge symmetry in nature indicates the independence of spacetime vectors to the position of the lifted vector in group space.

In the case of electromagnetism, with a structure group U(1), the transition function between two points in group space is $g_{ji}(x) = e^{-i\phi(x)}$ at a point x on the base space. The gauge transformation law for A is therefore (in a coordinate system):

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\phi(x), \tag{1.12}$$

where A corresponds to the lift i and A' corresponds to the lift j.

With a connection in the group space we can define parallel transport and the covariant derivative. The last piece we need is a representation of the group space, for the connection, and a corresponding vector space for the object to be translated. This additional structure provides an *associated vector bundle* to the principle fiber bundle. Then we can define a covariant derivative on an object ψ in the group vector space,

$$D\psi^i = \partial_\mu\psi^i + \omega^i_{\mu j}\psi^j, \tag{1.13}$$

in a particular coordinate system in spacetime (represented by μ) and in the group vector space (represented by i, j). In the case of electromagnetism the equation is $D\psi = \partial_\mu\psi - ieA_\mu$. Setting $D\psi = 0$ corresponds to parallel translating ψ along a lifted curve in the associated vector bundle.

1.4 Curvature

The curvature of a space can be quantified in terms of the change of a vector when transported around a closed path. As an example, for the parallelogram shown in Fig. 1.1 the change in the vector V^μ is

$$\Delta V^\mu = R^\mu_{\nu\alpha\beta}V^\nu\sigma^{\alpha\beta}, \tag{1.14}$$

where $\sigma^{\alpha\beta} = a^\alpha b^\beta$ is the area enclosed by the path and the multiplicative factor $R^\nu_{\mu\alpha\beta}$ is the curvature tensor

$$R^\mu_{\nu\alpha\beta} = \partial_\alpha\Gamma^\mu_{\nu\beta} - \partial_\beta\Gamma^\mu_{\nu\alpha} + \Gamma^\lambda_{\nu\beta}\Gamma^\mu_{\lambda\alpha} - \Gamma^\lambda_{\nu\alpha}\Gamma^\mu_{\lambda\beta}. \tag{1.15}$$

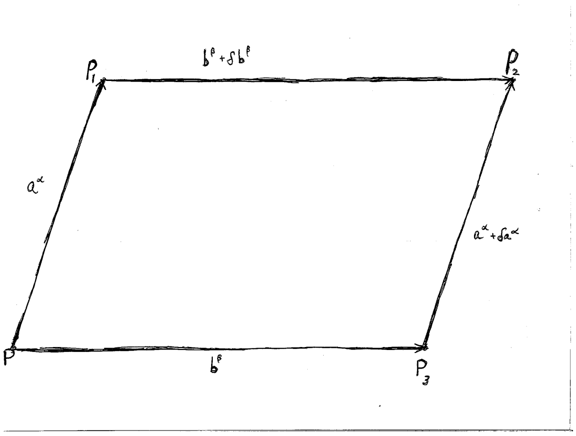


Figure 1.1 A parallelogram defined by vectors a^α and b^β at point P, with parallel sides given by their transported vectors at points P_1 and P_3.

Because of the symmetries of the curvature tensor, there is only one tensor that can be constructed by index contraction: the Ricci tensor $R_{\nu\beta} = g^\alpha_\mu R^\mu_{\nu\alpha\beta}$, whose trace $R = g^{\nu\beta} R_{\nu\beta}$ is the curvature scalar.

A vector in the fiber space also changes as it is transported around a closed curve horizontally lifted into the product bundle P:

$$\Delta\psi^i = F^i_{j\mu\nu}\psi^j\sigma^{\mu\nu}. \tag{1.16}$$

The curvature tensor is again related to the connection via:

$$F^i_{j\mu\nu} = \partial_\mu A^i_{j\nu} - \partial_\nu A^i_{j\mu} + [A_\mu, A_\nu]^i_j. \tag{1.17}$$

1.5 Principle of least action

Classically, the equations of motion for a particle of matter can be derived from the principle of least action, i.e. the requirement that the action is minimized. In general relativity, the action associated with spacetime is the integral of the scalar curvature:

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}d^4x. \tag{1.18}$$

The action of the fiber bundle is not a straightforward extension since the curvature scalar was defined by applying a metric to the curvature tensor, but there is no metric on the fibers. Instead the contraction takes place with another curvature tensor, giving an action that is quadratic in the fiber curvature. Including this action and that of matter (expressed as the Lagrangian $\mathcal{L}_M$) gives

$$S = \int\left(\frac{R}{16\pi G} - \frac{F^2}{4} + \mathcal{L}_M\right)\sqrt{-g}d^4x. \tag{1.19}$$

The classical equations of motion result from the minimization of the action with respect to variations of the metric tensor or the fiber connection:

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = 0, \tag{1.20}$$

$$\frac{\delta S}{\delta A^{\mu}} = \partial_\nu F^{\mu\nu} = 0. \tag{1.21}$$

Historical interlude (II): In a five-dimensional theory, the effective action in four dimensions can be derived using the five-dimensional action and integrating over the fifth dimension. The action in five dimensions is

$$I_5 = -\frac{1}{16\pi G_5}\int d^5x|\det g_5|^{1/2}R_5, \tag{1.22}$$

where G_5 is the gravitational constant in five dimensions, R_5 is the five-dimensional curvature scalar, and g_5 is the five-dimensional metric. In terms of the Ricci tensor, the curvature is

$$R_5 = g^{AB}R_{AB}. \tag{1.23}$$

The Ricci tensor is defined as

$$\begin{aligned} R_{BD} &= g^C_A R^A_{BCD} && (1.24)\\ &= g^C_A(\partial_C\Gamma^A_{BD} - \partial_B\Gamma^A_{CD} + \Gamma^A_{CE}\Gamma^E_{BD} - \Gamma^A_{BE}\Gamma^E_{CD}) && (1.25)\end{aligned}$$

in terms of the connection

$$\Gamma^C_{AB} = \frac{1}{2}g^{CD}(\partial_A g_{DB} + \partial_B g_{DA} - \partial_D g_{AB}). \tag{1.26}$$

Consider once again the metric expansion about the ground state [Eq. (1.3)]. Terms with ∂_5 and $\partial_\mu g_{55}$ are zero. The Ricci tensor with only θ components is $g^\nu_\beta R^\beta_{5\nu5}$, or

$$R_{55} = g^\nu_\beta(\partial_\nu\Gamma^\beta_{55} + \Gamma^\lambda_{55}\Gamma^\beta_{\lambda\nu} + \Gamma^5_{55}\Gamma^\beta_{\nu5} - \Gamma^\lambda_{5\nu}\Gamma^\beta_{\lambda5} - \Gamma^5_{5\nu}\Gamma^\nu_{55}) + ... \tag{1.27}$$

The additional terms are zero; the only non-zero term is the one with connections of the form

$$\Gamma^\nu_{5\mu} = \frac{1}{2}g^{\nu\lambda}(\partial_\mu g_{\lambda5} - \partial_\lambda g_{5\mu}). \tag{1.28}$$

The curvature from the $g^{55}R_{55}$ term becomes

$$\begin{aligned} g^{55}R_{55} &= -\frac{1}{4}g^{55}g^\nu_\beta g^{\lambda\mu}g^{\beta\rho}(\partial_\nu g_{\mu5} - \partial_\mu g_{5\nu})(\partial_\lambda g_{\rho5} - \partial_\rho g_{\lambda5})\\ &= -\frac{1}{4}g^{55}(\partial_\nu B_\mu\Phi - \partial_\mu B_\nu\Phi)(\partial_\mu B_\nu\Phi - \partial_\nu B_\mu\Phi)\\ &= -\frac{\Phi\xi^2}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\Phi^2\xi^2}{4}B_\alpha B^\alpha F_{\mu\nu}F^{\mu\nu}. && (1.29)\end{aligned}$$

The first term contributes to the action, while the second term is cancelled when adding to $g^{5\rho}R_{5\rho}$. It turns out that there is another term in $g^{\mu\nu}R_{\mu\nu}$ equal to $\frac{\Phi\xi^2}{2}F_{\mu\nu}F^{\mu\nu}$. Adding this term gives $R_5 = R_4 + \Phi\xi^2F^2/4$, resulting in the action

$$I_5 = -\frac{1}{16\pi G_5}\int d^5x|\det g_5|^{1/2}(R_4 + \Phi\xi^2F^2/4). \tag{1.30}$$

Integrating over θ gives 2π; we can also pull out $\sqrt{g_{55}} = R$ from the determinant to leave $|\det g_4|$. Writing $G_4 = G_5/(2\pi R)$ gives us the four-dimensional gravitational term we are looking for. Then, recalling that we are considering the ground state $\Phi = g_{55} = R^2$, we can obtain the desired electromagnetic term by setting $\xi^2 = 16\pi G_4/R^2$. We are left with the four-dimensional effective action,

$$I_4 = -\frac{1}{16\pi G_4}\int d^4x|\det g_4|^{1/2}R_4 - \frac{1}{4}\int d^4x|\det g_4|^{1/2}F_{\mu\nu}F^{\mu\nu}. \tag{1.31}$$

We can calculate the mass of a charged scalar field, which has a Fourier expansion

$$\phi(x,\theta) = \sum_{n=-\infty}^{\infty}\phi^n(x)e^{in\theta} \tag{1.32}$$

and satisfies the five-dimensional Klein-Gordon equation,

$$\left(\partial_\mu\partial^\mu - \frac{\partial^2}{R^2\partial\theta^2}\right)\phi = 0. \tag{1.33}$$

Writing the Klein-Gordon equation as

$$(\partial_\mu\partial^\mu + m_n^2)\phi^n(x) = 0, \tag{1.34}$$

we see that $m_n^2 = n^2/R^2$, defining the mass of a given Fourier mode. A translation of the extra dimension leads to the following change in mode n:

$$\phi^n(x) \to e^{in\xi\epsilon}\phi^n(x). \tag{1.35}$$

Equating the charge with $-n\xi$ leads to the following expression for R for $n = \pm 1$:

$$R = \sqrt{16\pi G_4}/\xi. \tag{1.36}$$

If we take U(1) to be that of QED and the unit charge to be $e/3 = \sqrt{4\pi\alpha_{EM}}/3$, the charge ± 1 particles have a mass

$$m = R^{-1} = 1.7 \times 10^{17}\ \text{GeV}, \tag{1.37}$$

while the chargeless mode will be massless. The large masses of the charged particles is a weakness of the theory.

1.6 Conservation laws

Noether's theorem states that any transformation that leaves the Lagrangian invariant results in a conserved charge. Invariance with respect to a translation in spacetime leads to the conservation of energy and momentum,

$$\partial_\mu T^{\mu\nu} = 0. \tag{1.38}$$

A transformation of coordinates in gauge group space leads to the conservation of charge,

$$\partial_\mu J^\mu = 0. \tag{1.39}$$

CHAPTER 2

PATH INTEGRALS AND FIELDS

The classical description of forces is valid in systems with large action ($S \gg \hbar$), characterized by high particle multiplicity where the effects of individual fluctuations are suppressed. In this limit, a force can be described by a smoothly curved spacetime or fiber, resulting in the acceleration of particles following a geodesic. To describe forces at a fundamental level, individual particle interactions must be considered. Quantum mechanically, a force is transmitted by the exchange of a quantized spacetime tensor particle (for gravity) or vector particle (for the other forces). Mathematically, the gravitational interaction is a quantized modification of the metric that shortens the distance between the particles, like pulling two points of a rubber sheet toward each other. The gauge interaction is a quantized modification of the connection that changes the trajectories, or momenta, or the particles. The physical description is the exchange of a graviton or gauge boson. The probability amplitude for any such exchange can be calculated, and the probability for any given interaction is the square of the sum of all possible amplitudes.

We thus determine the probability of an interaction by calculating the transition rate from the initial state to the final state. This transition rate depends on all possible interactions that produce the final state, given the initial state. In particular, it depends on the exchange of not just a single particle, but of multiple particles. Calculating such exchanges is non-trivial, but can be compactly summarized by a phase-weighted integral over all possible intermediate states. This integral is known as the path integral [8].

2.1 Non-relativistic path integral

To become familiar with the path integral approach, it is useful to first consider a non-relativistic transition of a particle from an initial state $|q,t\rangle$ to a final state $\langle q',t'|$. The Schrödinger equation provides the time dependence of the state, $|q,t\rangle = e^{iHt}|q\rangle$, so

$$\langle q',t'|q,t\rangle = \langle q'|e^{-iH(t'-t)}|q\rangle. \tag{2.1}$$

The amplitude can be split into short steps in time; considering each possible intermediate state q_i at time t_i gives

$$\langle q',t'|q,t\rangle = \int dq_1...dq_n \langle q',t'|q_n,t_n\rangle...\langle q_1,t_1|q,t\rangle. \tag{2.2}$$

The integrals reflect the fact that all possible intermediate states must be considered, since the states form a complete orthonormal basis. One can say that the amplitude integrates all possible paths between states, hence the name "path integral". Each intermediate state has a weight determined by the Hamiltonian.

To derive an expression for the integrals over intermediate states, consider the transition amplitude for an individual step $\tau = t_{i+1} - t_i$:

$$\langle q_{i+1},t_{i+1}|q_i,t_i\rangle \approx \langle q_{i+1}|1 - iH\tau|q_i\rangle, \tag{2.3}$$

where the approximation arises from ignoring higher order terms in $H\tau$. We now order the Hamiltonian such that position operators are symmetric about momentum operators and can act on the position states to give $f(\frac{q_{i+1}+q_i}{2})$ and leave a function of momentum operators $F(\hat{p})$ to act on the position states. This process is known as *Weyl ordering* and can be performed for any general Hamiltonian. We then insert a complete set of states in momentum space to get:

$$\langle q_{i+1},t_{i+1}|q_i,t_i\rangle \approx \int \frac{dp_i}{2\pi} e^{i[p_i(q_{i+1}-q_i)-H\tau]}, \tag{2.4}$$

where the first term in the exponential arises from switching from position to momentum bases, i.e. $\langle p|q\rangle = e^{ipq}$, and terms quadratic and higher in $H\tau$ are again assumed to be negligible. If we now consider the time steps to be infinitesimal, we obtain the path integral form for the transition amplitude:

$$\langle q',t'|q,t\rangle = \int \frac{\mathcal{D}q\mathcal{D}p}{2\pi} e^{i\int_t^{t'} dt[p\dot{q}-H(p,q)]}, \tag{2.5}$$

where $\mathcal{D}q$ and $\mathcal{D}p$ indicate that the integrands are functions of time.

For the non-relativistic Hamiltonian $H = p^2/2m + V(q)$, the integral over momentum functions can be performed. Making use of the relation

$$\int_{-\infty}^{\infty} e^{(-ax^2+bx+c)}dx = e^{\left(\frac{b^2}{4a}+c\right)} \left(\frac{\pi}{a}\right)^{1/2}, \tag{2.6}$$

the path integral becomes

$$\langle q',t'|q,t\rangle = N \int \mathcal{D}q \; e^{i\int_t^{t'} dt L(q,\dot{q})}, \tag{2.7}$$

where $N = \lim(n \to \infty)[m/(2\pi i\tau)]^{(n+1)/2}$ is a normalization factor and $L = m\dot{q}^2/2 - V(q)$ is the classical Lagrangian. This is just the integral over all possible paths, with a phase determined by the action of each path. The phases will cause a cancellation in amplitudes except near the minimum of the action, where "near" is defined in units of $\hbar$. The limit $\hbar \to 0$ reproduces the classical equations of motion.

2.2 Perturbation theory

For a general potential $V(q)$, the transition amplitude is not analytically calculable. If the potential is small, one can perform a perturbative expansion:

$$e^{-i\int_t^{t'} V(q,t_1)dt_1} = 1 - i\int_t^{t'} V(q,t_1)dt_1 + \frac{1}{2}\left[-i\int_t^{t'} V(q,t_1)dt_1\right]^2 + \ldots \tag{2.8}$$

Inserting the first term into the transition amplitude gives an integral over the free-particle Lagrangian, which is quadratic in $\dot{q}$. Splitting this into infinitesimal integrals over the path, and making use of the general result

$$\int_{-\infty}^{\infty} e^{-a(x'-x_i)^2} e^{-b(x_i-x)^2} dx_i = \sqrt{\frac{\pi}{a+b}}\, e^{-\frac{ab}{a+b}(x'-x)^2}, \tag{2.9}$$

the first step in the integral is

$$\frac{m}{2\pi i\tau}\int_{\infty}^{\infty} dx_1 e^{im(x_2-x_1)^2/(2\tau)} e^{im(x_1-x)^2/(2\tau)} = \sqrt{\frac{m}{2\pi i(2\tau)}}\, e^{im(x_2-x)^2/[2(2\tau)]}. \tag{2.10}$$

In the next step $2\tau \to 3\tau$, so after N steps we have $(N+1)\tau = t' - t$. This leads to the following expression for the free-particle transition function:

$$\langle q', t'|q, t\rangle_0 = \theta(t'-t)\sqrt{\frac{m}{2\pi i(t'-t)}}\, e^{im(q'-q)^2/[2(t'-t)]}. \tag{2.11}$$

The second term in the perturbative expansion contains a factor $e^{im\dot{q}^2/2}V(q,t)$ in the integral. To evaluate this we separate the free-particle factor into times before and after the application of the potential:

$$\langle q', t'|q, t\rangle_1 = -i\int_{-\infty}^{\infty} dt_1 \int dq_1 \langle q', t'|q_1, t_1\rangle_0 V(q_1, t_1)\langle q_1, t_1|q, t\rangle_0. \tag{2.12}$$

A similar procedure can be performed for the remaining terms, resulting in the Born series shown pictorially in Fig. 2.1. The series represents a sum over the number of possible intermediate interactions with the potential $V(q,t)$. The pictorial representation is simply a combination of lines (propagators) and vertices (interactions) represented respectively by $\langle q', t'|q, t\rangle_0$ and $V(q,t)$.

Thus far only initial and final position eigenstates have been considered. One can also consider initial and final momentum eigenstates. Then the transition, or scattering, ampli-

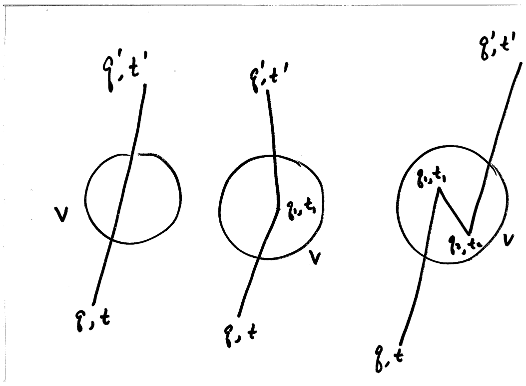


Figure 2.1 A pictorial representation of the non-relativistic perturbation series for the transition $\langle q', t'|q, t\rangle$ in the presence of potential $V(q, t)$.

tude is

$$\begin{aligned} S &= \langle p't'|pt\rangle \\ &= \int \langle p't'|q't'\rangle\langle q't'|qt\rangle\langle qt|pt\rangle dqdq' + ... \\ &= \delta(p'-p) + \\ &\quad -i\int \langle p't'|q't'\rangle\langle q't'|q_1t_1\rangle_0 V(q_1,t_1)\langle q_1t_1|qt\rangle_0\langle qt|pt\rangle dqdq'dq_1dt_1 + .. \end{aligned} \tag{2.13}$$

As before, the first term is simply the no-scattering case; additional terms describe the interactions. S is a matrix giving scattering amplitudes from p to p'.

2.2.1 Green's functions

The path integral approach is particularly useful for describing scattering experiments. One typically has initial and final states consisting of narrow gaussians of momentum and position eigenstates. The transition amplitudes are computed from Green's functions, which can be calculated using perturbative path-integral methods.

The two-point Green's function $G(t, t')$ is defined as the propagator between initial and final free-particle (ground) states. The propagation between an initial position to a final position can be represented by $\langle 0|\hat{q'}^\dagger(t')\hat{q}(t)|0\rangle$. We can relate the Green's function to the measurements of the particle's positions at times between those of the distant past and future:

$$\langle q', \infty|\hat{q}^\dagger(t')\hat{q}(t)|q, -\infty\rangle = \langle q', \infty|0\rangle\langle 0|\hat{q}^\dagger(t')\hat{q}(t)|0\rangle\langle 0|q, -\infty\rangle. \tag{2.14}$$

The expression on the right contains the Green's function and $\langle q', \infty|0\rangle\langle 0|q, -\infty\rangle$, which is $\langle q', \infty|q, -\infty\rangle$ since the particles are in the ground state in the infinite past and future. We can move this to the left-hand side to leave the Green's function on the right-hand side. Expressing the left-hand side in terms of a path integral, we have the following expression for the Green's function:

$$\begin{aligned} G(t,t') &= \frac{\langle q',\infty|\hat{q'}^{\dagger}(t')\hat{q}(t)|q,-\infty\rangle}{\langle q',\infty|q,-\infty\rangle} \quad (2.15) \\ &= \lim_{t_{(i)f}\to(-)\infty} \frac{1}{\langle q_f,t_f|q_i,t_i\rangle}\int \frac{\mathcal{D}q\mathcal{D}p}{2\pi} q(t')q(t)e^{i\int_{t_i}^{t_f} dt[p\dot{q}-H(p,q)+i\epsilon q^2/2]}. \end{aligned}$$

One can extend this to an $n-$point Green's function and express it more compactly by adding to the Lagrangian a "source" term Jq, which is removed by setting $J = 0$ at the end of the calculation:

$$G(t_1...t_n) = \frac{(-i)^n \delta^n Z[J]}{\delta J(t_1)...\delta J(t_n)}|_{J=0}, \quad (2.16)$$

where

$$Z[J] = \lim_{t_i\to-\infty, t_f\to\infty} \frac{1}{\langle q_f,t_f|q_i,t_i\rangle}\int \frac{\mathcal{D}q\mathcal{D}p}{2\pi} e^{i\int_{t_i}^{t_f} dt[p\dot{q}-H(p,q)+Jq+i\epsilon q^2/2]}. \quad (2.17)$$

The function $Z[J]$ is the analogue of the partition function of statistical mechanics, with the replacement $T \to it$. It represents the transition amplitude from the ground state at a time in the infinite past to a time in the infinite future. We have added a term $-i\epsilon q^2/2$ to the Hamiltonian in order to suppress the contributions from higher energy states at large values of absolute time. The integral over momentum functions has not been performed, since in general the integral does not lead to a path integral over the Lagrangian.

2.3 Path integral of a scalar field

To calculate a transition amplitude in relativistic quantum mechanics one has to consider not only all possible paths of a given particle, but all possible paths of all possible intermediate particles. To facilitate calculation, the wavefunction is elevated to an operator with a component that produces a particle when operating on the vacuum (the "creation" operator) and a component that eliminates a particle when operating on the vacuum (the "annihilation" operator). The wavefunction thus becomes a "field" operator whose excitations correspond to individual particles. Free scalar particles are represented by solutions to the Klein-Gordon equation,

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}[\hat{a}^{\dagger}(k)e^{-i(kx-\omega t)} + \hat{a}(k)e^{i(kx-\omega t)}], \quad (2.18)$$

where the annihilation and creation operators have the commutation relation

$$[\hat{a}(k), \hat{a}^{\dagger}(k')] = [(2\pi)^3 2\omega]^{1/2}\delta^3(k-k')$$

deriving from $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^3(x-y)$, where $\hat{\pi} = \partial\hat{\phi}/\partial t$ is the canonical field "momentum". A single-particle state is represented by $|k\rangle = [(2\pi)^3 2\omega]^{1/2}\hat{a}^{\dagger}(k)|0\rangle$.

Viewed from a path integral perspective, the relevant transition amplitude is no longer between initial and final positions or momenta of a single particle, but between initial and final field configurations. One can still consider single-particle initial and final states, but intermediate multiparticle states must also be included. The transition amplitude between initial and final field configurations of a single-particle scalar state is

$$\langle 0|\phi(x')\phi(x)|0\rangle = -\frac{\delta^2 Z}{\delta J(x)\delta J(x')}|_{J=0} \tag{2.19}$$

where

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x[\mathcal{L}(\phi)+J\phi+i\epsilon\phi^2/2]}. \tag{2.20}$$

This can be derived in a manner analogous to the single-particle function by slicing up both space and time into incremental steps and using a Lagrangian density for ϕ, e.g.

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - \frac{g}{4!}\phi^4. \tag{2.21}$$

In a field theory the ground state is the vacuum, so $Z[J]$ represents the vacuum-to-vacuum transition amplitude. It is also known as the generating functional.

2.3.1 Free-field transition amplitude

A succinct closed-form expression for the vacuum-to-vacuum transition amplitude can be derived in the case of a scalar free-field theory (i.e., the Lagrangian in equation 2.21 without the ϕ^4 term). Expressing $\partial_\mu\phi\partial^\mu\phi$ in terms of a total divergence, which vanishes as $x \to \infty$, the generating functional can be written

$$Z_0[J] = \int \mathcal{D}\phi e^{-i\int d^4x[\frac{1}{2}\phi(\partial_\mu\partial^\mu+m^2+i\epsilon)\phi - J\phi]}. \tag{2.22}$$

To perform this integration, express ϕ as the deviation from the field ϕ_0 that satisfies an effective Klein-Gordon equation:

$$(\partial_\mu\partial^\mu + m^2 - i\epsilon)\phi_0 = J. \tag{2.23}$$

Then the generating functional is

$$Z_0[J] = \int \mathcal{D}\phi e^{-i\int d^4x[\frac{1}{2}\phi(\partial_\mu\partial^\mu+m^2+i\epsilon)\phi - \frac{1}{2}J\phi_0]}. \tag{2.24}$$

This has two pieces: an integral over ϕ that doesn't contribute to the Green's functions, and a $\phi_0 J$ integral over space that does. Express ϕ_0 as

$$\phi_0 = -\int \Delta_F(x-y)J(y)d^4y, \tag{2.25}$$

where Δ_F satisfies

$$(\partial_\mu\partial^\mu + m^2 - i\epsilon)\Delta_F(x) = -\delta^4(x). \tag{2.26}$$

Now write the integral over ϕ as a constant N. To normalize the functional such that $Z_0[0] = 1$ we divide by N to obtain:

$$Z_0[J] = e^{-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)}. \tag{2.27}$$

From this functional it is straightforward to calculate the n-point Green's function:

$$G(x_1, ..., x_n) = \left(\frac{1}{i}\right)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]|_{J=0}. \tag{2.28}$$

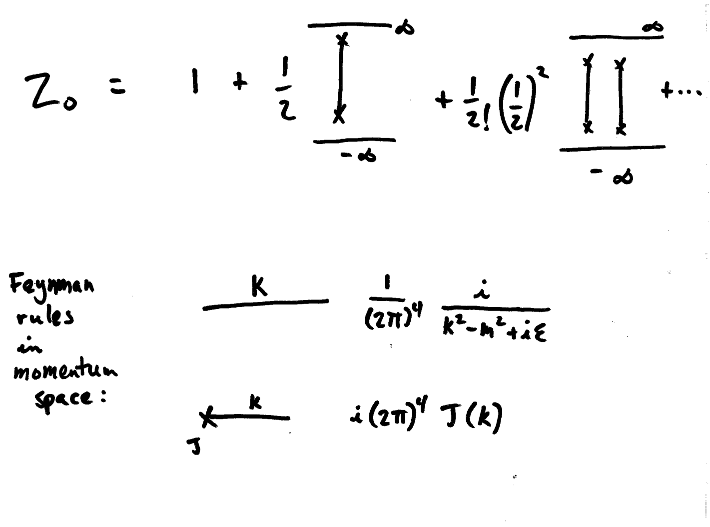


Figure 2.2 A pictorial representation of the vacuum-to-vacuum transition amplitude for the scalar free-field theory. Also shown are the "Feynman rules" for the source and the propagator.

The free-field generating functional thus gives a simple n-particle propagator: the 2-point Green's function corresponds to single-particle propagation, the 4-point Green's function corresponds to two-particle propagation, and so on. The n-th term in the expansion of the exponential gives the $2n$-point Green's function for n-particle propagation. The propagator satisfies the Klein-Gordon equation, with all other field configurations normalized away. It is straightforward to show that the propagator can be expressed as

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon}. \tag{2.29}$$

The expansion of the exponential in Z_0 is shown diagramatically in Fig. 2.2, along with the corresponding "Feynman rules".

2.3.2 Interacting-field transition amplitude

The generating functional for a self-interacting scalar field (Eq. 2.21) can be expressed in terms of the free-field functional $Z_0[J]$ as:

$$Z[J] = Ne^{i\int \frac{-g}{4!}(\frac{\delta}{i\delta J})^4 dx} Z_0[J]. \tag{2.30}$$

While this cannot be solved explicitly, it can be solved at each order in g. The zeroth-order term just gives the free-particle functional. The term linear in g is:

$$\begin{aligned} Z_g[J] &= N_g \frac{-ig}{4!}\int dz \left(\frac{\delta}{i\delta J(z)}\right)^4 e^{-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)} \\ &= N_g \frac{-ig}{4!}\int dz\{-3\left[\Delta_F(0)\right]^2 + 6i\Delta_F(0)\left[\int \Delta_F(z-x)J(x)dx\right]^2 + \\ &\quad [\Delta_F(z-x)J(x)dx]^4\} e^{-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)}. \end{aligned} \tag{2.31}$$

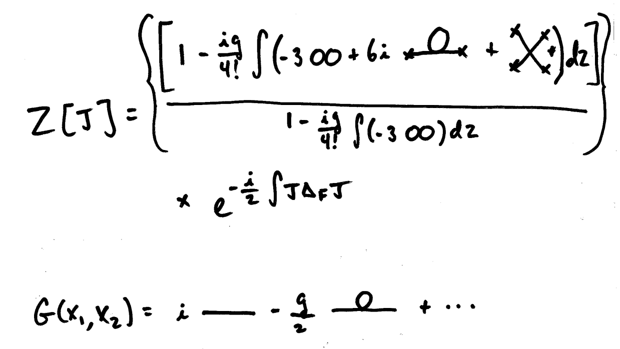


Figure 2.3 A pictorial representation of the vacuum-to-vacuum transition amplitude for the scalar interacting-field theory, to first order in g. Also shown is the first-order correction to the 2-point Green's function.

As in the free-field case, the normalization factor is chosen such that $Z[0] = 1$, and is just $Z^{-1}[J]|_{J=0}$. Expressing $\Delta_F(x-y)$ as a line and $\Delta_F(0)$ as a loop, the functional can be written as in Fig. 2.3 to first order in g. The $n-$point Green's functions are straightforward to derive; they simply pick out the factors with n factors of J. The 2-point function is shown in Fig. 2.3 to first order in g. In equation form it is

$$\begin{aligned} G(x_1,x_2) &= i\Delta_F(x_1-x_2) - \frac{g}{2}\Delta_F(0)\int dz \Delta_F(z-x_1)\Delta_F(z-x_2) + ... \\ &= \frac{i}{(2\pi)^4}\int d^4k \frac{e^{-ik(x_1-x_2)}}{k^2-m^2+i\epsilon}\left[1+\frac{ig\Delta_F(0)/2}{k^2-m^2+i\epsilon}+...\right] \\ &\approx \frac{i}{(2\pi)^4}\int d^4k \frac{e^{-ik(x_1-x_2)}}{k^2-m^2-ig\Delta_F(0)/2+i\epsilon}. \end{aligned} \tag{2.32}$$

The interaction has a rather remarkable effect: it produces loops in the propagator that shift the particle's mass. The result is that a particle's effective mass is not simply the mass term in the Lagrangian, but rather the combination of such a term with all the possible loops in the propagator. For an interacting field theory the shift at order g is $\delta m^2 = ig\Delta_F(0)/2$. By itself such a shift is problematic, since $\Delta_F(0)$ is quadratically divergent. This can be handled by a renormalization procedure that fixes the effective mass to the measured value and uses the mass term in the Lagrangian (the "bare" mass) to cancel the divergence.

2.4 Path integral of a fermion field

The propagator of a Dirac fermion field can be derived in a similar manner, with two important differences to a scalar field: (1) the field now has internal degrees of freedom, and (2) it satisfies the anticommutation relations $\{\hat{\psi}_\alpha(x), \hat{\psi}^\dagger_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y)$. The field operators can be expressed as

$$\psi(\hat{x}) = \sum_{s=1,2} \int \frac{d^3k}{(2\pi)^3 2E} \left[\hat{b}(k,s)u(k,s)e^{-ikx} + \hat{d}^\dagger(k,s)v(k,s)e^{ikx}\right] \tag{2.33}$$

and $\hat{\bar{\psi}} \equiv \hat{\psi}^\dagger \gamma^0$, where $\hat{b}^{(\dagger)}$ and $\hat{d}^{(\dagger)}$ are the annihilation (creation) operators for particles (u) and antiparticles (v). The operators satisfy the anticommutation relations $\{\hat{b}(k,s), \hat{b}^\dagger(k',s')\} = \{\hat{d}(k,s), \hat{d}^\dagger(k',s')\} = (2\pi)^3 2E\delta_{ss'}\delta(k-k')$.

The generating functional for a free fermion field is

$$Z_0(\bar{\eta}, \eta) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i\int d^4x[\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]}, \tag{2.34}$$

where η and $\bar{\eta}$ are source fields and $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. To extract the propagator, we write $\psi = \psi_0 + \psi'$ and $\bar{\psi} = \bar{\psi}_0 + \bar{\psi}'$, where

$$\begin{aligned} (i\partial\!\!\!/ - m)\psi_0 &= -\eta \\ \bar{\psi}_0(i\partial\!\!\!/ - m) &= -\bar{\eta}. \end{aligned} \tag{2.35}$$

The solution ψ_0 can be written as

$$\psi_0 = -\int d^4y \Delta_F(x-y)\eta(y), \tag{2.36}$$

where $(i\partial\!\!\!/ - m)\Delta_F(x-y) = \delta^4(x-y)$. Writing the delta function as

$$\delta^4(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}, \tag{2.37}$$

we find the fermion propagator,

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k\!\!\!/ - m + i\epsilon}\right) e^{-ik(x-y)}, \tag{2.38}$$

In terms of the Feynman propagator, the generating functional becomes

$$\begin{aligned} Z_0(\bar{\eta}, \eta) &= \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' e^{i\int d^4x[\bar{\psi}'(i\partial\!\!\!/ - m)\psi' + \frac{1}{2}\bar{\psi}_0\eta + \frac{1}{2}\bar{\eta}\psi_0]}. \\ &= e^{-i\int d^4x d^4y \bar{\eta}(x)\Delta_F(x-y)\eta(y)}, \end{aligned} \tag{2.39}$$

where the integral that does not depend on the source has been divided out in the normalization. The Feynman propagator can be derived from the generating functional:

$$i\Delta_F(x-y) = \left(\frac{\delta}{i\delta\bar{\eta}(x)}\right)\left(-\frac{\delta}{i\delta\eta(y)}\right) Z_0[\eta, \bar{\eta}]|_{\eta=\bar{\eta}=0}. \tag{2.40}$$

2.5 Path integral of a gauge field

Interactions between charged matter, i.e. between vectors in the gauge fiber, must include all intermediate fiber configurations. These fiber configurations must be classified according to their connections A_μ, since the covariant derivatives of the vectors depend explicitly on these connections. In a perturbative theory, an expansion of the configurations in increasing orders of the connection can be performed to provide a good approximation to the complete calculation. Thus, to first approximation an interaction can be thought of as the exchange of a single quantized connection, or a *gauge boson*.

Care must be taken when considering all intermediate fiber configurations. Two configurations are not distinct if they are simply translations along the fiber, i.e. if they are related by a gauge transformation. These configurations can be made explicitly equivalent by changing the coordinates of the fiber. Thus, when integrating over intermediate configurations, one must first fix the coordinate system. Within this coordinate system the connection is uniquely defined and can be quantized to allow an expansion of the intermediate states.

The coordinates can be set by inserting a delta function in the integration over the coordinate variable. Recall that a general gauge transformation takes the form:

$$[\tau^a A'^a_\mu]^j_m = \left[e^{-i\tau^a\theta^a(x)}\right]^j_k [\tau^b A^b_\mu]^k_l \left[e^{i\tau^c\theta^c(x)}\right]^l_m - \frac{i}{g}\left[e^{i\tau^a\theta^a(x)}\right]^j_k \left[\partial_\mu e^{-i\tau^b\theta^b(x)}\right]^k_m . \tag{2.41}$$

For small rotations this reduces to the following transformations of connection coefficents in the basis τ^a:

$$A'^a_\mu = A^a_\mu + f^a_{bc}\theta^b A^c_\mu - \frac{1}{g}\partial_\mu\theta^a \tag{2.42}$$

where $f^a_{bc}\tau_c = [\tau^a, \tau_b]$ and indices for the group elements have been dropped. Now choose a gauge using functions of the connection coefficients, $f^i(A_\mu) = 0$, and integrate over the gauge transformation coefficients θ^j for each basis vector:

$$\int [d\theta^j(x)]\delta[f_i(A^i_\mu)] = \left[\det\left(\frac{\partial f^i}{\partial\theta^j}\right)\right]^{-1}. \tag{2.43}$$

Now, to do the path integral, we include the delta function that chooses a particular coordinate system and divide out the extra volume factor that arises from this integration. The generating functional is then

$$Z[J] = \int \mathcal{D}A^j_\mu \det\left(\frac{\partial f_i}{\partial\theta_j}\right)\delta[f^i(A_\mu)]e^{i\int d^4x[\mathcal{L}+J^\mu A_\mu]}. \tag{2.44}$$

This procedure of fixing the coordinate system is known as the Faddeev-Popov ansatz.

To perform calculations it is useful to recast the additional factors as extra terms in the Lagrangian. The determinant can be written as

$$\det\left(\frac{\partial f^i}{\partial\theta^j}\right) = \int \mathcal{D}\bar\eta\mathcal{D}\eta e^{-i\int\bar\eta^i M^{ij}\eta^j dx}, \tag{2.45}$$

where $M^{ij} = \frac{\partial f^i}{\partial\theta^j}$ and the new fields $\bar\eta^i$ and η^j are non-physical fermion "fields", typically referred to as "ghost" fields. The delta function can also be expressed as an exponential by writing it as $\delta[f_i(A_\mu) - B(x)]$ and multiplying by a constant integration $\int\mathcal{D}B\exp[-i/(2\alpha)\int d^4xB^2(x)]$, which just affects the normalization. Then the integral over $\mathcal{D}B$ replaces B with f_i (due to the delta function) and the generating functional becomes

$$Z[J] = N\int\mathcal{D}A_\mu\mathcal{D}\bar\eta\mathcal{D}\eta e^{-i\int\mathcal{L}_{\text{eff}}d^4x}, \tag{2.46}$$

where

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \mathcal{L} - \frac{f_i f^i}{2\alpha} - \bar{\eta}^i M^{ij} \eta^j \\ &= \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{FPG}\end{aligned} \tag{2.47}$$

(GF refers to the gauge-fixing term and FPG refers to the Faddeev-Popov ghost term). This is the starting point from which the Green's functions, and hence the gauge boson propagators, can be derived.

2.5.1 Free-field generating functional

To derive the free-field generating functional from the effective Lagrangian, it is convenient to use the class of gauges where $f^i = \partial^\mu A^i_\mu$. To find the matrix M, we take the derivative of $\partial_\mu A^\mu_i$ with respect to θ_j in Eq. (2.42). Combining this with the other terms in the Lagrangian, we obtain the following generating functional:

$$\begin{aligned}Z[J, c, c^\dagger] &= N \int \mathcal{D}A_\mu \mathcal{D}\eta \mathcal{D}\eta^\dagger \exp\{i \int d^4x [-\frac{1}{4}(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu)^2 - \frac{1}{2\alpha}(\partial^\mu A^i_\mu)^2 + \\ &\quad \eta^{i\dagger} \partial^\mu (\delta^{ij}\partial_\mu - g f^{ijk} A^k_\mu)\eta^j + J^{i\mu} A^i_\mu + c^{i\dagger}\eta^i + c^i \eta^{i\dagger}]\},\end{aligned} \tag{2.48}$$

where J^μ, c and $c^\dagger$ are sources of the fields A_μ, $\eta^\dagger$, and η, respectively (note the normalizations have been redefined to take out a factor of $1/g$). As in the scalar field case, we can separate this into a quadratic free-field term and an interaction term at higher order in the fields. Then we can write the generating functional as

$$Z[J, c, c^\dagger] = e^{iS_I\left[\frac{\delta}{i\delta J_\mu}, \frac{\delta}{i\delta c}, \frac{\delta}{i\delta c^\dagger}\right]} Z_0[J] Z_0[c, c^\dagger], \tag{2.49}$$

where S_I contains only interaction terms. The propagators can be extracted from the free-field functionals $Z_0[J]$ and $Z_0[c, c^\dagger]$. We can express $Z_0[J]$ as

$$\int \mathcal{D}A_\mu e^{i\int d^4x \{\frac{1}{2}A^i_\mu [g^{\mu\nu}\partial_\lambda \partial^\lambda - (1-\frac{1}{\alpha})\partial^\mu \partial^\nu] A^i_\nu + J^{i\mu} A^i_\mu\}} \tag{2.50}$$

after removing surface terms (the normalization is implicit here and in the following). The Gaussian integral can be evaluated using

$$\int \mathcal{D}A e^{-\frac{1}{2}AKA + JA} \propto (\det K)^{-\frac{1}{2}} e^{JK^{-1}J}, \tag{2.51}$$

to obtain:

$$Z_0[J] = e^{-\frac{i}{2}\int d^4x d^4y J^{i\mu}(x) G^{ij}_{\mu\nu}(x-y) J^{j\nu}(y)}, \tag{2.52}$$

where

$$G^{ij}_{\mu\nu}(x-y) = -\delta^{ij} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \left[g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right]. \tag{2.53}$$

Common gauges are $\alpha = 0$ (Lorenz gauge) and $\alpha = 1$ (Feynman gauge). One can verify that this is the inverse of K by integrating $d^4y K^{\lambda ab}_\mu(x-y) G^{bc}_{\lambda\nu}(y-z)$. A similar procedure can be applied to obtain the free-field generating functional for ghosts:

$$Z_0[c, c^\dagger] = e^{-i\int d^4x d^4y c^{i\dagger}(x) G^{ij}(x-y) c^j(y)}, \tag{2.54}$$

where

$$G^{ij}(x-y) = -\delta^{ij} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2+i\epsilon}. \tag{2.55}$$

Interaction terms can be determined using S_I and the derivatives of the free-field generating functional. We will return to the interaction terms after considering the modification of the propagator due to the Higgs mechanism.

CHAPTER 3

THE HIGGS MECHANISM

Fundamental scalar fields are a special form of matter. They have explicit self-couplings and provide potential terms to the fermions, without any known geometrical origin (unlike the gauge bosons). In addition, the form of the scalar potential can create a charged vacuum that hides the gauge symmetry and gives mass to the gauge bosons.

3.1 Self-interacting scalar field theory

It is instructive to investigate the theories of a single self-interacting scalar field described by various Lagrangians with different ground states. Recasting the Lagrangian into a combination of physical states and vacuum expectation values demonstrates how simple changes to the Lagrangian can lead to a rich set of physical phenomena. The most basic case to consider is the real scalar field. With a complex scalar field a number of additional phenomena can appear, such as the presence of massless Goldstone bosons resulting from an invariance in the Lagrangian with respect to rotations of the field.

3.1.1 Real scalar field

A basic Lagrangian for an interacting real scalar field theory is

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2) - \frac{\lambda}{4}\phi^4.$$

As $\lambda \to 0$, this approaches a free field theory with kinematics described by the Klein-Gordon equation:

$$[\partial_\mu \partial^\mu - m^2]\phi = 0. \quad (3.1)$$

This equation represents the physical situation of a ground state at $\phi = 0$ and harmonic oscillations of the field about this ground state. The solution is given by Eq. (2.18):

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} [e^{ik_\mu x^\mu} \hat{a}^\dagger(k) + e^{-ik_\mu x^\mu} \hat{a}(k)], \quad (3.2)$$

where $\hat{a}(k)$ is the destruction operator that removes a particle with momentum k and $\hat{a}^\dagger(k)$ is the creation operator that produces a particle with momentum k. Incorporating a non-zero self-coupling λ creates perturbations that affect the propagation of a particle through emission and reabsorption of additional particles.

Now consider a change in sign of the mass term, taking the mass parameter to be real. This case corresponds to a potential with a local maximum at $\phi = 0$ rather than a local minimum. A field starting at $\phi = 0$ will radiate energy until it reaches the ground state at $\phi = \pm\sqrt{m^2/\lambda}$. The ground state, or equivalently the vacuum, has a non-zero eigenvalue for the field operator $\hat{\phi}$. Expressing the operator as $\hat{\phi}_0 + \hat{\delta}$, with $\hat{\phi}_0|0\rangle = \pm\sqrt{m^2/\lambda}|0\rangle$, the Lagrangian is:

$$\begin{aligned}\mathcal{L}(\delta) &= \frac{1}{2}(\partial_\mu \delta \partial^\mu \delta + m^2\phi_0^2 + 2m\phi_0\delta + m^2\delta^2) - \frac{\lambda}{4}(\phi_0^4 + 4\phi_0^3\delta + 6\phi_0^2\delta^2 + 4\phi_0\delta^3 + \delta^4) \\ &= \frac{1}{2}(\partial_\mu \delta \partial^\mu \delta - 2m^2\delta^2) \mp m\sqrt{\lambda}\delta^3 - \frac{\lambda}{4}\delta^4 + \frac{m^4}{4\lambda}, \end{aligned} \quad (3.3)$$

where in the second line we have assumed operation on a vacuum-to-vacuum transition and replaced ϕ_0 with $\pm\sqrt{m^2/\lambda}$. This Lagrangian describes an interacting scalar field theory of a particle oscillating about the ground state with a mass of $\sqrt{2}m$. The theory contains both δ^3 and δ^4 self-interaction terms and the ground state corresponds to a potential well with a minimum $V_0 = -m^4/(4\lambda)$. The δ^3 term breaks the inherent Z_2 symmetry of the Lagrangian. The expansion about this ground state allows a physically transparent expression for the Lagrangian, including the symmetries of the Lagrangian with respect to the physical states.

3.1.2 Complex scalar field

It is straightforward to extend this investigation from real to complex scalar fields. Writing the complex field as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and the Lagrangian as

$$\mathcal{L}(\phi) = (\partial_\mu \phi^* \partial^\mu \phi + \mu^2 \phi^* \phi) - \lambda(\phi^*\phi)^2, \quad (3.4)$$

it is clear that the ground state will occur for a field value of $|\phi_0| = \mu/\sqrt{\lambda}$. This corresponds to a circle in the complex ϕ plane with radius $|\phi_0|$. As before, a field starting at $\phi = 0$ will radiate until it reaches this circle in ϕ space, and we can rewrite the Lagrangian by considering variations about the ground state. The ground state will be some point on the circle and we can choose axes such that it is in the direction of the positive axis of the real field. Then $\phi = \phi_0 + (\delta + i\epsilon)/\sqrt{2}$ and the Lagrangian becomes:

$$\mathcal{L}_\phi(\delta, \epsilon) = \frac{1}{2}(\partial_\mu \delta \partial^\mu \delta - 2\mu^2\delta^2) + \frac{1}{2}(\partial_\mu \epsilon \partial^\mu \epsilon) - \mu\sqrt{\lambda}\delta(\delta^2 + \epsilon^2) - \frac{\lambda}{4}(\delta^2 + \epsilon^2)^2 + \frac{\mu^4}{4\lambda}. \quad (3.5)$$

There are two types of oscillations, one along the radial direction and the other along the azimuthal direction. Since there is no quadratic term for the field in the azimuthal direction, the corresponding particle is massless. This massless particle arising from an invariance of the Lagrangian with respect to the ground state is known as a Goldstone boson.

In this derivation we have chosen axes such that the ground state is along the real axis. However, if the phase of the vacuum changes at different points in spacetime, such an axis cannot be chosen globally and a phase will appear in the $\delta(\delta^2 + \epsilon^2)$ term; the symmetry has been broken by the vacuum. This changes when one considers a gauge field theory.

3.2 Gauged scalar field theory

The most interesting phenomena occur when a scalar field is a vector on a fiber and has a non-zero vacuum expectation value. Then the vacuum itself is a fiber vector with a current producing a fiber curvature. The connection transports the vacuum vector through spacetime along the curved fiber space. This transport leads to a mass term for the connection. The fundamental Lagrangian maintains gauge symmetry—any fiber coordinate system can be used to describe the vacuum fiber curvature—but the charged vacuum hides this symmetry in physical interactions.

To study the mechanism explicitly, we consider charged scalar fields under an Abelian U(1) gauge group and a non-Abelian SU(2) gauge group.

3.2.1 U(1)-charged scalar field

The simplest gauge group is U(1), which can be represented by a phase or a location on a circle. A single gauge boson, or connection, A_μ, describes the parallel transport of the U(1) fiber vector of a field with charge $-e$ along its path:

$$D\phi = (\partial_\mu + ieA_\mu)\phi dx^\mu. \tag{3.6}$$

There are no group indices, since it is a one-dimensional space. The scalar field has no direction in space, only a position in the fiber.

The Lagrangian for this field is simply the interacting scalar-field Lagrangian with derivatives given by Eq. (3.6), plus a curvature term:

$$\mathcal{L}(\phi) = D_\mu\phi^* D^\mu\phi + \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.7}$$

The minimum of $V(\phi)$ has not changed, so again we expand around the ground state of the vacuum and obtain the terms in Eq. (3.5) $[\mathcal{L}_\phi(\delta,\epsilon)]$ plus terms with A_μ from the covariant derivative:

$$\begin{aligned}
\mathcal{L}(\delta,\epsilon,A_\mu) &= \mathcal{L}_\phi(\delta,\epsilon) + \mathcal{L}_{A_\mu}(\delta,\epsilon,A_\mu) \\
&= \mathcal{L}_\phi(\delta,\epsilon) + \frac{e^2\mu^2}{2\lambda}A_\mu A^\mu - \frac{e\mu}{\sqrt{\lambda}}\partial_\mu\epsilon A^\mu + e[\delta\partial_\mu\epsilon - \epsilon\partial_\mu\delta]A^\mu + \\
&\quad \frac{e^2\mu}{2\sqrt{\lambda}}\delta A_\mu A^\mu + \frac{e^2}{2}(\epsilon^2+\delta^2)A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.
\end{aligned} \tag{3.8}$$

There are a number of remarkable phenomena in this Lagrangian. First, consider the term $\frac{e^2\mu^2}{2\lambda}A_\mu A^\mu = e^2\langle\phi_0\rangle^2 A_\mu A^\mu$. The non-zero expectation value $\langle\phi_0\rangle$ has a particular location in group space, i.e. it has a specific phase. The $e^2\langle\phi_0\rangle^2 A_\mu A^\mu$ term transmits the local curvature in this region of group space along the spacetime manifold, with a characteristic distance $|\langle\phi_0\rangle|^{-1}$. One can picture the situation as follows: a particle has a particular U(1) phase, which is parallel-transported via A_μ. But the phase associated with $\langle\phi_0\rangle$ is in a potential well, so the phase "falls" in this direction over a spacetime distance $|\langle\phi_0\rangle|^{-1}$. Oscillations in the phase are thus damped out over distances of this scale.

The parallel-transport of the phase can be seen in the bilinear term $\frac{e\mu}{\sqrt{\lambda}}\partial_\mu\epsilon A^\mu$, which corresponds to a vertex with one $\partial_\mu\epsilon$ line and one A_μ line. Recalling Eq. (3.2) for a free field, $\partial_\mu\epsilon = k_\mu\epsilon$, the vertex projects A_μ along the direction of propagation of ϵ. The connection A_μ parallel-transports ϵ into the potential well. One can use a coordinate system, or equivalently a gauge, where this term is zero. Changing to this coordinate system requires the following changes in the fields:

$$\begin{aligned} \phi' &= e^{i\epsilon/\phi_0}\phi \\ A'_\mu &= A_\mu + \frac{1}{e\phi_0}\partial_\mu\epsilon. \end{aligned} \tag{3.9}$$

In this gauge the connection follows the scalar field oscillations about the vacuum in group space; this component of the scalar field is absorbed by the connection and the Lagrangian becomes:

$$\begin{aligned} \mathcal{L}(\delta, A'_\mu) \quad = \quad & \left[\frac{1}{2}(\partial_\mu\delta\partial^\mu\delta - 2\mu^2\delta^2)\right] + \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2\mu^2}{2\lambda}A'_\mu A'^\mu\right] + \\ & \frac{e^2\mu}{2\sqrt{\lambda}}\delta A'_\mu A'^\mu + \frac{e^2}{2}\delta^2 A_\mu A^\mu + \mu\sqrt{\lambda}\delta^3 - \frac{\lambda}{4}\delta^2 + \frac{\mu^4}{4\lambda}. \end{aligned} \tag{3.10}$$

We see that oscillations in ϵ have moved to oscillations in the connection; the two-component gauge field has acquired a third field, i.e. another degree of freedom, along its direction of motion. This is the longitudinal component of the massive vector field. This choice of coordinates is known as the "unitary", or "physical" gauge.

3.2.2 SU(2)-charged scalar field

A scalar charged under SU(2) can have any half-integer charge. We consider the case of a charge 1/2 scalar, which can be represented as a complex doublet ϕ_i, $i = 1, 2$. The analysis continues along the lines of the Abelian case, though now with group indices on ϕ, A_μ and $F_{\mu\nu}$. The set of connections again describes the parallel-transport of the momentum of the field, but now with the possibility of changing the SU(2) direction of the field:

$$D\phi_i = [\delta_i^j\partial_\mu + ig(\tau^a A^a)_{i\mu}^j]\phi_j dx^\mu. \tag{3.11}$$

where τ^l can be represented by the usual Pauli matrices. Taking the usual potential

$$V(\phi) = -\mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2, \tag{3.12}$$

the ground state corresponds to an expectation value of $\langle\phi^\dagger\phi\rangle = \mu^2/2\lambda$. Now we choose axes such that this expectation value is real, positive, and in the "down" state:

$$\langle\phi\rangle_0 = \begin{pmatrix} 0 \\ \mu/\sqrt{2\lambda} \end{pmatrix}. \tag{3.13}$$

Expanding the scalar fields about the vacuum expectation value gives a Lagrangian

$$\mathcal{L} = \left[\left(\partial_\mu - ig\frac{\tau^a}{2}A^a_\mu\right)(\phi' + \langle\phi\rangle_0)\right]^\dagger \left[\left(\partial_\mu - ig\frac{\tau^b}{2}A^b_\mu\right)(\phi' + \langle\phi\rangle_0)\right] + V(\phi') - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \tag{3.14}$$

where $\phi' = \phi - \langle\phi\rangle_0$. To obtain the gauge boson masses, we take the term combining two A^a_μ with two $\langle\phi\rangle_0$:

$$\begin{aligned} &\tfrac{g^2}{4}(\tau^a A^a_{ij\mu}\langle\phi\rangle^i_0)^\dagger(\tau^b A^{b\mu}_{jk}\langle\phi\rangle^k_0) = \\ &\tfrac{g^2}{4}\langle\phi\rangle^\dagger_0 \begin{pmatrix} A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu \end{pmatrix} \begin{pmatrix} A^{3\mu} & A^{1\mu} - iA^{2\mu} \\ A^{1\mu} + iA^{2\mu} & -A^{3\mu} \end{pmatrix} \langle\phi\rangle_0 \\ &= \tfrac{g^2}{4}\tfrac{\mu^2}{2\lambda}\begin{pmatrix} A^1_\mu + iA^2_\mu & -A^3_\mu \end{pmatrix}\begin{pmatrix} A^{1\mu} - iA^{2\mu} \\ -A^{3\mu} \end{pmatrix} \\ &= \tfrac{g^2}{4}\tfrac{\mu^2}{2\lambda}(A^1_\mu A_{1\mu} + A^2_\mu A^{2\mu} + A^3_\mu A^{3s\mu}). \end{aligned} \tag{3.15}$$

This is of the form $\frac{1}{2}M^2_A A^a_\mu A^{a\mu}$, where $M_A = g\mu/(2\sqrt{\lambda}) = g\langle\phi\rangle_0/\sqrt{2}$. Thus, all three gauge bosons have equal mass in the theory.

To obtain the scalar field masses, the terms quadratic in ϕ' must be calculated. We have

$$\begin{aligned} &\mu^2\phi'^\dagger\phi' - \lambda(\phi'^\dagger\phi' + \phi'^\dagger\langle\phi\rangle_0 + \langle\phi\rangle^\dagger_0\phi' + \langle\phi\rangle^\dagger_0\langle\phi\rangle_0)^2 = \\ &\mu^2\phi'^\dagger\phi' - \lambda[\phi'^\dagger\phi'\langle\phi\rangle^\dagger_0\langle\phi\rangle_0 + (\phi'^\dagger\langle\phi\rangle_0)^2 + (\langle\phi\rangle^\dagger_0\phi')^2 + 2(\phi'^\dagger\langle\phi\rangle_0)(\langle\phi\rangle^\dagger_0\phi')]. \end{aligned} \tag{3.16}$$

We now write

$$\phi' = \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} \tag{3.17}$$

and expand equation 3.16:

$$\mu^2(|\phi'_1|^2 + |\phi'_2|^2) - \frac{\mu^2}{2}(2|\phi_1|^2 + 2|\phi'_2|^2 + \phi'^{*2}_2 + \phi'^2_2 + 2|\phi'_2|^2) = -\frac{\mu^2}{2}(\phi'_2 + \phi'^*_2)^2. \tag{3.18}$$

Thus, the real component of ϕ_2 has a mass $\sqrt{2}\mu$ and the other three scalar fields remain massless (the Goldstone bosons). The three massless fields are “eaten” by the gauge fields in the unitary gauge.

3.2.3 Propagators after symmetry breaking

In a scalar field theory with a Lagrangian given by Eq. (3.7) and a U(1) gauge symmetry, the field has a vacuum expectation value that can be defined as $\langle\phi\rangle_0 = \mu/\sqrt{2\lambda}$. Expanding the scalar field about $\langle\phi\rangle_0$ gives Eq. (3.8). We parametrize the gauge-fixing condition as

$$f = \partial^\mu A_\mu + \frac{\alpha e\mu}{\sqrt{\lambda}}\epsilon, \tag{3.19}$$

which is set to zero with a delta function in the generating functional. With this gauge-fixing term and neglecting ghost terms, the free-field Lagrangian is

$$\begin{aligned} \mathcal{L}(\delta, \epsilon, A'_\mu) &= \frac{1}{2}\left(\partial_\mu\delta\partial^\mu\delta - 2\mu^2\delta^2\right) + \frac{1}{2}\left(\partial_\mu\epsilon\partial^\mu\epsilon - \frac{\alpha\mu^2}{\lambda}\epsilon^2\right) + \\ &\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2\mu^2}{2\lambda}A'_\mu A'^\mu - \frac{1}{2\alpha}(\partial_\mu A'^\mu)^2\right]. \end{aligned} \tag{3.20}$$

The propagators can be calculated from the generating functional using the expression for a Gaussian integral [Eq. (2.51)]. The propagators are:

$$\begin{aligned} i\Delta_\delta(k) &= \frac{i}{k^2 - 2\mu^2 + i\epsilon}, \\ i\Delta_\epsilon(k) &= \frac{i}{k^2 - \alpha\mu^2/\lambda + i\epsilon}, \\ i\Delta_{\mu\nu}(k) &= \frac{i}{k^2 - \mu^2/\lambda + i\epsilon}\left[g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2 - \alpha\mu^2/\lambda}\right]. \end{aligned} \tag{3.21}$$

There are several useful gauges. The unitary gauge corresponds to $\alpha \to \infty$, where the ϵ propagator disappears and is included as the longitudinal component of the gauge boson propagator. The Landau gauge corresponds to $\alpha = 0$, where the ϵ propagator is the original massless Goldstone boson. The 't Hooft-Feynman gauge corresponds to $\alpha = 1$, where the ϵ has the same mass as the gauge boson. In these cases the gauge field has two degrees of freedom and the ϵ propagator is separated out as the third degree of freedom.

CHAPTER 4

THE ELECTROWEAK THEORY

The Electroweak theory is based on a simple Lagrangian built from the following components: an SU(2)×U(1) fiber structure; a single scalar field; and three massless generations of fermion fields that interact with the scalar field. The scalar and fermion fields have representations in the SU(2)×U(1) fiber. The scalar field has a potential with a non-zero vacuum expectation value, resulting in an effective Lagrangian of massive gauge bosons and fermions, with a residual U(1) gauge symmetry.

4.1 The Electroweak Lagrangian

Fundamentally, the Electroweak Lagrangian has a remarkably simple structure. It is composed of a fiber curvature, a complex scalar field, and massless fermions:

$$\mathcal{L} = \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm fermion}. \tag{4.1}$$

The fiber structure governs the interactions between fermions. The curvature of the fiber affects particle trajectories through the connections; each fermion has charges that determine how its motion is affected by this curvature. The electroweak fiber structure is $\rm SU(2)_L \times U(1)_Y$ and the curvature of the fiber can be described by the Lagrangian

$$\mathcal{L}_{\rm gauge} = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \frac{1}{4} F'_{\mu\nu} F'^{\mu\nu}, \tag{4.2}$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^j_\mu A^k_\nu \tag{4.3}$$

is the curvature tensor of the $\mathrm{SU(2)_L}$ gauge group and

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu \tag{4.4}$$

is the curvature tensor of the $\mathrm{U(1)_Y}$ group. The charge associated with the $\mathrm{SU(2)_L}$ group is called "weak" charge and that associated with the $\mathrm{U(1)_Y}$ group is called "hypercharge".

There is one scalar field in the Electroweak theory: a complex doublet under SU(2) transformations, with hypercharge equal to 1. Its Lagrangian is:

$$\mathcal{L}_{\mathrm{scalar}} = (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2, \tag{4.5}$$

where the covariant derivative is

$$D\phi = \left(\partial_\mu - ig\frac{\tau^i}{2}A^i_\mu - i\frac{g'}{2}A'_\mu\right)\phi dx^\mu. \tag{4.6}$$

The fermion fields are massless in the fundamental Lagrangian, so they can be separated into right- and left-handed helicity doublets:

$$\psi_{R,L} = \frac{1}{2}(1 \pm \gamma_5)\psi, \tag{4.7}$$

where the positive (negative) sign corresponds to the right-handed (left-handed) helicity state. The fermion Lagrangian is

$$\mathcal{L}_{\mathrm{fermion}} = i\bar{\psi}_L\gamma^\mu D_\mu\psi_L + i\bar{\psi}_R\gamma^\mu D_\mu\psi_R - (y^d_{ij}\bar{\psi}_{iL}\phi\psi^d_{jR} + y^u_{ij}\bar{\psi}_{iL}\tilde{\phi}\psi^u_{jR} + h.c.) \tag{4.8}$$

where

$$\begin{aligned} D\psi_L &= \left(\partial_\mu - ig\frac{\tau^i}{2}A^i_\mu - iY\frac{g'}{2}A'_\mu\right)\psi_L dx^\mu, \\ D\psi_R &= \left(\partial_\mu - iY\frac{g'}{2}A'_\mu\right)\psi_R dx^\mu, \end{aligned} \tag{4.9}$$

$\tilde{\phi} = i\tau_2\phi$, Y are the fermion hypercharges, and y^u_{ij} and y^d_{ij} are Yukawa fermion-scalar couplings that are different for each pair of fermions (i, j are generation indices). The right-handed partners to the down-type and up-type left-handed fermions are ψ^d_R and ψ^u_R, respectively. There are three generations of fermions separated into quarks and leptons, and the Yukawa couplings are not diagonal with respect to these generations. These are the only couplings in the model that are generation-dependent. The hypercharges are 1/3 (left-handed quarks), 4/3 (right-handed up quarks), -2/3 (right-handed down quarks), -1 (left-handed leptons), and -2 (right-handed charged leptons). If neutrinos were massless, no right-handed neutrinos would be required; even with massive neutrinos there may not be right-handed neutrinos since the mass terms could be of Majorana rather than Dirac form. If right-handed neutrinos exist they have no weak charge or hypercharge—they are singlets in the gauge groups of the Standard Model.

To complete the Standard Model, one simply needs to add another gauge field to $\mathcal{L}_{\mathrm{gauge}}$. It transforms under the group $\mathrm{SU(3)_c}$; only quarks are charged under this group.

4.2 Electroweak symmetry breaking

In the Electroweak theory the scalar field has a non-zero vacuum expectation value. This significantly complicates the effective Lagrangian: three of the four gauge bosons have non-zero masses, fermions have non-zero masses, and the residual massless gauge boson is a linear combination of the original "neutral" gauge bosons.

4.2.1 Scalar field Lagrangian

The scalar field potential has a minimum at $\langle\phi^\dagger\phi\rangle_0 = \mu^2/(2\lambda)$. Choosing the coordinate axes such that

$$\langle\phi\rangle_0 = \begin{pmatrix} 0 \\ \mu/\sqrt{2\lambda} \end{pmatrix}, \tag{4.10}$$

we can expand $\mathcal{L}_{\mathrm{scalar}}$ about $\phi' = \phi - \langle\phi\rangle_0$:

$$\begin{aligned}\mathcal{L}_{\mathrm{scalar}} &= \left[\left(\partial_\mu - ig\frac{\tau^a}{2}A^a_\mu - i\frac{g'}{2}A'_\mu\right)(\phi' + \langle\phi\rangle_0)\right]^\dagger \left(\partial^\mu - ig\frac{\tau^b}{2}A^{b\mu} - i\frac{g'}{2}A'^\mu\right)(\phi' + \langle\phi\rangle_0) \\ &\quad +\mu^2(\phi' + \langle\phi\rangle_0)^\dagger(\phi' + \langle\phi\rangle_0) - \lambda[(\phi' + \langle\phi\rangle_0)^\dagger(\phi' + \langle\phi\rangle_0)]^2. \end{aligned} \tag{4.11}$$

Expanding the covariant derivatives gives:

$$\begin{aligned}\mathcal{L}_{\mathrm{scalar}} &= D_\mu\phi' D^\mu\phi' + \left[\left(\partial_\mu - ig\frac{\tau_l}{2}A^l_\mu - i\frac{g'}{2}A'_\mu\right)\phi'\right]^\dagger \left(-ig\frac{\tau_l}{2}A^{l\mu} - i\frac{g'}{2}A'^\mu\right)\langle\phi\rangle_0 + \\ &\quad \left[\left(-ig\frac{\tau_l}{2}A^l_\mu - i\frac{g'}{2}A'_\mu\right)\langle\phi\rangle_0\right]^\dagger \left(\partial^\mu - ig\frac{\tau_l}{2}A^{l\mu} - i\frac{g'}{2}A'^\mu\right)\phi' + \\ &\quad \left[\left(-ig\frac{\tau_l}{2}A^l_\mu - i\frac{g'}{2}A'_\mu\right)\langle\phi\rangle_0\right]^\dagger \left(-ig\frac{\tau_l}{2}A^{l\mu} - i\frac{g'}{2}A'^\mu\right)\langle\phi\rangle_0 + \ldots \end{aligned} \tag{4.12}$$

The first term is the kinetic term for the physical ϕ' field. The second and third terms give coefficients for $\phi' A_\mu A^\mu$ (a three-point vertex) and $\partial_\mu\phi' A^\mu$ (a two-point mixing term). The last term gives the gauge field mass coefficients:

$$\begin{aligned}&\begin{pmatrix} \frac{\mu g}{2\sqrt{2\lambda}}(iA^1_\mu - A^2_\mu) & -i\frac{\mu g}{2\sqrt{2\lambda}}A^3_\mu + i\frac{\mu g'}{2\sqrt{2\lambda}}A'_\mu \end{pmatrix} \begin{pmatrix} \frac{\mu g}{2\sqrt{2\lambda}}(-iA^{1\mu} - A^{2\mu}) \\ i\frac{\mu g}{2\sqrt{2\lambda}}A^{3\mu} - i\frac{\mu g'}{2\sqrt{2\lambda}}A'^\mu \end{pmatrix} = \\ &\frac{\mu^2}{8\lambda}\left[g^2(A^1_\mu A_{1\mu} + A^2_\mu A^{2\mu} + A^3_\mu A^{3\mu}) + g'^2 A'_\mu A'^\mu - gg'(A^3_\mu A'^\mu + A^{3\mu}A'_\mu)\right]. \end{aligned} \tag{4.13}$$

We see a new feature in this matrix: a mixing term between the SU(2) and U(1) "neutral" fields. Now the transport of a fiber vector purely along either direction increases the potential. However, there is a flat direction along which the potential does not increase and the propagator remains massless. This is the direction where the $\mathrm{U(1)_Y}$ connection is weighted by the coupling constant g and the $\mathrm{SU(2)_L}$ τ^3 connection is weighted by the coupling constant g':

$$B_\mu = \frac{g' A_{3\mu} + gA'_\mu}{\sqrt{g^2 + g'^2}}. \tag{4.14}$$

This combination of connections is the massless photon (the combination is represented pictorially in Fig. 4.1 [10]). The remaining eigenstates are massive:

$$\begin{aligned} W^\pm_\mu &= \frac{A_{1\mu} \mp iA_{2\mu}}{\sqrt{2}}, \\ Z^0_\mu &= \frac{gA_{3\mu} - g'A'_\mu}{\sqrt{g^2 + g'^2}}, \end{aligned} \tag{4.15}$$

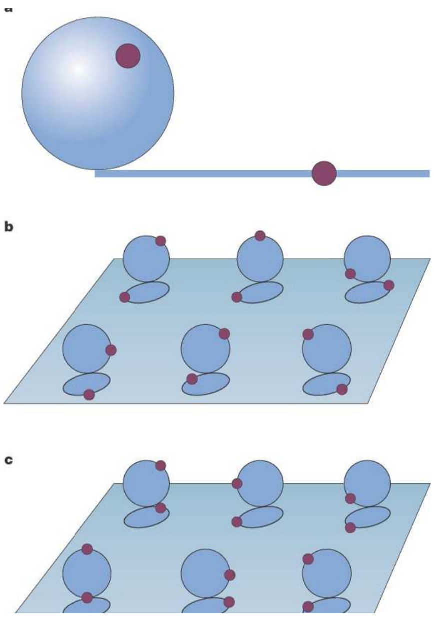


Figure 4.1 (a) A representation of points on $SU(2)_L \times U(1)_Y$. (b) A simplification of $U(1)\times U(1)$. (c) The points along the direction determined by the vacuum expectation value of a scalar field. A common rotation does not change the potential, but a change in the relative positions does.

where the masses are extracted by equating the coefficient of $W_\mu^+ W^{\mu-} + W_\mu^- W^{\mu+}$ to $m_{W^\pm}^2/2$ and the coefficient of $Z_\mu^- Z^{\mu-}$ to $m_{Z^0}^2/2$:

$$\begin{aligned} m_{W^\pm} &= \frac{\mu g}{2\sqrt{\lambda}}, \\ m_{Z^0} &= \frac{\mu\sqrt{g^2+g'^2}}{2\sqrt{\lambda}}. \end{aligned} \tag{4.16}$$

It is common to express the relative values of g and g' in terms of an angle, with the $SU(2)_L$ direction τ_3 along the x-axis and the $U(1)_Y$ direction along the y-axis, so that $m_{W^\pm} = m_{Z^0}\cos\theta_W$.

The residual massless connection is related to the number of Goldstone bosons. The scalar terms in the Lagrangian lead to three massless scalar fields and one field with mass $\sqrt{2}\mu$. The three massless scalar fields combine with the massive vector bosons to produce the spin-0 longitudinal components of these vector bosons. The fourth scalar field describes a massive physical particle with oscillations up the sides of the potential well. With four vector bosons and only three Goldstone bosons, one vector boson must remain massless. This vector boson has the same quantum numbers as the massive scalar.

4.2.2 Fermion field Lagrangian

As with the scalar Lagrangian, we can expand the fermion Lagrangian about the vacuum expectation value, $\phi' = \phi - \langle\phi\rangle_0$:

$$\begin{aligned} \mathcal{L}_{\text{fermion}} &= i\bar{\psi}_L\gamma^\mu D_\mu\psi_L + i\bar{\psi}_R\gamma^\mu D_\mu\psi_R - (y^d_{ij}\bar{\psi}_{iL}\phi'\psi^d_{jR} + \frac{\mu y^d_{ij}}{\sqrt{2\lambda}}\bar{\psi}_{iL}\psi^d_{jR} + \\ & y^u_{ij}\bar{\psi}_{iL}\tilde{\phi}'\psi^u_{jR} + \frac{\mu y^u_{ij}}{\sqrt{2\lambda}}\bar{\psi}_{iL}\psi^u_{jR} + h.c.), \end{aligned} \tag{4.17}$$

where i and j are generational indices. The fermions have received mass terms through the vacuum expectation value of the scalar field and the Yukawa couplings, which together produce a fermion potential well. The terms are complicated by the off-diagonal couplings of the Yukawa matrix, which lead to a difference between the mass eigenstates and the weak eigenstates. We can parameterize the relationship between eigenstates with a set of rotations as follows.

Since none of the other interactions contain off-diagonal couplings, the generation eigenstates of ψ_R can be defined by these couplings. That is, only the Yukawa couplings are sensitive to the rotation

$$\psi'^i_R = U_{ij}\psi^j_R. \tag{4.18}$$

The generation eigenstates of ψ_L can also be rotated, but because they are weak doublets, the up-type and down-type rotations cannot be made independently. Defining their rotation matrix as V, the combined rotation results in the following transformation of the Yukawa couplings:

$$\begin{aligned} y'^d &= V^\dagger y^d U^d \\ y'^u &= V^\dagger y^u U^u, \end{aligned} \tag{4.19}$$

where U^d, U^u and V are unitary matrices. It is in general possible to choose U^u and V such that y'^u is diagonal and therefore the mass matrix of up-type fermions:

$$\bar{\psi'}_L y'^u \psi'^u_R = (\bar{\psi}_L V)(V^\dagger y^u U^u)(U^{u\dagger}\psi^u_R) \tag{4.20}$$

We then have

$$\bar{\psi'}_L y'^d \psi'^d_R = (\bar{\psi}_L V)(V^\dagger y^d U^d)(U^{d\dagger}\psi^d_R). \tag{4.21}$$

Now we can write $V = V^d V'$, where $V^{\dagger d} y^d U^d$ is the diagonal mass matrix for down-type fermions. Then

$$\bar{\psi'}_L y'^d \psi'^d_R = (\bar{\psi}_L V_d V')V'^\dagger(V^{\dagger d} y^d U^d)(U^{d\dagger}\psi^d_R). \tag{4.22}$$

We see that the difference between mass and weak eigenstates is the matrix V'. We can redefine the right-handed eigenstates so that the Yukawa couplings can be expressed in terms of the mass matrix as:

$$y'^d = V'^\dagger(V^{\dagger d} y^d U^d)V'. \tag{4.23}$$

The relationship can be expressed in terms of eigenstates as $\psi'^{mass}_L = V'\psi^{weak}_L$.

A general $N \times N$ unitary matrix has N^2 parameters, of which $N(N-1)/2$ can be parameterized as real Euler angles. There are therefore three real angles and six phases in the matrix V'. We can freely choose the phases of the right-handed eigenstates. The ψ^u_R

phases will be compensated by a choice for the $\bar{\psi}_L V$ states, providing diagonal phases to multiply the $V'^{\dagger}$ matrix. The ψ_R^d phases will be compensated by the $V^{d\dagger}$ to maintain a real mass matrix. These diagonal phases multiply the $V'^{\dagger}$ matrix on the right. This leads to a general matrix of

$$V' = \begin{pmatrix} V_{11}e^{i(\phi_1-\theta_1)} & V_{12}e^{i(\phi_1-\theta_2)} & V_{13}e^{i(\phi_1-\theta_3)} \\ V_{21}e^{i(\phi_2-\theta_1)} & V_{22}e^{i(\phi_2-\theta_2)} & V_{23}e^{i(\phi_2-\theta_3)} \\ V_{31}e^{i(\phi_3-\theta_1)} & V_{32}e^{i(\phi_3-\theta_2)} & V_{33}e^{i(\phi_3-\theta_3)} \end{pmatrix}. \tag{4.24}$$

Since only phase differences appear in the matrix, five of the six phases can be used to rotate away the phases in the matrix. The sixth is linearly related to the others:

$$\begin{aligned} \phi_1 - \theta_1 = \quad & \tfrac{1}{2}[\phi_1 - \theta_2 + \phi_1 - \theta_3 + \phi_2 - \theta_1 + \phi_3 - \theta_1 - \\ & \tfrac{1}{2}(\phi_2 - \theta_2 + \phi_2 - \theta_3 + \phi_3 - \theta_2 + \phi_3 - \theta_3)]. \end{aligned} \tag{4.25}$$

With three measurable angles and one measureable phase, the "CKM" matrix can be written as:

$$V' = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \tag{4.26}$$

where $c_{12} = \cos\theta_{12}$, $s_{12} = \sin\theta_{12}$, and so on. This is the CKM matrix for quarks, and if there are no Majorana neutrinos there is an equivalent matrix for leptons.

It is instructive to make the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$ to obtain the following parametrization:

$$V' \approx \begin{pmatrix} 1-(\theta_{12}^2+\theta_{13}^2)/2 & \theta_{12} & \theta_{13}e^{-\delta_{13}} \\ -\theta_{12} & 1-(\theta_{12}^2+\theta_{23}^2)/2 & \theta_{23} \\ -\theta_{13}e^{i\delta_{13}} & -\theta_{23} & 1-(\theta_{23}^2+\theta_{13}^2)/2 \end{pmatrix}. \tag{4.27}$$

We see that for small angles the phase only affects the $ij = 13$ and 31 elements of the CKM matrix. In the Wolfenstein parametrization $\theta_{12} \propto \lambda$, $\theta_{23} \propto \lambda^2$, and $\theta_{13} \propto \lambda^3$. Experimentally $\lambda = 0.226$ and the proportionality constant is 0.814, so there is a hierarchy in the angles. This demonstrates the challenge in experimentally accessing the phase δ.

4.3 Electroweak propagators

We now have the tools to write down the propagators for the Electroweak Lagrangian. We define the general gauge-fixing terms

$$\begin{aligned} f^a &= \partial_\mu A^{a\mu} + ig\alpha\left(\phi'^{\dagger}\frac{\tau^a}{2}\langle\phi\rangle_0 - \langle\phi^{\dagger}\rangle_0\frac{\tau^a}{2}\phi'\right), \\ f &= \partial_\mu A'^{\mu} + ig'\frac{\alpha}{2}\left(\phi'^{\dagger}\langle\phi\rangle_0 - \langle\phi^{\dagger}\rangle_0\phi'\right), \end{aligned} \tag{4.28}$$

for $\mathrm{SU(2)_L}$ and $\mathrm{U(1)}_Y$ respectively. Inserting these terms into the Lagrangian and solving the generating functional for the propagators gives:

$$\begin{aligned}
\Delta^{W^\pm}_{\mu\nu} &= \frac{-i}{k^2 - m_W^2 + i\epsilon}\left[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2 - \alpha m_W^2}\right] \\
\Delta^{Z}_{\mu\nu} &= \frac{-i}{k^2 - m_Z^2 + i\epsilon}\left[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2 - \alpha m_Z^2}\right] \\
\Delta^{A}_{\mu\nu} &= \frac{-i}{k^2 + i\epsilon}\left[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2}\right] \\
\Delta_{\phi^\pm} &= \frac{i}{k^2 - \alpha m_W^2 + i\epsilon} \\
\Delta_{\phi_1} &= \frac{i}{k^2 - 2\mu^2 + i\epsilon} \\
\Delta_{\phi_2} &= \frac{i}{k^2 - \alpha m_Z^2 + i\epsilon} \\
\Delta_{\omega_{W^\pm}} &= \frac{-i}{k^2 - \alpha m_W^2 + i\epsilon} \\
\Delta_{\omega_Z} &= \frac{-i}{k^2 - \alpha m_Z^2 + i\epsilon} \\
\Delta_{\omega_A} &= \frac{-i}{k^2 + i\epsilon}.
\end{aligned} \tag{4.29}$$

From the complete Electroweak Lagrangian, the full set of interaction terms can be extracted. Before listing the interaction terms, we discuss the calculation of cross sections from the matrix element, and the representation of the matrix element in terms of Feynman diagrams.

CHAPTER 5

CROSS SECTIONS AND FEYNMAN DIAGRAMS

To link the underlying theory to experimental observations we need to translate Lagrangian densities into differential cross-section predictions. The procedure involves producing a scattering matrix based on the Green's functions and relating it to a cross section measurement. The matrix can be divided into a phase space component and a fundamental interaction component, which can be calculated in a straightforward way using Feynman diagrams and rules.

5.1 Scattering matrix

A cross-section or lifetime calculation begins with the scattering matrix $S = \langle f|i\rangle$, where i represents an initial state and f represents a final state. The LSZ reduction formula expresses an n-particle to n'-particle scattering matrix in terms of the Green's function in the position basis as

$$\langle x_{1'}, ..., x_{n'}|x_1, ..., x_n\rangle = \prod_i \delta^4(x_{i'} - x_i) + \left[\prod_{i,i'} \phi(x_i)\phi^\dagger(x_{i'})(\partial_\mu\partial^\mu + m^2)_{i,i'}\right] G(x_1, ..., x_{n'}). \tag{5.1}$$

The states ϕ are the incoming and outgoing free-particle plane waves with creation and annihilation operators that act on the vacuum. In the momentum basis the differential operators become the particle momenta:

$$\begin{aligned}\langle p_{1'}, ..., p_{n'}|p_1, ..., p_n\rangle &= \prod_i \delta^4(p_{i'} - p_i) + (-i)^{n+n'} \prod_{i,i'} (p_{i'}^2 - m^2)(p_i^2 - m^2) \\ &\times G(-p_{1'}, ..., -p_{n'}, p_1, ..., p_n), \end{aligned} \tag{5.2}$$

where

$$G(p_1, ..., p_n) = \int \left[\prod_i d^4x e^{-ip_ix^i}\right] G(x_1, ..., x_n). \tag{5.3}$$

The prefactors to the Green's functions (e.g. $p_i^2 - m^2$) cancel the external propagators that don't contribute to the scattering probability.

Example: In an interacting scalar field theory with a $-\frac{g}{4!}\phi^4$ term, the four-point Green's function of two incoming and two outgoing particles is

$$G(x_1, x_2, x_{1'}, x_{2'}) = -ig \int dz \Delta_F(x_1 - z)\Delta_F(x_2 - z)\Delta_F(z - x_{1'})\Delta_F(z - x_{2'}). \tag{5.4}$$

Since we typically have approximate momentum eigenstates, it is most appropriate to calculate the scattering matrix in the momentum basis. Using the Fourier representation of the propagator,

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon} \tag{5.5}$$

the integral over x in Eq. (5.3) will give a delta function requiring $k_i = p_i$; then the integral over k_i just replaces k_i with p_i. The factors of $p_i^2 - m^2$ cancel, leaving only exponentials in the integral over z. This last integral gives $(2\pi)^4$ times a delta function that enforces momentum conservation:

$$\begin{aligned}\langle p_{1'}, p_{2'}|p_1, p_2\rangle = &\quad \delta^4(p_1 - p_{1'})\delta^4(p_2 - p_{2'}) - ig(2\pi)^4\delta^4(p_{1'} + p_{2'} - p_1 - p_2) \\ \equiv &\quad \delta^4(p_{1'} - p_1)\delta^4(p_{2'} - p_2) + i(2\pi)^4\delta^4(p_{1'} + p_{2'} - p_1 - p_2)\mathcal{M},\end{aligned}$$

where $\mathcal{M}$ is a Lorentz-invariant quantity generally referred to as the "matrix element" of the process. In this example, the matrix element is simply equal to $-g$. In general, every 4-point vertex will contribute a factor of $-g$ to the matrix element; this is an example of a "Feynman rule".

5.2 Cross sections and lifetimes

The scattering matrix represents the probability amplitude for a process. To get a probability one needs to square the amplitude. Since we are generally interested in probabilities of scattering processes or decays, we consider only the interaction term in the scattering matrix (the "transition" amplitude). We further factor out momentum conservation to get the matrix element capturing the dynamics of the process:

$$|\langle p_{1'}, ..., p_{n'}|p_1, ..., p_n\rangle|^2 = \delta_{nn'} \prod_i |\delta^4(p_{i'} - p_i)|^2 + (2\pi)^8 \left[\delta^4\left(\sum_{i'} p_{i'} - \sum_i p_i\right)\right]^2 |\mathcal{M}|^2. \tag{5.6}$$

A typical scattering experiment will have two colliding particles in the initial state. These particles will have a spread of momentum with wavefunctions represented by

$$|\phi_1, \phi_2\rangle = \int \frac{d^3p_1 d^3p_2}{(2\pi)^6 2E_1 2E_2} |p_1, p_2\rangle\langle p_1, p_2|\phi_1, \phi_2\rangle. \tag{5.7}$$

This gives a spread of matrix elements:

$$\int \tfrac{d^3p_1 d^3p_2}{(2\pi)^6 2E_1 2E_2} |\langle p_{1'}, ..., p_{n'}||p_1, p_2\rangle\langle p_1, p_2|\phi_1, \phi_2\rangle|^2 =$$
$$\int \tfrac{d^4x}{(2\pi)^2} |\phi_1(x)|^2 |\phi_2(x)|^2 [\textstyle\prod_i \delta^4(p_{i'} - p_i) + (2\pi)^4 \delta^4(\sum_{i'} p_{i'} - \sum_i p_i)|\mathcal{M}|^2], \tag{5.8}$$

where we have used one of the delta functions in each term to Fourier transform $|\phi_i(p)|^2$ to $|\phi_i(x)|^2$ for colliding particles 1 and 2. We focus on the interaction term that changes the state, defined as the transition probability P:

$$P = \int d^4x \prod_i |\phi_i(x)|^2 (2\pi)^4 \delta^4 \left(\sum_{i'} p_{i'} - \sum_i p_i \right) |\mathcal{M}|^2. \tag{5.9}$$

For a two-particle collision, the cross section is defined as the transition rate per unit volume divided by the “incident” particle flux and the “target” particle density:

$$\frac{dP}{dV dt} = \text{flux} \times \text{density} \times d\sigma. \tag{5.10}$$

The flux is given by the density of particles per unit volume ($\phi_i(x)|^2 \times 2E_i$) multiplied by the relative velocity of the initial state particles:

$$|\vec{v}| = \frac{\left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{\frac{1}{2}}}{E_1 E_2}. \tag{5.11}$$

The final-state particles are measured in a finite momentum range with a density of states $d^3p_{i'}/(2\pi)^3$ (the “phase space”). Combining all the pieces gives the following differential cross section:

$$d\sigma(p_1, p_2 \to p_{1'}, ..., p_{n'}) = \frac{(2\pi)^4 |\mathcal{M}|^2}{4\left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{\frac{1}{2}}} \delta^4(\sum_{i'} p_{i'} - \sum_i p_i) \prod_{i'} \frac{d^3p_{i'}}{(2\pi)^3 2E_{i'}}. \tag{5.12}$$

We have assumed that the final-state particles are distinguishable; if there are n indistinguishable final-state particles then we need to divide by $n!$ to remove the combinatoric factor.

The expression for inverse lifetime, or width, is nearly equivalent to that of the cross section; the difference is merely to move one particle from the initial state to the final state:

$$d\Gamma(p_1 \to p_{1'}, ..., p_{n'}) = \frac{(2\pi)^4 |\mathcal{M}|^2}{2E_1} \delta^4 \left(\sum_{i'} p_{i'} - p_1 \right) \prod_{i'} \frac{d^3p_{i'}}{(2\pi)^3 2E_{i'}}. \tag{5.13}$$

Example: Given the differential equation for the cross section, we can apply it to our simple example of $\phi\phi$ scattering in a scalar field theory with a $-\frac{g}{4!}\phi^4$ term. Assuming highly relativistic initial and final states, the differential cross section becomes

$$d\sigma(p_1, p_2 \to p_{1'}, p_{2'}) = \frac{(2\pi)^4 g^2}{2 \times 4 p_1 p_2} \delta^4(p_{1'} + p_{2'} - p_1 - p_2) \frac{d^3p_{1'}}{(2\pi)^3 2E_{1'}} \frac{d^3p_{2'}}{(2\pi)^3 2E_{2'}}, \tag{5.14}$$

where the factor of $1/2$ is the combinatoric factor for indistinguishable particles. In the case of scattering in the center of mass frame of the initial particles, $\vec{p}_1 = -\vec{p}_2$ and $p_1 p_2 = 2E_i^2$ (where E_i is the energy of each initial-state particle). Then the cross section equation is

$$d\sigma(p_1, p_2 \to p_{1'}, p_{2'}) = \frac{(2\pi)^4 g^2}{16E_i^2}\delta^3(\vec{p}_{1'} + \vec{p}_{2'})\delta(E_{1'} + E_{2'} - 2E_i)\frac{d^3p_{1'}}{(2\pi)^3 2E_{1'}}\frac{d^3p_{2'}}{(2\pi)^3 2E_{2'}}. \tag{5.15}$$

Performing the integral over $d^3p_{2'}$ sets $\vec{p}_{2'} = -\vec{p}_{1'}$. The integral over $d^3p_{1'} = E_{1'}^2 dE_{1'} d\Omega$ then cancels the $E_{1'}^2$ in the denominator, and $\delta(2E_{1'} - 2E_i)$ contributes a factor of $\frac{1}{2}$. We are left with:

$$\sigma(p_1, p_2 \to p_{1'}, p_{2'}) = \frac{g^2}{128\pi E_i^2}. \tag{5.16}$$

5.3 Feynman rules

The four-point vertex in the scalar field theory gives a simple contribution to the matrix element: g. This is an example of a Feynman rule. Any matrix element can be constructed by collecting all diagrams contributing to a process and inserting coupling factors at each vertex and propagators for each internal line. The coupling factors are straightforward to extract from the Lagrangian: they correspond to the coefficient in front of a set of field operators that determine the particle lines emanating from the vertex, times a combinatoric factor for indistinguishable external lines. In the case of the $-\frac{g}{4!}\phi^4$ term, there are four scalar operators emanating from a vertex with a coefficient of $-\frac{g}{4!}$ and a combinatoric factor of 4! for associating each particle momentum with a given line.

Including Lagrangian terms for the fermion fields with covariant derivatives for the $\mathrm{U(1)_{EM}}$ gauge fields, we can construct a complete set of rules for determining the matrix element for electromagnetic processes from any Feynman diagram:

1. apply a factor of $-iqe\gamma^\mu$ for a vertex emitting or absorbing a photon;
2. enforce conservation of momentum at each vertex with a delta function;
3. integrate over a propagator for each internal line,

$$\int \frac{d^4p}{(2\pi)^4}\frac{i}{\gamma_\mu p^\mu - m} \tag{5.17}$$

   for a fermion,

$$\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2} \tag{5.18}$$

   for a scalar, and

$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 + i\epsilon}\left[g_{\mu\nu} - (1-\alpha)\frac{k^\mu k^\nu}{k^2}\right] \tag{5.19}$$

   for a photon;

4. include arrows for fermion particle flow and consistently order terms in the same particle-flow direction (conventionally backwards in particle flow);
5. include spin factors of $u(p)$ [$\bar{u}(p)$] and $\bar{v}(p)$ [$v(p)$] for external initial-state [final-state] fermions and anti-fermions, respectively;

6. include polarization factors of ϵ^{μ} [$\epsilon^{\mu *}$] for external initial-state [final-state] photons;

7. and include a factor of -1 for each fermion loop.

The extension of these rules to the complete Electroweak theory is straightforward; one just needs to add the propagators for the electroweak gauge bosons and ghosts, and include all electroweak vertices (shown in Figs. 5.1, 5.2, 5.3, and 5.4 [11]). Note that the gauge parameter does not enter any of the interaction terms.

The Feynman rules allow the calculation of any electroweak process. However, loop diagrams appear beyond the leading order, giving a divergent integral over the loop momentum. To systematically remove such divergences and allow physical predication one needs a program of *renormalization*.

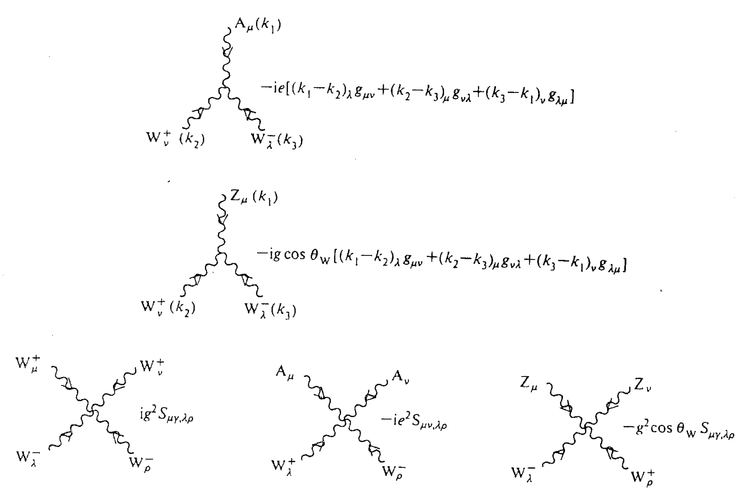


Figure 5.1 The self-interaction vertices of the gauge bosons resulting from the non-Abelian gauge group SU(2). The quartic vertices contain a common factor $S_{\mu\nu,\lambda\rho} = 2g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}$. Not shown is the $WWAZ$ vertex with a factor of $-ieg\cos\theta_W S_{\mu\nu,\lambda\rho}$.

Example: To familiarize ourselves with the Electroweak Feynman rules, we construct the matrix element for ZH production in e^+e^- collisions. The Feynman diagram for $e^+e^- \to ZH$ production in the s-channel contains an e^+e^-Z vertex, a Z-boson propagator, and a ZZH vertex (in the unitary gauge). The e^+e^-Z vertex contributes a factor:

$$v_{e^+e^-Z} = \frac{ig}{4\cos\theta_W}\gamma_\mu[(-1+4\sin^2\theta_W) - \gamma_5]. \tag{5.20}$$

The Z-boson propagator in the unitary gauge is

$$\Delta^Z_{\mu\nu} = \frac{-i}{k^2 - m_Z^2 + i\epsilon}\left[g_{\mu\nu} + \frac{k_\mu k_\nu}{m_Z^2}\right]. \tag{5.21}$$

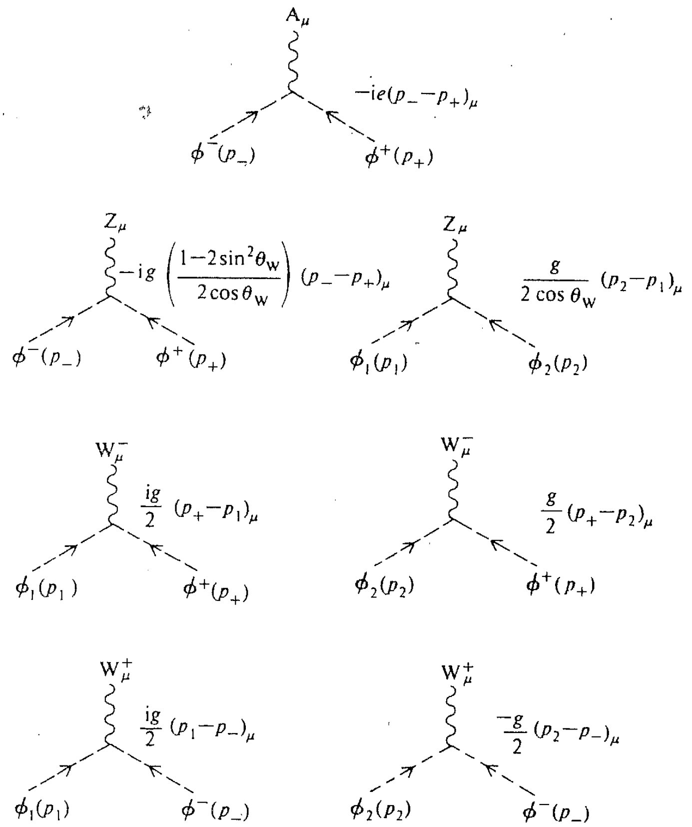


Figure 5.2 The $\phi\phi A_\mu$ vertices from the $A_\mu \phi \partial^\mu \phi$ terms in the Lagrangian.

The last piece of the matrix element is the ZZH vertex, whose factor is:

$$v^{\mu\nu}_{ZZH} = \frac{igm_Z}{\cos\theta_W} g^{\mu\nu}. \tag{5.22}$$

Putting the pieces together, and including the external fermion spinors and external gauge boson polarization (ϵ^μ), gives:

$$\frac{ig}{4\cos\theta_W}\bar{v}_2\gamma_\mu[(-1+4\sin^2\theta_W)-\gamma_5]u_1 \left[\frac{-i}{k^2 - m_Z^2 + i\epsilon}\left(g^{\mu\lambda} + \frac{k^\mu k^\lambda}{m_Z^2}\right)\right]\frac{igm_Z}{\cos\theta_W} g_{\lambda\nu}\epsilon^\nu. \tag{5.23}$$

Combining terms, we obtain:

$$\mathcal{M} = \frac{ig^2 m_Z}{4\cos^2\theta_W}\bar{v}_2\gamma_\mu\left[(-1+4\sin^2\theta_W)-\gamma_5\right]u_1\left(\frac{g^{\mu\nu} + k^\mu k^\nu/m_Z^2}{k^2 - m_Z^2 + i\epsilon}\right)\epsilon_\nu. \tag{5.24}$$

Higher-order corrections to the propagator will add a width to the propagator, $im_Z\Gamma_Z$.

Figure 5.3 The $\phi A_\mu A^\mu$ and $\phi\phi A_\mu A^\mu$ vertices from the $A_\mu A^\mu \phi\phi$ terms in the Lagrangian.

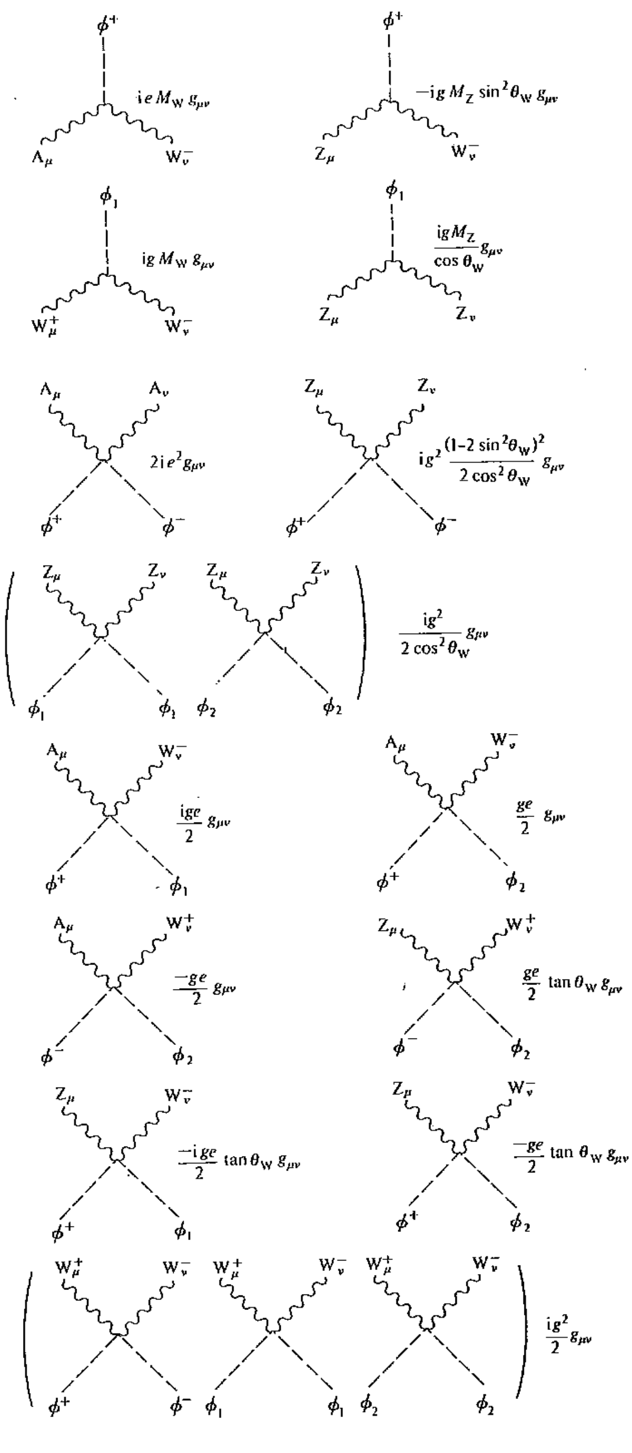


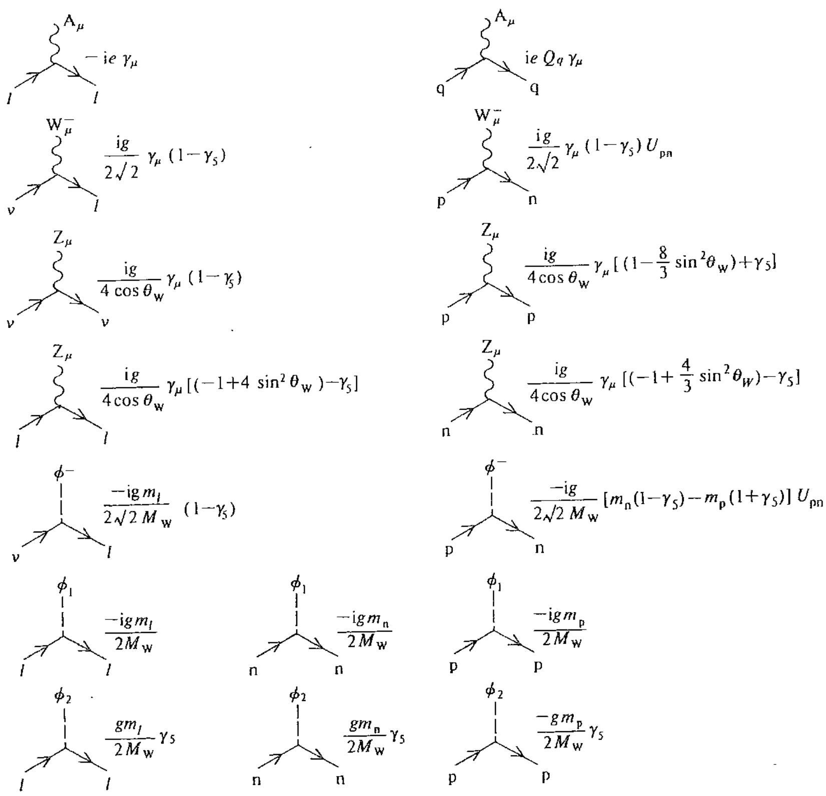
Figure 5.4 The fermion propagator, the fermion-fermion-gauge boson coupling and the fermion-fermion-scalar field coupling from the $i\bar{\psi}\gamma^\mu\partial_\mu\psi$, $\bar{\psi}A_\mu\psi$ and $\psi\phi\phi$ terms in the Lagrangian, respectively.

Inclusion of leptons and quarks

Propagator: $\xrightarrow{\;p\;}$ $\quad \dfrac{\mathrm{i}}{\not p - m_i + \mathrm{i}\varepsilon}$.

Vertices for leptons: $l = (\mathrm{e}, \mu, \tau)$, $\nu_l = (\nu_e, \nu_\mu, \nu_\tau)$

for quarks q: $\mathrm{p} = (\mathrm{u}, \mathrm{c}, \mathrm{t})$, $\mathrm{n} = (\mathrm{d}, \mathrm{s}, \mathrm{b})$ with the CKM mixing matrix U_{pn} of eqn (12.39).



There are also t- and u-channel diagrams, involving an e^+e^-H vertex and an electron propagator. These diagrams are negligible by comparison. To see this consider the e^+e^-H vertex,

$$v_{e^+e^-H} = \frac{-igm_e}{2m_W}. \tag{5.25}$$

The vertex will contribute a factor proportional to $m_e/m_W = (1.6 \times 10^5)^{-1}$. For completeness we can write down the corresponding matrix element,

$$\mathcal{M} = \frac{g^2 m_e}{8m_W \cos\theta_W} \bar{v}_2 \gamma_\mu [(-1 + 4\sin^2\theta_W) - \gamma_5] \frac{i}{\gamma_\alpha p^\alpha - m_e} u_1 \epsilon^\mu. \tag{5.26}$$

There will be two such terms with $p = p_1 - p_4$ and $p = p_1 - p_3$ in the denominator, corresponding to t- and u-channel production.

CHAPTER 6

SCALAR RENORMALIZATION

The calculation of physical processes is straightforward in the approximation of a single particle exchange. However, loop diagrams at the next order in perturbation theory lead to divergences in the calculations. In a *renormalizable* theory, the divergences can be removed by fixing the Lagrangian parameters such that the effective values (including loop corrections) are fixed by measurement. One can imagine splitting the Lagrangian parameters into arbitrary finite values and divergent *counterterms* to cancel the divergent loop factors. In practice, we set the finite value using a measurement at a particular mass scale and "run" the parameter to the mass scale of the desired process.

6.1 Renormalized Lagrangian

Recall the two-point Green's function for a scalar-field Lagrangian $\mathcal{L} = \partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{m_0^2}{2}\phi_0^2 - \frac{g_0}{4!}\phi_0^4$:

$$\begin{aligned} G(x_1, x_2) &= i\Delta_F(x_1 - x_2) - \frac{g_0}{2}\Delta_F(0)\int d^4z \Delta_F(x_1 - z)\Delta_F(x_2 - z) + ... \\ &= i\Delta_F(x_1 - x_2) - \int d^4z \Delta_F(x_1 - z)\left[\frac{g_0}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - m_0^2 + i\epsilon}\right] \times \\ & \quad \Delta_F(x_2 - z) + ... \end{aligned} \tag{6.1}$$

Replacing d^4q with $q^3 dq d\Omega$, it is clear that the loop integral will be quadratically divergent (i.e., $\propto q^2$ as $q \to \infty$). Its effect is to modify the pole of the propagator. To see this, define the correction term as $i\Delta_F[-i\Sigma(p^2)]i\Delta_F$, where $\Sigma(p^2) = i\Delta(0)$ for this loop, and p is the propagator momentum into and out of the vertex. Combining terms with consecutive loops at all orders gives

$$\begin{aligned} i\Delta_F(p) &= \frac{i}{p^2 - m_0^2 + i\epsilon} + \frac{i}{p^2 - m_0^2 + i\epsilon}[-i\Sigma(p^2)]\frac{i}{p^2 - m_0^2 + i\epsilon} + ... \\ &= \frac{i}{p^2 - m_0^2 + i\epsilon}\left[\frac{1}{1 + i\Sigma(p^2)\frac{i}{p^2 - m_0^2 + i\epsilon}}\right] \\ &= \frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\epsilon}. \end{aligned} \tag{6.2}$$

The function $\Sigma(p^2)$ can be extended to include all irreducible loops, i.e. all diagrams that cannot be separated into propagator subdiagrams by cutting one of the diagram lines (the reducible diagrams enter the infinite sum). This more general function $\Sigma(p^2)$ can be expanded in a Taylor series about a finite m^2:

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\frac{\partial}{\partial p^2}\Sigma(p^2)|_{p^2=m^2} + ... \tag{6.3}$$

The divergent contribution from the momentum-independent loop can be absorbed completely by the first term $\Sigma(m^2)$, and we can choose m_0^2 such that the divergence is cancelled and $m^2 = m_0^2 + \Sigma(m^2)$. Then the propagator is

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\frac{\partial}{\partial p^2}\Sigma(p^2)|_{p^2=m^2} + ... \tag{6.4}$$

where $\Sigma'(p^2) = \frac{\partial}{\partial p^2}\Sigma(p^2)|_{p^2=m^2}$. There is one more divergent loop contributing to the propagator, arising from the splitting of the initial line into three internal lines, which then reconnect at the outgoing line. This divergence is momentum-dependent, leading to a divergent $\Sigma'(p^2)$. We are thus forced to renormalize the fields to cancel the $Z \equiv [1 - \Sigma'(p^2)]$ factor and obtain a finite propagator. Writing the inverse of the propagator as a renormalized two-point function,

$$i\Gamma^{(2)}(p^2) = p^2 - m^2, \tag{6.5}$$

we can fix m^2 using a measurement at a scale $p^2 = \mu^2$. Note that m^2 is in general complex, producing a finite lifetime of the particle. We can similarly redefine the four-point coupling to remove the divergence arising from an intermediate loop; the resulting function is

$$i\Gamma^{(4)}(p^2) = g. \tag{6.6}$$

We can again fix g using a measurement at a scale $p^2 = \mu^2$. Then the Lagrangian can be split into a compontent corresponding to the physical mass and coupling, and a divergent component countering the divergences: the counterterms. Explicitly,

$$\mathcal{L} = \partial_\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4 + \delta Z\partial_\mu\phi\partial^\mu\phi - \frac{\delta m^2}{2}\phi^2 - \frac{\delta g}{4!}\phi^4, \tag{6.7}$$

where

$$\begin{aligned} \phi &= Z^{-1/2}\phi_0, \\ \delta Z &= Z - 1, \\ \delta m^2 &= m_0^2 Z - m^2, && (6.8) \\ \delta g &= g_0 Z^2 - g. && (6.9) \end{aligned}$$

For this Lagrangian, all divergent loop can be absorbed by measurements of the Lagrangian parameters. Such a Lagrangian is renormalizable. The presence of infinities arises from the naive extension of the theory to infinite momenta. In reality there are effects at high momentum scales that will modify the effective vertex and render it finite (specifically gravity and other new physics beyond the Standard Model). The counterterms can be thought of as integrating out these high-momentum degrees of freedom, leaving a predictive low-energy theory.

We can define additional Feynman rules for the counterterms to facilitate loop calculations. There is a propagator counterterm that comes with a factor of $i(p^2\delta z - \delta m^2)$ and a four-point counterterm with a factor of $-i\delta g$. We can then perform loop calculations, keeping track of the divergences using the process of *regularization*.

6.2 Regularization

There are a number of possibilities for regularizing an integral. The most straightforward is to simply define a cutoff Λ, with the integral reproduced when $\Lambda \to \infty$. However, some care needs to be taken to make this prescription gauge invariant. It is more common to use the *dimensional regularization* procedure, where the integral is performed in d dimensions. For $d < 4$, loop integrals are generally finite. We first study the simple case of the one-loop correction to the scalar propagator, then consider the more challenging computation of the one-loop correction to the four-point vertex, which requires a non-trivial renormalization condition.

6.2.1 Propagator loop corrections

Since the action is dimensionless and is the integral of the Lagrangian over space, the Lagrangian has d^{-1} space dimensions, or d mass dimensions. From the $\partial_\mu\phi_0\partial^\mu\phi_0$ term one can see that the ϕ field must have $d/2-1$ dimensions. In order to keep g dimensionless, a factor μ^{4-d} is multiplied to the ϕ^4 term. In d dimensions the divergent loop can be calculated using

$$\int \frac{d^d q}{(2\pi)^d(q^2+2pq-m^2)^\alpha} = \frac{(-1)^\alpha i}{(4\pi)^2}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)}\frac{1}{(p^2+m^2)^{\alpha-d/2}}, \tag{6.10}$$

where $\Gamma(\alpha)$ is the mathematical gamma function, which takes the value $(n-1)!$ for integer arguments n. For the loop integral in Eq. (6.1), $p = 0$ and $\alpha = 1$. Then

$$\begin{aligned} -i\Sigma(p^2) &= \frac{g\mu^{4-d}}{2}\int \frac{d^d q}{q^2-m^2+i\epsilon} \\ &= -i\frac{gm^2}{32\pi^2}\left[\frac{4\pi\mu^2}{m^2}\right]^{2-d/2}\Gamma(1-d/2). \end{aligned} \tag{6.11}$$

This expression is cancelled by the counterterm when the physical pole of the propagator is defined to be m^2; this is the *on-shell* scheme. Other schemes are possible, for example the *minimal subtraction* scheme where the counterterm removes only the divergent component of the loop correction. Then m^2 is related to the physical pole through finite loop correction terms.

We can explicitly separate the divergent component of the loop by expanding the gamma function in terms of the deviation from four dimensions, $\epsilon = 4 - d$. Then

$$\begin{aligned} \Gamma(1-d/2) &= \Gamma(\epsilon/2-1) \\ &= -\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \end{aligned} \tag{6.12}$$

using

$$\Gamma(\epsilon - n) = (-1)^{n-1}\frac{\Gamma(-\epsilon)\Gamma(1+\epsilon)}{\Gamma(n+1-\epsilon)} \tag{6.13}$$

and

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \tag{6.14}$$

where γ is the Euler-Mascheroni constant, 0.5772157. Now we use the expansion $a^\epsilon \approx 1 + \epsilon \ln a$ for the factor raised to $2 - d/2$ in the integral in Eq. (6.11). Then $-i\Sigma(p^2)$ becomes

$$-i\frac{gm^2}{32\pi^2}\left[\frac{-2}{\epsilon} - \gamma\right]\left[1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] = i\frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon} + 1 - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]. \tag{6.15}$$

The divergent term can be removed using a counterterm $\delta m^2 = -gm^2/(16\pi^2\epsilon)$ in the minimal subtraction (MS) scheme. In the modified minimal subtraction scheme ($\bar{M}S$), the additional γ and logarithmic factors are included in the counterterm, since these appear universally in dimensional regularization.

At $\mathcal{O}(g^2)$ an additional divergent contribution to the propagator depends on p^2 and is removed by two diagrams with counterterms: the four-point counterterm, where two lines are connected (a similar loop to one in the leading loop diagram for the propagator); and the momentum-dependent piece of the two-point counterterm (δZ).

6.2.2 Vertex loop correction

There is a loop in the middle of the s, t and u channel diagrams, with a similar divergent form to the propagator. This one-loop correction to the four-point coupling has the following form in the s-channel:

$$-i\Delta\Gamma^{(4)} = \frac{(-ig\mu^\epsilon)^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m^2 + i\epsilon}\frac{i}{(k - p_1 - p_2)^2 - m^2 + i\epsilon}. \tag{6.16}$$

The first step is to combine the denominator using the standard Feynman integral,

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}. \tag{6.17}$$

Then the denominator becomes

$$\begin{aligned} (k^2 - m^2)x + [(k - p_1 - p_2)^2 - m^2](1-x) = \\ k^2 - m^2 + [-2k(p_1+p_2) + 2p_1p_2 + p_1^2 + p_2^2](1-x) = \\ [k - (p_1+p_2)(1-x)]^2 + (p_1+p_2)^2x(1-x) - m^2. \end{aligned} \tag{6.18}$$

We shift the integrand to $k' = k - (p_1 + p_2)(1-x)$ and use the standard integral equation to get

$$-i\Delta\Gamma^{(4)} = \frac{(g\mu^\epsilon)^2}{2(2\pi)^d}\left[i\pi^{d/2}\frac{\Gamma(2-d/2)}{\Gamma(2)}\right]\int_0^1 dx \frac{1}{[-(p_1+p_2)^2 x(1-x)+m^2]^{2-d/2}}. \tag{6.19}$$

Writing $2 - d/2 = \epsilon/2$ and using $\Gamma(\epsilon/2) \approx \frac{2}{\epsilon} - \gamma$ and $a^\epsilon \approx 1 + \epsilon \ln a$,

$$-i\Delta\Gamma^{(4)} = \frac{ig^2}{2(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma\right]\left[1 - \frac{\epsilon}{2}\int_0^1 dx \ln\left(\frac{-(p_1+p_2)^2 x(1-x) + m^2}{4\pi\mu^2}\right)\right]. \tag{6.20}$$

The integral is soluble and can be expressed as an arctan or a ratio of logarithms. Leaving it in integral form, we have the s-channel result; the t and u channel diagrams add terms that differ only by $p_1 + p_2 \to p_1 - p_3$ and $p_1 + p_2 \to p_1 - p_4$. We can now define the following renormalization condition: the two-to-two scattering is measured with zero outgoing momentum. Then $s = (p_1 + p_2)^2 = 4m^2$ and $t = u = 0$. The counterterm is then:

$$\begin{aligned}\delta g &= \Delta\Gamma^{(4)}(s = 4m^2, t = u = 0)\\ &= \frac{3g^2}{(4\pi)^2\epsilon} - \frac{g^2}{2(4\pi)^2}\left[3\gamma + \int_0^1 dx \ln\left(\frac{-4m^2x(1-x)+m^2}{4\pi\mu^2}\right) + 2\ln\left(\frac{m^2}{4\pi\mu^2}\right)\right].\end{aligned}$$

Once we fix the coupling for final-state particles with zero momentum, we can calculate the effect of the loop on particles with non-zero momentum. The one-loop correction to the coupling is:

$$\begin{aligned}&[-i\Delta\Gamma^{(4)}(s,t,u)] - [-i\Delta^{(4)}(s=4m^2, t=u=0)] =\\ &-i\tfrac{g^2}{2(4\pi)^2}\left[\int_0^1 dx \ln\left(\tfrac{-sx(1-x)+m^2}{-4m^2x(1-x)+m^2}\right) + \ln\left(\tfrac{-tx(1-x)+m^2}{m^2}\right) + \ln\left(\tfrac{-ux(1-x)+m^2}{m^2}\right)\right].\end{aligned}$$

This is an important result: the effective coupling is no longer a constant, but increases logarithmically with momentum transfer. The mass of the particle in the loop sets the scale of the enhancement; the smaller the mass, the larger the correction at a given momentum transfer. We can parameterize the coupling as a function of momentum transfer, $g(Q^2)$; this is done systematically using *renormalization group* equations.

6.3 The renormalization group

We have seen that after renormalization we have finite loop corrections to interactions. However, since we are working within a perturbative framework, we must also ensure that the loop corrections do not become larger than the leading-order approximation. Since the correction has a logarithmic momentum dependence, divergent behavior is possible if the parameter is set at a scale far from the process under study. In complicated processes there may be different scales relevant to different vertices. Using the momentum dependence of the loop correction, one can define an effective coupling at the relevant scale of the process, such that higher order logarithmic terms are small.

A change in the renormalization scale will generally affect not only the coupling parameter, but also the mass parameter and the field normalization. The Green's function does not depend on the scale, so the vertex function generally satisfies $\mu d\Gamma^{(4)}/d\mu = 0$, where

μ is the renormalization scale. Generalizing to an n-point vertex and using appropriate partial derivatives gives

$$\left[-n\mu\frac{\partial}{\partial\mu}\ln\sqrt{Z_\phi}+\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}+\mu\frac{\partial m}{\partial\mu}\frac{\partial}{\partial m}\right]\Gamma^{(n)}=0. \tag{6.21}$$

We simplify the expression by defining

$$\begin{aligned}\alpha&=\mu\frac{\partial}{\partial\mu}\ln\sqrt{Z_\phi},\\ \beta&=\mu\frac{\partial g}{\partial\mu},\\ m\gamma&=\mu\frac{\partial m}{\partial\mu}.\end{aligned} \tag{6.22}$$

The result is the renormalization group equation,

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-n\alpha+m\gamma\frac{\partial}{\partial m}\right]\Gamma^{(n)}=0. \tag{6.23}$$

The parameters α, β, and γ must be scale-invariant and can therefore only depend on g. The renormalization group equation describes the changes in parameters with changes in renormalization procedure. We can see how this works in practice using the ϕ^4 theory. Set the vertex renormalization scale to $s=t=u=-Q^2$ and assume $Q^2\gg m^2$. Then the four-point function is:

$$-i\Gamma^{(4)}=-ig-ig^2[V(s)+V(t)+V(u)]-i\delta g, \tag{6.24}$$

where the counterterm can be written

$$\delta g=\frac{3g^2}{(4\pi)^2\epsilon}-\frac{3g^2}{2(4\pi)^2}\left[\gamma+\int_0^1 dx\ln\left(\frac{Q^2}{\mu^2}\right)+\ln\left(\frac{x(1-x)}{4\pi}\right)\right]. \tag{6.25}$$

We can now calculate the β function to $\mathcal{O}(g^2)$ with the renormalization group equation. We neglect the mass term, which does not have a momentum dependence, and the field renormalization, which has a mass dependence at $\mathcal{O}(g^2)$ and multiplies g in the four-point function. Using the renormalization group equation and taking derivatives with respect to Q gives

$$\begin{aligned}-i\beta&=i\left(-\frac{3g^2}{2(4\pi)^2}\times Q\frac{2Q}{Q^2}\right);\\ \beta&=\frac{3g^2}{(4\pi)^2}.\end{aligned} \tag{6.26}$$

The β function determines the momentum dependence of the coupling; since it is positive, the coupling g increases as the scale increases. The sign of the β function is generally used to determine the region of perturbativity of a given theory. We can put momenta on one side and renormalized couplings on the other side to get

$$\frac{\partial\mu}{\mu}=\frac{\partial g}{\left(\frac{3g^2}{16\pi^2}\right)}. \tag{6.27}$$

We now integrate both sides starting from the renormalization scale $\mu = Q$ to get an expression for the effective coupling g as a function of μ:

$$\begin{aligned} \ln(\mu/Q) &= \frac{16\pi^2}{3}\left(\frac{1}{g(Q)} - \frac{1}{g}\right) \\ \frac{1}{g} &= \frac{1}{g(Q)} - \frac{3}{16\pi^2}\ln(\mu/Q), \end{aligned} \tag{6.28}$$

or

$$g(\mu) = \frac{g(Q)}{1 - \frac{3}{16\pi^2} g_0(Q)\ln(\mu/Q)}. \tag{6.29}$$

The denominator decreases as μ increases; thus g increases as μ increases. Starting from a renormalized value of the coupling, its value at other scales can be used in a calculation to capture the effect of one-loop corrections at different vertices.

CHAPTER 7

QED RENORMALIZATION

The introduction of renormalization into QED converted many field-theory skeptics, particularly once it was used to predict the anomalous magnetic moment of the electron. Today the prediction and measurement of the anomalous moment are the most precise of any fundamental physics quantity. The most precise measurement is typically used to fix the value of the electromagnetic coupling constant in the renormalization of the Lagrangian.

Because QED is an unbroken Abelian gauge theory, it is among the simplest physical theories to renormalize. It involves the renormalization of fermion wavefunctions and masses, the photon wavefunction, and the fermion-fermion-photon vertex coupling.

7.1 QED divergences

In QED there are three divergent diagrams at the one-loop level that must be renormalized: photon emission and reabsorption in a fermion propagator, a fermion loop in the photon propagator, and photon emission by one fermion and absorption by the other fermion at a fermion-fermion-photon vertex. These divergences can be calculated using dimensional regularization, and then removed with counterterms in the Lagrangian.

The calculations use the QED Lagrangian in d dimensions:

$$\mathcal{L} = i\bar{\psi}_0\gamma^\mu\partial_\mu\psi_0 - m_0\bar{\psi}_0\psi_0 - e_0\mu^{2-d/2}A_0^\mu\bar{\psi}_0\gamma_\mu\psi_0 - \frac{1}{4}(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu})^2 - \frac{1}{2}(\partial_\mu A_0^\mu)^2, \tag{7.1}$$

where the last term is the gauge-fixing term with $\alpha = 1$. The parameter μ with dimensions of mass is introduced to keep e dimensionless, and is required because A_0^μ has dimension $d/2 - 1$ and ψ_0 has dimension $(d-1)/2$.

7.2 Fermion self energy

The one-loop fermion self-energy diagram contains a loop where a photon is emitted and then absorbed. In d dimensions, the expression for the loop in Feynman gauge is

$$-i\Sigma(p) \equiv (-ie_0)^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \gamma_\mu \frac{i}{\gamma_\alpha p^\alpha - \gamma_\alpha k^\alpha - m_0} \gamma_\nu \frac{-ig^{\mu\nu}}{k^2}, \tag{7.2}$$

where we are taking the charge to be that of the electron for simplicity. To calculate this factor, we first multiply the numerator and denominator by $\gamma_\alpha p^\alpha - \gamma_\alpha k^\alpha + m_0$ to move all gamma matrices to the numerator. We then separate the propagator factors with the Feynman integral

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}. \tag{7.3}$$

Applying this to the loop integral gives

$$\Sigma(p) = -ie_0^2 \mu^{4-d} \int_0^1 dz \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu(\gamma_\alpha p^\alpha - \gamma_\alpha k^\alpha + m_0)\gamma^\mu}{[(p-k)^2 z - m_0^2 z + k^2(1-z)]^2}. \tag{7.4}$$

The integration variable can now be isolated by defining it as $k' = k - pz$:

$$\Sigma(p) = -ie_0^2 \mu^{4-d} \int_0^1 dz \int \frac{d^d k'}{(2\pi)^d} \frac{\gamma_\mu[\gamma_\alpha p^\alpha(1-z) - \gamma_\alpha k'^\alpha + m_0]\gamma^\mu}{[k'^2 - m_0^2 + p^2 z(1-z)]^2}. \tag{7.5}$$

The term linear in k' will integrate to 0 so we have an integral of the familiar form

$$\int \frac{d^d q}{(2\pi)^d (q^2 - m_0^2)^\alpha} = \frac{(-1)^\alpha i}{(4\pi)^2} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \left(\frac{4\pi}{m_0^2}\right)^{\alpha - d/2}. \tag{7.6}$$

For the loop integral in Eq. (7.5), $p = 0$ and $\alpha = 2$. The loop factor becomes

$$e_0^2 \mu^{4-d} \frac{\Gamma(2 - d/2)}{(4\pi)^2} \int_0^1 dz \gamma_\mu [\gamma_\alpha p^\alpha (1-z) + m_0] \gamma^\mu \left[\frac{m_0^2 z - p^2 z(1-z)}{4\pi}\right]^{d/2-2}. \tag{7.7}$$

Now we use $\gamma_\mu \gamma^\mu = d$, $\gamma_\mu \gamma_\nu \gamma^\mu = (2-d)\gamma_\nu$, and $\epsilon = 4 - d$ to get

$$\begin{aligned}
\Sigma(p) &= \frac{e_0^2}{16\pi^2} \Gamma(\epsilon/2) \int_0^1 dz \{\epsilon[\gamma_\alpha p^\alpha (1-z) - m_0] + 4m_0 - 2\gamma_\alpha p^\alpha (1-z)\} \times \\
&\quad \left(\frac{m_0^2 z - p^2 z(1-z)}{4\pi\mu^2}\right)^{-\epsilon/2} \\
&= \frac{e^2}{8\pi^2 \epsilon} (-\gamma_\alpha p^\alpha + 4m_0) + \frac{e_0^2}{16\pi^2} \{\gamma_\alpha p^\alpha (1+\gamma) - 2m_0(1+2\gamma) + \\
&\quad 2\int_0^1 dz [\gamma_\alpha p^\alpha (1-z) - 2m_0] \ln\left(\frac{m_0^2 z - p^2 z(1-z)}{4\pi\mu^2}\right)\}.
\end{aligned} \tag{7.8}$$

where we have used $\Gamma(2-d/2) \approx \frac{2}{\epsilon} - \gamma$. The divergent term has two components that can be removed with if we separate out the following pieces of the bare fermion terms in the Lagrangian:

$$\Delta\mathcal{L}_{ct-\psi} = -\frac{ie^2}{8\pi^2\epsilon}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{me^2}{2\pi^2\epsilon}\bar{\psi}\psi. \tag{7.9}$$

These counterterms correspond to the renormalization of the fermion wave function and of the fermion mass. It is more physically intuitive to work in the "on-shell" scheme, where the Lagrangian is split into the finite physical field and mass terms with ψ and m, and counterterms $i\delta Z\bar{\psi}\gamma^\mu\partial_\mu\psi$ and $-i\delta m\bar{\psi}\psi$ ($\psi = \psi_0/\sqrt{\delta Z}$, $\delta m = \sqrt{\delta Z}m_0 - m$). The counterterm propagator is $i(\delta Z\gamma^\mu p_\mu - \delta m)$, with the on-shell constraints

$$\begin{aligned} m\delta Z - \delta m - \Sigma(m) &= 0, \\ \delta Z - \frac{\partial}{\partial p_\mu \gamma^\mu}\Sigma(\gamma^\mu p_\mu|_{\gamma^\mu p_\mu = m} &= 0. \end{aligned} \tag{7.10}$$

We can use the one-loop expression for $\Sigma(p)$ to solve for the counterterms δm and δZ.

7.3 Vacuum polarization

There is a divergent loop in the photon propagator similar to that of the fermion propagator. It occurs when the photon splits into a fermion-antifermion pair, which subsequently annihilates into a photon. It is referred to as the vacuum polarization because the correction to the t-channel exchange of a photon can be viewed as an interaction with a fermion-antifermion pair appearing from the vacuum and subsequently disappearing into the vacuum. The polarization screens the "bare" electromagnetic charge in analogy to charge screening in a dielectric medium.

The vacuum polarization contribution to the photon propagator can be represented as

$$i\Delta_{F\mu\nu} = i\Delta^0_{F\mu\nu} - \Delta^0_{F\mu\alpha}i\Pi^{\alpha\beta}\Delta^0_{F\beta\nu}$$

Higher-order corrections do not affect the pole of the propagator, as can be seen by expressing it in the Landau gauge as

$$\begin{aligned} i\Delta_{F\mu\nu} &= \frac{-i}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(g_{\mu\alpha} - \frac{q_\mu q_\alpha}{q^2}\right)i\Pi^{\alpha\beta}\frac{-i}{q^2}\left(g_{\beta\nu} - \frac{q_\beta q_\nu}{q^2}\right) + ... \\ &= i\Delta^0_{F\mu\nu}[1 + \Pi + ...] \\ &= \frac{i\Delta^0_{F\mu\nu}}{1 - \Pi}. \end{aligned} \tag{7.11}$$

where $\Pi^{\mu\nu} = q^2(g^{\mu\nu} - q^\mu q^\nu/q^2)\Pi$. In a general gauge the second (gauge-dependent) term in the propagator will be modified by higher-order corrections.

The one-loop correction is

$$i\Pi_{\mu\nu}(k) = -\mu^{4-d}(ie_0)^2\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}\left[\gamma_\mu\frac{i}{\gamma_\alpha p^\alpha - m_0}\gamma_\nu\frac{i}{\gamma_\beta p^\beta - \gamma_\beta k^\beta - m_0}\right], \tag{7.12}$$

where the minus sign is due to the fermion loop. To evaluate this integral, we move all gamma matrices to the numerator by multiplying numerator and denominator by the same factor, and we separate the propagators with the Feynman integral:

$$i\Pi_{\mu\nu} = -e_0^2\mu^{4-d}\int_0^1 dz \int \frac{d^d p}{(2\pi)^d}\frac{\mathrm{Tr}[\gamma_\mu(\gamma_\alpha p^\alpha + m_0)\gamma_\nu(\gamma_\beta p^\beta - \gamma_\beta k^\beta + m_0)]}{\{(p^2-m_0^2)z + [(p-k)^2 - m_0^2](1-z)\}^2}. \quad (7.13)$$

Again redefining the integrand to $p' = p - kz$ and recalling that integrals over terms linear in p' give zero, the numerator becomes

$$[p'^\alpha p'^\beta - k^\alpha k^\beta z(1-z)]\mathrm{Tr}(\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta) + m_0^2\mathrm{Tr}(\gamma_\mu\gamma_\nu). \quad (7.14)$$

The traces can be evaluated in d dimensions using

$$\begin{aligned}\mathrm{Tr}(\gamma_\mu\gamma_\nu) &= f(d)g_{\mu\nu},\\ \mathrm{Tr}(\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta) &= f(d)(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha}),\end{aligned} \quad (7.15)$$

where $f(d)$ is some function with the property $f(4) = 4$. The numerator becomes

$$f(d)\{2p'_\mu p'_\nu - 2z(1-z)(k_\mu k_\nu - k^2 g_{\mu\nu}) - g_{\mu\nu}[p'^2 - m_0^2 + k^2 z(1-z)]\}. \quad (7.16)$$

Putting this back into the integral gives

$$\begin{aligned} i\Pi_{\mu\nu} &= -e_0^2\mu^{4-d}f(d)\int_0^1 dz\int\frac{d^dp}{(2\pi)^d}\{\frac{2p_\mu p_\nu}{[p^2-m_0^2+k^2z(1-z)]^2} - \\ &\quad \frac{2z(1-z)(k_\mu k_\nu - k^2 g_{\mu\nu})}{[p^2-m_0^2+k^2z(1-z)]^2} - \frac{g_{\mu\nu}}{[p^2-m_0^2+k^2z(1-z)]}\}.\end{aligned} \quad (7.17)$$

The last term is again of the form of Eq. (6.10), and it can be shown that the first term gives the same result [$p_\mu p_\nu$ simply contributes $m_0^2 - k^2z(1-z)g_{\mu\nu}/2$], so the two terms cancel. For the middle term we again use Eq. (6.10) to obtain

$$i\Pi_{\mu\nu} = \frac{ie_0^2}{2\pi^2}\left(k_\mu k_\nu - g_{\mu\nu}k^2\right)\left\{\frac{1}{3\epsilon} - \frac{\gamma}{6} - \int_0^1 dz(1-z)z\ln\left[\frac{m_0^2 - k^2z(1-z)}{4\pi\mu^2}\right]\right\}. \quad (7.18)$$

There is a divergent term and several finite terms. In the minimal subtraction renormalization scheme, the counterterm is only the divergent part of $-\Pi_{\mu\nu}$:

$$\Delta\mathcal{L}_{ct-A} = \frac{e^2}{24\pi^2\epsilon}F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{12\pi^2\epsilon}(\partial_\mu A^\mu)^2. \quad (7.19)$$

In the on-shell scheme, the propagator is fixed at $k^2 = 0$ such that the sum of $\Pi_{\mu\nu}$ and the counterterm factor δA ($= -\frac{e^2}{6\pi^2\epsilon}$ in the minimal subtraction scheme) give zero. The factor δA renormalizes the photon field according to $A_\mu = A_{\mu 0}/\sqrt{\delta A}$. At a given k^2 the correction is

$$i[\Pi_{\mu\nu}(k^2) - \Pi_{\mu\nu}(0)] = \frac{ie^2}{2\pi^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\int_0^1 dz(1-z)z\ln\left(1 - \frac{k^2z(1-z)}{m^2}\right). \quad (7.20)$$

For small spacelike momentum transfer ($-k^2 \ll m^2$ and $k^2 < 0$), the logarithm can be approximated by $\ln(1+x) \approx x$, giving

$$i[\Pi_{\mu\nu} - \Pi_{\mu\nu}(0)] = \frac{ie^2}{60\pi^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left(-\frac{k^2}{m^2}\right). \quad (7.21)$$

We can also consider the limit of high momentum transfer ($-k^2 \gg m^2$ and $k^2 < 0$, where the integral gives

$$i[\Pi_{\mu\nu} - \Pi_{\mu\nu}(0)] = \frac{ie^2}{12\pi^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left[\ln\left(-\frac{k^2}{5m^2}\right) - \frac{5}{3}\right]. \tag{7.22}$$

Recall that we wrote the connection as $\omega = -iQeA$ in the covariant derivative. When the field A is renormalized via $A_\mu = A_{\mu 0}/\sqrt{\delta A}$, the coupling factor e must compensate with $e = e_0\sqrt{\delta A}$. This is because the covariant derivative can be written $i\bar{\psi}D_\mu\psi$, so it receives just the renormalization factor from the wavefunction $\psi = \psi_0/\sqrt{deltaZ}$. We will see this explicitly when we calculate the vertex loop correction. Because of this compensating effect on the coupling, we can view the propagator loop as a correction to the electromagnetic coupling, and associate the k^2 dependence with the coupling.

To see the impact of the loop on the photon propagator, consider the exchange of a photon between a pair of electrons. The matrix element will have the structure

$$(-ie)\frac{i\Delta_F^{\mu\nu}}{1 - \Pi(k^2)}(-ie). \tag{7.23}$$

At high momentum transfer, $|k^2| \gg m^2$, the effective electromagnetic coupling is

$$\alpha(k^2) = \frac{\alpha(0)}{1 - \frac{\alpha}{3\pi}\ln\left(\frac{-k^2}{m^2 e^{5/3}}\right)} \tag{7.24}$$

where $\alpha = e^2/(4\pi)$. We see that the electromagnetic coupling increases in strength logarithmically with increasing momentum transfer. Higher momentum probes shorter distances and the interaction is therefore more sensitive to the bare charge of the electron. The typical interpretation is that the loops represent electron-positron pairs in the vacuum that screen the charges of the interacting electrons.

Application: A practical example of the impact of a running coupling is its correction to the Lamb shift of the energy states in the hydrogen atom, whose high experimental precision provides sensitivity to higher-order loop effects. At leading order, the Dirac equation of an electron orbiting a nucleus can be written

$$\left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m - \frac{Z\alpha}{r}\right)\psi = E\psi. \tag{7.25}$$

The solution to the equation gives a series of hypergeometric functions with energy eigenvalues

$$E_{nj} = m\left[1 - \frac{Z^2\alpha^2}{2n^2} - \frac{(Z^2\alpha^2)^2}{2n^4}\left(\frac{n}{j+1/2} - 3/4\right) + ...\right]. \tag{7.26}$$

According to this expression, the states $2P_{1/2}$ and $2S_{1/2}$ should have the same energy, since they have the same total spin $j = 1/2$ and radial number $n = 2$. However, their different values of orbital momentum ($l = 0, 1$) lead to slightly different energy levels when higher order corrections are taken into account. The higher order corrections affect the $l = 0$ state, whose wavefunction is substantial at low r ($< m_e^{-1}$) where the corrections are relevant. The energy splitting of these states is known as the *hyperfine* structure of the hydrogen energy levels.

The corrections to the potential from loops in the photon propagator can be expressed as

$$\Delta V(r) = Q_e Q_p \int \frac{d^3q}{(2\pi)^3} e^{iqr} \frac{\Pi(q^2)}{q^2}. \tag{7.27}$$

Inserting the one-loop correction relevant at low momentum gives

$$\begin{aligned} \Delta V(r) &= (-e^2) \frac{e^2}{60m^2\pi^2(2\pi)^3} \int d^3q e^{iqr} \\ &= \frac{-e^4}{60m^2\pi^2} \delta(r). \end{aligned} \tag{7.28}$$

The corresponding shift in the energy state is

$$\begin{aligned} \Delta E &= |\psi(0)|^2 \int d^3 \Delta V(r) \\ &= -\frac{1}{\pi} \left(\frac{\alpha m}{2}\right)^3 \frac{e^4}{60m^2\pi^2} \\ &= -\frac{\alpha^5 m}{30\pi}, \end{aligned} \tag{7.29}$$

where we have used the wavefunction at the origin

$$\psi(0) = \frac{2}{\sqrt{4\pi}} \left(\frac{\alpha m}{n}\right)^{3/2}. \tag{7.30}$$

We can evaluate the correction numerically using $\alpha = 1/137$ and $m = 5.11 \times 10^5$ eV:

$$\Delta E = -1.12 \times 10^{-7} \text{ eV}, \tag{7.31}$$

corresponding to a frequency shift of $\omega = -27.2$ MHz. This is the *Uehling* term in the hyperfine splitting, and is a small fraction of the total predicted splitting of 1051 MHz. However, its inclusion in the prediction provides good agreement with the measured total splitting of 1054 MHz.

Useful fact: Since QED is a theory based on a U(1) gauge symmetry, the photon does not interact with itself at tree-level. But an intermediate fermion loop can cause photon self-interactions, though only with an even number of photon lines. The exclusion of odd numbers of lines is a result of the charge conjugation symmetry of QED, and can be seen as follows.

The matrix element for the three-photon vertex has two components corresponding to the two directions of the internal lines:

$$\begin{aligned} \Pi^{\mu\nu\lambda} &= (iQ_f e\mu^{2-d/2})^3(-1) \int \frac{d^dq}{(2\pi)^d} \\ & \mathrm{Tr}[\gamma^\mu \frac{i}{\gamma_\alpha q^\alpha + \gamma_\alpha k_2^\alpha - m_f} \gamma^\nu \frac{i}{\gamma_\beta q^\beta - m_f} \gamma^\lambda \frac{i}{\gamma_\rho q^\rho - \gamma_\rho k_1^\rho - m_f} + \\ & \gamma^\mu \frac{i}{-\gamma_\alpha q^\alpha + \gamma_\alpha k_1^\alpha - m_f} \gamma^\lambda \frac{i}{-\gamma_\beta q^\beta - m_f} \gamma^\nu \frac{i}{-\gamma_\rho q^\rho - \gamma_\rho k_2^\rho - m_f} \end{aligned} \tag{7.32}$$

where $\mathcal{M} = \Pi^{\mu\nu\lambda}\epsilon^\mu(-k_1 - k_2)\epsilon^{*\nu}(k_2)\epsilon^{*\lambda}(k_1)$, the factor of -1 comes from the internal fermion loop, and the momentum of the fermion connecting the ν and λ vertices is q. The sum of these terms is zero. Write the sum as:

$$\begin{aligned}\Pi^{\mu\nu\lambda} &= (iQ_f e\mu^{2-d/2})^3(-1)\int \frac{d^dq}{(2\pi)^d} \\ &\quad \mathrm{Tr}[\gamma^\mu \frac{i(\gamma_\alpha q^\alpha + \gamma_\alpha k_2^\alpha + m_f)}{(q+k_2)^2 - m_f^2}\gamma^\nu \frac{i(\gamma_\beta q^\beta + m_f)}{q^2 - m_f^2}\gamma^\lambda \frac{i(\gamma_\rho q^\rho - \gamma_\rho k_1^\rho + m_f)}{(q-k_1)^2 - m_f^2} + \\ &\quad \gamma^\mu \frac{i(-\gamma_\alpha q^\alpha + \gamma_\alpha k_1^\alpha + m_f)}{(-q+k_1)^2 - m_f^2}\gamma^\lambda \frac{i(-\gamma_\beta q^\beta + m_f)}{q^2 - m_f^2}\gamma^\nu \frac{i(-\gamma_\rho q^\rho - \gamma_\rho k_2^\rho + m_f)}{(-q-k_2)^2 - m_f^2}].\end{aligned}$$

The denominators of the two terms are the same, so we add the numerators. The terms with even factors of momenta have an odd number of gamma matrices, so their traces are zero. For the remainder, the momenta in the sum have opposite sign but the gamma matrices are in a different order. By inserting charge conjugation factors $C^\dagger C = 1$ (where $C\gamma_\mu C^\dagger = -\gamma_\mu^T$), the signs of the terms even in the number of gamma matrices stay the same and the gamma matrices can be put in the same order. The sum then cancels.

7.4 Vertex correction

To calculate the loop correction to the vertex, we start with the QED Lagrangian in d dimensions:

$$\mathcal{L} = i\bar{\psi}_0\gamma^\mu\partial_\mu\psi_0 - e_0\mu^{2-\frac{d}{2}}A_0^\mu\bar{\psi}_0\gamma_\mu\psi_0 - m_0\bar{\psi}_0\psi_0 - \frac{1}{4}(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu})^2 - \frac{1}{2}(\partial_\mu A_0^\mu)^2, \tag{7.33}$$

where the last term is the gauge-fixing term with $\alpha = 1$ (Feynman gauge), and μ is a parameter with dimensions of mass.

The Feynman rules give the following expression for the loop at the fermion-fermion-photon vertex:

$$-ie\mu^{2-\frac{d}{2}}\Lambda_\mu \equiv (-ie\mu^{2-\frac{d}{2}})^3 \int \frac{d^dk}{(2\pi)^d}\frac{-ig_{\lambda\nu}}{k^2}\gamma^\lambda \frac{i}{\gamma_\alpha(p'-k)^\alpha - m}\gamma_\mu \frac{i}{\gamma_\beta(p-k)^\beta - m}\gamma^\nu, \tag{7.34}$$

where we have switched to the renormalized parameters that will be used to split up the Lagrangian once the divergent factor is extracted.

We remove the gamma matrices from the denominator with an appropriate multiplication of numerator and denominator, and separate the terms in the denominator using the two-parameter Feynman integral:

$$\frac{1}{abc} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{[a(1-x-y) + bx + cy]^3}. \tag{7.35}$$

The one-loop vertex becomes

$$\Lambda_\mu = \frac{-2ie^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx \int_0^{1-x} dy \int d^dk \frac{\gamma_\nu[\gamma_\alpha(p'^\alpha - k^\alpha) + m]\gamma_\mu[\gamma_\beta(p^\beta - k^\beta) + m]\gamma^\nu}{[k^2 - m^2(x+y) - 2k(px + p'y) + p^2x + p'^2y]^3}. \tag{7.36}$$

We next remove the term linear in k from the denominator by shifting the integration over k to an integration over $k - px - p'y$:

$$\Lambda_\mu = -\frac{2ie^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx \int_0^{1-x} dy \int d^dk \qquad (7.37)$$
$$\frac{\gamma_\nu\{\gamma_\alpha[p'(1-y) - px - k]^\alpha + m\}\gamma_\mu\{\gamma_\beta[p(1-x) - p'y - k]^\beta + m\}\gamma^\nu}{[k^2 - m^2(x+y) + p^2x(1-x) + p'^2y(1-y) - 2pp'xy]^3}.$$

From counting the factors of k in the numerator and the denominator, we see that the divergent term is the one with $k^\alpha k^\beta$ in the numerator. Separating out this term and using the general expression for an integral over $k^\mu k^\nu$,

$$\int d^dk \frac{k^\mu k^\nu}{(k^2 - m^2)^n} = (-1)^{n-2}\frac{i\pi^{d/2}}{\Gamma(n)}\frac{1}{(m^2)^{n-2-d/2}}\frac{g^{\mu\nu}}{2}\Gamma(n-1-d/2), \qquad (7.38)$$

we get

$$\Lambda_\mu^{div} = \frac{e^2}{2}\mu^{4-d}\frac{1}{(4\pi)^{d/2}}\Gamma(2-d/2)\int_0^1 dx\int_0^{1-x} dy \times$$
$$\frac{\gamma_\nu\gamma_\alpha\gamma_\mu\gamma^\alpha\gamma^\nu}{[m^2(x+y)^2]^{2-d/2}}, \qquad (7.39)$$

where we have simplified the denominator using $p^2 = p'^2 = m_0^2$ and $(p - p')^2 = q^2 = 0$, with q_μ the external photon momentum. The numerator can be calculated using expressions for gamma matrices in d dimensions,

$$\gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma\gamma^\nu = (2-d)\gamma_\rho\gamma_\mu\gamma_\sigma + 2(\gamma_\mu\gamma_\sigma\gamma_\rho - \gamma_\rho\gamma_\sigma\gamma_\mu). \qquad (7.40)$$

This gives a numerator of $(2-d)^2\gamma_\mu$, or equivalently $(\epsilon - 2)^2\gamma_\mu$. Combining the components of the divergent piece (including a factor of 1/2 from the Feynman integral) gives

$$\Lambda_\mu = \frac{e_0^2}{8\pi^2\epsilon}\gamma_\mu + F(p,p')\gamma_\mu, \qquad (7.41)$$

where $F(p,p')$ is finite. The divergence can be removed with a counterterm $-\Lambda_\mu$ applied to the interaction term in the Lagrangian (in analogy with the scalar vertex renormalization):

$$\Delta\mathcal{L}_{ct-vtx} = \frac{e^2}{8\pi^2\epsilon}\bar{\psi}\gamma^\mu A_\mu\psi. \qquad (7.42)$$

Now recall the counterterm associated with the fermion propagator:

$$\Delta\mathcal{L}_{ct-f} = -\frac{ie^2}{8\pi^2\epsilon}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{me^2}{2\pi^2\epsilon}\bar{\psi}\psi. \qquad (7.43)$$

We see that the renormalization of the propagator has the same divergent factor as the vertex. This is in fact required to maintain the covariant form of the Lagrangian, and has been explicitly demonstrated through the Ward identities. The loop is the same in both diagrams; the vertex diagram simply involves the radiation of a photon along the propagator (i.e. ∂_μ is replaced with A_μ). Including counterterms, the full QED Lagrangian becomes:

$$
\begin{aligned}
\mathcal{L} &= \left(1 - \frac{e^2}{8\pi^2\epsilon}\right) i\bar{\psi}\gamma^\mu D_\mu \psi - \left(1 - \frac{e^2}{2\pi^2\epsilon}\right) m\bar{\psi}\psi - \\
&\quad \left(1 - \frac{e^2}{6\pi^2\epsilon}\right) \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2\right] \\
&= i\bar{\psi}_0 \gamma^\mu D_\mu \psi_0 - m_0 \bar{\psi}_0 \psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2}(\partial_\mu A_0^\mu)^2. \qquad (7.44)
\end{aligned}
$$

The form of the Lagrangian is thus maintained at one-loop level by renormalizing the fermion and photon wavefunctions, the fermion mass, and the electromagnetic charge.

CHAPTER 8

ELECTRON MAGNETIC MOMENT

The measurement of the anomalous magnetic moment of the electron is among the greatest successes of QED. Within the context of the Electroweak theory, it determines the electromagnetic coupling constant α_{EM}, fixing one of the three required inputs to the theory for describing interactions between fermions and gauge bosons. The most precise measurement has an uncertainty of 0.28 parts per trillion [21]; it can be used to fix α_{EM} to 0.25 parts per billion. This value of the electromagnetic coupling is confirmed by other measurements to an accuracy of 0.66 parts per billion [22]. These are the most precise tests of quantum field theory.

To achieve the required level of theoretical accuracy, five orders in perturbation theory must be calculated. At next-to-leading order, divergent loop diagrams appear and must be removed with a renormalization procedure. The divergence relevant for the anomalous magnetic moment is the loop in the fermion-fermion-photon vertex.

8.1 Leading order

The magnetic moment of a fermion arises non-relativistically from the interaction between the spin and the magnetic field. This can be seen by applying the covariant Hamiltonian operator to an energy eigenstate:

$$[(\gamma^\mu D_\mu)^2/2m + m]\psi_n = E_n\psi_n. \tag{8.1}$$

In a momentum basis, the operator D_μ is $p_\mu - eA_\mu$, so the equation becomes

$$\left\{\frac{1}{2m}\frac{1}{2}\left(\{\gamma^\mu,\gamma^\nu\}+[\gamma^\mu,\gamma^\nu]\right)(p_\mu-eA_\mu)(p_\nu-eA_\nu)+m\right\}\psi_n = E_n\psi_n. \tag{8.2}$$

Using the Dirac commutation relations $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$ and $[\gamma_\mu,\gamma_\nu] = -2i\sigma_{\mu\nu}$, with $\sigma_{ij} = \epsilon_{ijk}\sigma^k \otimes I$,

$$\left[\frac{1}{2m}(p_\mu - eA_\mu)^2 - \frac{i\sigma^{\mu\nu}}{2m}(p_\mu - eA_\mu)(p_\nu - eA_\nu) + m\right]\psi_n = E_n\psi_n. \tag{8.3}$$

Now use the operator form of $p_\mu (= -i\partial_\mu)$ and the definition of the magnetic field $B_i = \epsilon_{ijk}\partial_j A_k$ to get

$$\left\{\frac{1}{2m}\left[(p-eA)^2 - e\sigma B\right] + m\right\}\psi_n = E_n\psi_n. \tag{8.4}$$

Comparing the $e\sigma B$ term to the expression for a magnetic moment of a particle with angular momentum l ($\mu = el/2m$, with $E = \mu B$) we see that the effective magnetic moment of the electron is $e\sigma/2m$. In terms of the electron spin $s = \sigma/2$, the magnetic moment is $g_e es/2m$, where $g_e = 2$. The factor g_e is known as the "anomalous magnetic moment", since it is twice as large as would be expected if spin were simply an angular momentum.

8.2 Next-to-leading order

The non-relativistic expression for the magnetic moment is affected by the higher order corrections of QED. Including the external spinors, one can expand the leading-order vertex factor using the Gordon identity,

$$\bar{u}(p')\gamma_\mu u(p) = \frac{1}{2m}\bar{u}(p')\left[(p+p')_\mu + i\sigma_{\mu\nu}(p-p')^\nu\right]u(p), \tag{8.5}$$

where the second factor gives the magnetic moment. The next order of corrections is simply the convergent component of the vertex factor contribution given by Eq. (7.37). The momentum dependence in the divergent component does not contribute to the anomalous magnetic moment after renormalization. Using the equation with renormalized quantities, we can apply the Dirac equation to replace $\gamma_\alpha p^\alpha$ ($\gamma_\alpha p'^\alpha$) with m when acting on the right (left) and move momentum factors using $\gamma_\alpha p^\alpha \gamma_\mu = 2p_\mu - \gamma_\mu\gamma_\alpha p^\alpha$. The convergent part of the integral is then

$$\begin{aligned}\Lambda_\mu^{conv} &= -\frac{2ie^2}{(2\pi)^4}\int_0^1 dx\int_0^{1-x} dy\int d^4k\,\times \\ &\quad \gamma_\nu\frac{\{[\gamma_\alpha p^\alpha(1-y) - x\gamma_\alpha p^\alpha + m]\gamma_\mu[\gamma_\beta p^\beta(1-x) - y\gamma_\beta p'^\beta + m]}{[k^2 - m^2(x+y)^2]^3}\gamma^\nu.\end{aligned}$$

We now integrate over k:

$$\begin{aligned}\Lambda_\mu^{conv} &= -\frac{2ie^2}{(2\pi)^4}\int_0^1 dx\int_0^{1-x} dy\frac{-i\pi^2}{2m^2(x+y)^2}\,\times \\ &\quad [2p'_\nu(1-y) - x\gamma_\nu\gamma_\alpha p^\alpha + y\gamma_\nu m]\gamma_\mu[2p^\nu(1-x) - y\gamma_\beta p'^\beta\gamma^\nu + x\gamma^\nu m].\end{aligned}$$

Multiplying the pairs of brackets gives nine terms:

$$\begin{array}{lll}
(a) & 4p'_\nu p^\nu(1-y)(1-x)\gamma_\mu & = 4m^2(1-y)(1-x)\gamma_\mu; \\
(b) & -2p'_\nu(1-y)\gamma_\mu y\gamma_\beta p'^\beta\gamma^\nu & = -2m^2y(1-y)\gamma_\mu; \\
(c) & 2p'_\nu(1-y)\gamma_\mu x\gamma^\nu m & = (2p'_\mu - m\gamma_\mu)2mx(1-y); \\
(d) & -x\gamma_\nu\gamma_\alpha p^\alpha\gamma_\mu 2p^\nu(1-x) & = -2m^2x(1-x)\gamma_\mu; \\
(e) & x\gamma_\nu\gamma_\alpha p^\alpha\gamma_\mu y\gamma_\beta p'^\beta\gamma^\nu & = -2m^2xy\gamma_\mu; \\
(f) & -x\gamma_\nu\gamma_\alpha p^\alpha\gamma_\mu x\gamma^\nu m & = -4x^2p_\mu m; \\
(g) & y\gamma_\nu m\gamma_\mu 2p^\nu(1-x) & = (2p^\mu - m\gamma_\mu)2my(1-x); \\
(h) & -y\gamma_\nu m\gamma_\mu y\gamma_\beta p'^\beta\gamma^\nu & = -4y^2p'_\mu m; \\
(i) & y\gamma_\nu m\gamma_\mu x\gamma^\nu m & = -2xym^2\gamma_\mu.
\end{array} \tag{8.6}$$

Summing the terms gives

$$\begin{gathered}
\{4m^2(1-y)(1-x) - 2m^2(x+y)[(1-y)+(1-x)] - 4xym^2\}\gamma_\mu + \\
4(x(1-y)-y^2)mp'_\mu + 4(y(1-x)-x^2)mp_\mu = \\
2m^2\gamma_\mu[2(1-2x-2y)+(x^2+y^2+2xy)] + \\
4m[(x-xy-y^2)p'_\mu + (y-yx-x^2)p_\mu].
\end{gathered} \tag{8.7}$$

We next integrate Eq. (8.6) over y, focusing on the p'_μ and p_μ terms:

$$\begin{aligned}
& -\frac{\pi^2e^2}{(2\pi)^4m^2}\int_0^1 dx\int_0^{1-x} dy\frac{4m[(x-xy-y^2)p'_\mu + (y-yx-x^2)p_\mu]}{(x+y)^2} \\
= \;& -\frac{e^2}{4\pi^2m}\int_0^1 dx\int_0^{1-x} dy\left(-1+\frac{x^2+xy+x}{(x+y)^2}\right)p'_\mu + \frac{y(1-x)-x^2}{(x+y)^2}p_\mu \\
= \;& -\frac{e^2}{4\pi^2m}\int_0^1 dx[\left(-y-\frac{x^2+x}{x+y}+\frac{x^2}{x+y}+x\ln(x+y)\right)p'_\mu + \\
& \left(\frac{x-x^2}{x+y}+(1-x)\ln(x+y)+\frac{x^2}{x+y}\right)p_\mu]_0^{1-x} \\
= \;& -\frac{e^2}{4\pi^2m}\int_0^1 dx(-x\ln x)p'_\mu + [x-1-(1-x)\ln x]p_\mu. \\
= \;& -\frac{e^2}{16\pi^2m}(p'_\mu + p_\mu).
\end{aligned} \tag{8.8}$$

Finally we substitute for $p'_\mu + p_\mu$ using the Gordon identity [Eq. (8.5)] to obtain

$$-\frac{e^2}{16\pi^2m}\bar{u}(p')\left[2m\gamma_\mu u(p) - i\sigma_{\mu\nu}(p-p')^\nu\right]u(p). \tag{8.9}$$

The term with γ_μ cancels the term neglected above. The total renormalized vertex factor is

$$\bar{u}(p')(\gamma_\mu + \Lambda_\mu)u(p) = \frac{1}{2m}\bar{u}(p')\left[(p+p')_\mu + \left(1+\frac{e^2}{8\pi^2}\right)i\sigma_{\mu\nu}(p-p')^\nu\right]u(p). \tag{8.10}$$

This is the $\mathcal{O}(\alpha)$ expression for the anomalous magnetic moment; the correction from the vertex loop is $\alpha/2\pi$.

8.3 Status

The most precise experimental measurement of the anomalous moment is based on energy transitions of a single-electron cyclotron, with a value

$$g/2 = 1.00115965218073(28). \tag{8.11}$$

The calculation of the relationship between $g/2$ and α_{EM} has been calculated to fifth order in α_{EM}, along with the two-loop weak and hadronic corrections [23].

CHAPTER 9

MUON DECAY

The muon lifetime was the earliest calculation of a weak process, made possible by the description of the decay as an effective four-point interaction because of the relatively large W-boson mass. The first radiative corrections were calculcated in the 1950s, and the latest corrections (calculated more than forty years later) have an uncertainty of 0.3 parts per million. Experimentally, a precision of $\approx 0.01\%$ had been achieved by the early 1970s. Since then the precision has been improved by a factor of 100. Given the precise theoretical and experimental knowledge of the muon lifetime, it is used as one of the three input parameters to the Electroweak theory of interactions between fermions and gauge bosons.

9.1 Tree-level prediction

Using the standard Feynman rules in the unitary gauge, the leading-order matrix element for muon decay $\mu(p_1) \to \nu_\mu(p_2) + e(p_{1'}) + \bar{\nu}_e(p_{2'})$ is

$$\begin{aligned}\mathcal{M} \quad = \quad & \int \frac{d^4k}{(2\pi)^4}\delta^4(p_1 - p_2 - k)\bar{u}_2\frac{ig}{2\sqrt{2}}\gamma_\mu(1-\gamma_5)u_1\frac{-i}{k^2 - m_W^2 + i\epsilon}\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{m_W^2}\right] \times \\ & \delta^4(k - p_{1'} - p_{2'})\bar{u}_{1'}\frac{ig}{2\sqrt{2}}\gamma_\nu(1-\gamma_5)v_{2'}. \end{aligned} \tag{9.1}$$

The integral over k enforces overall momentum conservation through the delta functions enforcing momentum conservation at each vertex. Performing these integrals and using the following Dirac equations to simplify the second term,

$$\begin{aligned}(\gamma_\mu p^\mu - m)u &= 0\\ \bar{u}(\gamma_\mu p^\mu - m) &= 0\\ (\gamma_\mu p^\mu + m)v &= 0,\end{aligned} \tag{9.2}$$

the matrix element becomes

$$\begin{aligned}\mathcal{M} &= \frac{ig^2}{8[(p_1-p_2)^2 - m_W^2]}\{\bar{u}_2\gamma_\mu(1-\gamma_5)u_1\bar{u}_{1'}\gamma^\mu(1-\gamma_5)v_{2'} - \\ & \frac{m_\mu m_e}{m_W^2}\bar{u}_2(1+\gamma_5)u_1\bar{u}_{1'}(1-\gamma_5)v_{2'}\},\end{aligned} \tag{9.3}$$

where neutrino masses have been neglected. The second term can also be neglected, since $\frac{m_\mu m_e}{m_W^2} = 8\times10^{-9}$. Finally, we neglect the $(p_1-p_2)^2$ term in the propagator, though we will see that its contribution is of the same order as the theoretical and experimental uncertainties on the muon lifetime.

With these approximations, the square of the matrix element is

$$|\mathcal{M}|^2 = \frac{g^4}{64m_W^4}[\bar{u}_2\gamma_\mu(1-\gamma_5)u_1\bar{u}_{1'}\gamma^\mu(1-\gamma_5)v_{2'}][\bar{v}_{2'}\gamma^\nu(1-\gamma_5)u_{1'}\bar{u}_1\gamma_\nu(1-\gamma_5)u_2]. \tag{9.4}$$

The multiplication of two spinor states with an intermediate gamma matrix produces a scalar number, so these combinations can be shifted around in the equation. Moving the $\bar{u}_1\gamma_\nu(1-\gamma_5)u_2$ factor to the front of the calculation,

$$|\mathcal{M}|^2 = \frac{g^4}{64m_W^4}[\bar{u}_1\gamma_\nu(1-\gamma_5)u_2\bar{u}_2\gamma_\mu(1-\gamma_5)u_1][\bar{u}_{1'}\gamma^\mu(1-\gamma_5)v_{2'}\bar{v}_{2'}\gamma^\nu(1-\gamma_5)u_{1'}], \tag{9.5}$$

the matrices within brackets are the fermion lines at each vertex. We can move the spinor at the front of the first bracket to the end of the bracket, and vice versa for the second bracket, and take the trace of each bracket:

$$|\mathcal{M}|^2 = \frac{g^4}{64m_W^4}\mathrm{Tr}[u_2\bar{u}_2\gamma_\mu(1-\gamma_5)u_1\bar{u}_1\gamma_\nu(1-\gamma_5)]\mathrm{Tr}[u_{1'}\bar{u}_{1'}\gamma^\mu(1-\gamma_5)v_{2'}\bar{v}_{2'}\gamma^\nu(1-\gamma_5)]. \tag{9.6}$$

We will perform a differential calculation for the final state electron energy and angular distributions, so we do not average over the muon spins or sum over the electron spins. Keeping spin and mass information, the spinor combinations are in general:

$$u\bar{u} = (\gamma_\mu p^\mu + m)(1+\gamma_5\gamma_\nu s^\nu)/2,$$

where s^ν is the spin direction. The first trace becomes:

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}\left[\gamma_\rho p_2^\rho\gamma_\mu(1-\gamma_5)(\gamma_\lambda p_1^\lambda + m_\mu)(1+\gamma_5\gamma_\delta s_\mu^\delta)\gamma_\nu(1-\gamma_5)\right] = \\ &\quad\frac{1}{2}\mathrm{Tr}\left[\gamma_\rho p_2^\rho\gamma_\mu(1-\gamma_5)(\gamma_\lambda p_1^\lambda + m_\mu\gamma_5\gamma_\lambda s_\mu^\lambda)\gamma_\nu(1-\gamma_5)\right],\end{aligned} \tag{9.7}$$

where we have used the fact that the trace of an odd number of gamma matrices is zero. Moving the first $(1-\gamma_5)$ to the right gives $(1-\gamma_5)^2 = (1-\gamma_5)$ (recall that γ_5 anti-commutes with γ_μ). Moving the residual γ_5 to the right gives $\gamma_5(1-\gamma_5) = -(1-\gamma_5)$

(since $\gamma_5^2 = 1$). Using a similar procedure for the second trace, we get a squared matrix element of

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{g^4}{64m_W^4}\{p_2^\rho(p_1^\lambda - m_\mu s_\mu^\lambda)(p_{1'\alpha} - m_e s_{e\alpha})p_{2'\beta}\mathrm{Tr}\left[\gamma_\rho\gamma_\mu\gamma_\lambda\gamma_\nu(1-\gamma_5)\right] \\ &\quad \mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu(1-\gamma_5)]\} \\ &= \frac{g^4}{m_W^4}\left[p_2\cdot(p_{1'} - m_e s_e)\right]\left[(p_1 - m_\mu s_\mu)\cdot p_{2'}\right]. \end{aligned} \tag{9.8}$$

In the last step we have used the following relation:

$$\begin{aligned} g^{\mu\nu}g^{\alpha\beta}\mathrm{Tr}[\gamma_\delta\gamma_\mu\gamma_\phi\gamma_\beta(C_1 - C_2\gamma_5)]\mathrm{Tr}[\gamma_\lambda\gamma_\nu\gamma_\rho\gamma_\alpha(C_3 - C_4\gamma_5)] &= \\ 32[C_1C_3(\delta_{\delta\lambda}\delta_{\phi\rho} + \delta_{\delta\rho}\delta_{\phi\lambda}) + C_2C_4(\delta_{\delta\lambda}\delta_{\phi\rho} - \delta_{\delta\rho}\delta_{\phi\lambda})]. \end{aligned} \tag{9.9}$$

We now combine this matrix element with the phase space for muon decay:

$$d\Gamma(p_1 \to p_2p_{1'}p_{2'}) = \frac{(2\pi)^4|\mathcal{M}|^2}{2E_1}\delta^4(p_2+p_{1'}+p_{2'}-p_1)\frac{d^3p_2}{(2\pi)^32E_2}\frac{d^3p_{1'}}{(2\pi)^32E_{1'}}\frac{d^3p_{2'}}{(2\pi)^32E_{2'}}. \tag{9.10}$$

Inserting the matrix element gives the completely differential cross section for muon decay. Since neutrinos are not observed in a muon decay experiment, we integrate over the neutrino momenta (note however that there can be applications for an expression with the neutrino energy dependence, e.g. a neutrino beam generated from decaying muons). The integral is

$$\begin{aligned} \int \frac{d^3p_2}{(2\pi)^32E_2}\frac{d^3p_{2'}}{(2\pi)^32E_{2'}}\delta^4(p_2 + p_{1'} + p_{2'} - p_1)p_2^\alpha p_{2'}^\beta &= \tfrac{\pi}{24}[g^{\alpha\beta}(p_1 - p_{1'})^2 + \\ &\quad 2(p_1 - p_{1'})^\alpha(p_1 - p_{1'})^\beta] \end{aligned} \tag{9.11}$$

The differential decay rate is then

$$\begin{aligned} d\Gamma &= \frac{g^4d^3p_{1'}}{192(2\pi)^4m_W^4E_1E_{1'}}[(p_{1'} - p_1)^2(p_1 - m_\mu s_\mu)\cdot(p_{1'} - m_e s_e) + \\ &\quad 2(p_{1'} - p_1)\cdot(p_1 - m_\mu s_\mu)(p_{1'} - p_1)\cdot(p_{1'} - m_e s_e)]. \end{aligned} \tag{9.12}$$

This is the Lorentz-invariant decay rate. We now choose a coordinate system where the muon is at rest and the angle between the muon spin and electron momentum is θ. Neglecting the electron mass and summing over its spin states, we obtain

$$\begin{aligned} d\Gamma &= \tfrac{2g^4d^3p_{1'}}{192(2\pi)^4m_W^4m_\mu E_{1'}}[(m_\mu^2 - 2E_{1'}m_\mu)(m_\mu E_{1'} + m_\mu E_{1'}\cos\theta) + \\ &\quad 2(m_\mu E_{1'} - m_\mu^2 + m_\mu E_{1'}\cos\theta)(-m_\mu E_{1'})] \\ &= \tfrac{2g^4E_{1'}dE_{1'}d\cos\theta d\phi}{192(2\pi)^4m_W^4m_\mu}[3m_\mu^3E_{1'} - 4m_\mu^2E_{1'}^2 + m_\mu^3E_{1'}\cos\theta - 4m_\mu^2E_{1'}^2\cos\theta] \end{aligned} \tag{9.13}$$

This expression can be further simplified by recasting $E_{1'}$ into a ratio $x = E_{1'}/(m_\mu/2)$, since the kinematic upper bound on $E_{1'}$ is half the muon's mass. Then we have:

$$\begin{aligned} d\Gamma &= \frac{g^42x^2dxd\cos\theta d\phi}{192\times16\pi^4m_W^4}\frac{m_\mu^5}{8}[3 - 2x - \cos\theta + 2x\cos\theta] \\ &= \frac{g^4}{32m_W^4}\frac{dxd\cos\theta d\phi}{4\pi}\frac{m_\mu^5}{192\pi^3}\left[2x^2(3-2x)\right]\left[1 + \frac{1-2x}{3-2x}\cos\theta\right]. \end{aligned} \tag{9.14}$$

This equation has several pieces: the first is $g^4/(32m_W^4)$, which is the original definition of the Fermi coupling G_F^2. The correction for the momentum transfer in the propagator $[(p_1 - p_2)^2]$ is included in the definition of G_F. The second piece is the final-state electron phase space, which integrates to one. The third piece contains a factor of the fifth power of the muon mass; this power is general to particles decaying weakly, as can be seen from dimensional arguments. The fourth piece describes the energy distribution of the final-state electron (or, equivalently, the muon neutrino). It peaks at $x = 1$ and integrates to one. The final piece describes the angular distribution of the electron with respect to the muon spin direction. At $x = 1$ the distribution is $1 - \cos\theta$; the electron momentum is opposite to the muon spin direction due to the $V - A$ coupling of the weak charge. Figure 9.1 shows the energy distribution of the electron $[n(x)]$ and electron antineutrino $[n'(x)]$, and the $\cos\theta$ coefficient $[\alpha(x)]$. The figure also shows the momentum and spin directions of the decay for $\cos\theta = -1$.

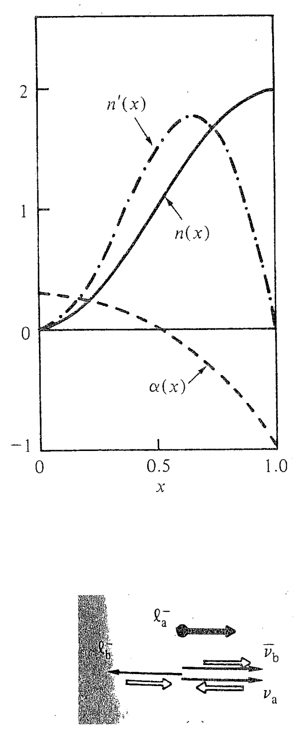


Figure 9.1 Top: Distributions of the electron energy fraction $[n(x)]$, antineutrino energy fraction $[n'(x)]$, and the electron angular coefficient $[\alpha(x)]$. Bottom: A pictorial representation of the decay.

Starting from equation 9.14, it is trivial to calculate the differential decay rate with respect to electron energy and the total decay rate. The integrals equal one, so

$$\frac{d\Gamma}{dx} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[2x^2(3-2x)\right], \tag{9.15}$$

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}. \tag{9.16}$$

This is the total decay width of the muon. A measurement of the muon lifetime, or $1/\Gamma$, determines G_F^2. Additional corrections to Γ from the electron mass and from higher order diagrams are included when calculating G_F^2 from the muon lifetime.

9.2 Lifetime corrections

Additional corrections to the lifetime include the effect of accounting for the mass of the electron, higher-order QED corrections, and the neglected momentum transfer in the propagator. We show this latter calculation in detail.

We are only concerned with the lifetime, so we average over initial spin states and sum over final spin states; the sum gives a factor of 2. We additionally expand the denominator using $1/(1-x) \approx 1+x$ to obtain:

$$|\mathcal{M}|^2 = \frac{2g^4[1+2(m_\mu^2 - 2m_\mu E_2)/m_W^2]}{m_W^4}(p_2 \cdot p_{1'})(p_1 \cdot p_{2'}). \tag{9.17}$$

In the muon rest frame, $p_1 \cdot p_{2'} = m_\mu E_{2'}$. The factor $p_2 \cdot p_{1'}$ can be calculated from the conservation of energy and momentum, $p_1 = p_2 + p_{1'} + p_{2'}$:

$$(p_1 - p_{2'})^2 = (p_2 + p_{1'})^2$$
$$m_\mu^2 - 2m_\mu E_{2'} = 2p_2 \cdot p_{1'}. \tag{9.18}$$

The matrix element is then

$$|\mathcal{M}|^2 = \frac{g^4[1+2(m_\mu^2 - 2m_\mu E_2)/m_W^2]}{m_W^4}(m_\mu E_{2'})\left(m_\mu^2 - 2m_\mu E_{2'}\right). \tag{9.19}$$

We can now insert the matrix element into the lifetime expression,

$$d\Gamma(p_1 \to p_2 p_{1'} p_{2'}) = \frac{(2\pi)^4|\mathcal{M}|^2}{2E_1}\delta^4(p_2+p_{1'}+p_{2'}-p_1)\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{d^3\vec{p}_{1'}}{(2\pi)^3 2E_{1'}}\frac{d^3\vec{p}_{2'}}{(2\pi)^3 2E_{2'}}. \tag{9.20}$$

To simplify the expression, we separate out the energy and momentum delta functions, and integrate over $\delta^3(\vec{p}_2 + \vec{p}_{1'} + \vec{p}_{2'})d^3\vec{p}_2$:

$$d\Gamma(p_1 \to p_2 p_{1'} p_{2'}) = 2\pi\frac{g^4[1+2(m_\mu^2 - 2m_\mu E_2)/m_W^2]}{4E_1 E_2 m_W^4}\left(m_\mu^2 - 2m_\mu E_{2'}\right) \times$$
$$m_\mu E_{2'}\delta(E_2 + E_{1'} + E_{2'} - m_\mu)\frac{d^3\vec{p}_{1'}}{(2\pi)^3 2E_{1'}}\frac{d^3\vec{p}_{2'}}{(2\pi)^3 2E_{2'}}, \tag{9.21}$$

where $E_2 = |\vec{p}_{1'} + \vec{p}_{2'}|$. At this point we separate the correction factor

$$\Delta d\Gamma(p_1 \to p_2 p_{1'} p_{2'}) = 2\pi\frac{g^4 2(m_\mu^2 - 2m_\mu E_2)/m_W^2}{4E_1 m_W^4}m_\mu E_{2'}\left(m_\mu^2 - 2m_\mu E_{2'}\right) \times$$
$$\delta(E_2 + E_{1'} + E_{2'} - m_\mu)\frac{d^3\vec{p}_{1'}}{(2\pi)^3 2E_{1'}}\frac{d^3\vec{p}_{2'}}{(2\pi)^3 2E_{2'}}. \tag{9.22}$$

Now we want to integrate over $d^3\vec{p}_{2'}$. The delta function includes a constraint on this vector through

$$E_2^2 = E_{1'}^2 + E_{2'}^2 + 2E_{1'}E_{2'}\cos\theta, \tag{9.23}$$

where θ is the angle between the electron and electron neutrino. Defining the z-axis along the direction of the electron neutrino, the integral over $d^3\vec{p}_{2'}$ becomes

$$E_{2'}^2 dE_{2'} d\phi d\cos\theta = E_{2'}^2 dE_{2'} d\phi \frac{2E_2 dE_2}{2E_{1'}E_{2'}}. \tag{9.24}$$

Integrating over ϕ and E_2, we get

$$\begin{aligned}\Delta d\Gamma(p_1 \to p_2 p_{1'} p_{2'}) &= (2\pi)^2 \frac{2g^4(m_\mu^2/m_W^2)}{4E_1 m_W^4} E_{2'}\left(m_\mu^2 - 2m_\mu E_{2'}\right) \times \\ & \quad \frac{m_\mu - 2(m_\mu - E_{1'} - E_{2'})}{E_{1'}E_{2'}} \frac{d^3\vec{p}_{1'}}{(2\pi)^3 2E_{1'}} \frac{E_{2'}^2 dE_{2'}}{(2\pi)^3 2E_{2'}}\end{aligned} \tag{9.25}$$

The integral over $E_{2'}$ ranges from $m_\mu/2 - E_{1'}$ to $m_\mu/2$; this range is a result of the integral over the delta function and E_2. We get

$$\begin{aligned}\Delta d\Gamma(p_1 \to p_2 p_{1'} p_{2'}) &= \frac{2g^4 m_\mu^2/m_W^2}{16\pi E_1 m_W^4} \frac{d^3\vec{p}_{1'}}{(2\pi)^3 2E_{1'}^2} \int_{m_\mu/2-E_{1'}}^{m_\mu/2} E_{2'}\left(m_\mu^2 - 2m_\mu E_{2'}\right) \\ & \quad \times(2E_{1'} + 2E_{2'} - m_\mu) dE_{2'} \\ &= \frac{2g^4 m_\mu^2/m_W^2}{16\pi E_1 m_W^4} \frac{d^3\vec{p}_{1'}}{(2\pi)^3 2E_{1'}^2} \int_{m_\mu/2-E_{1'}}^{m_\mu/2} [(2m_\mu^2 E_{1'} - m_\mu^3)E_{2'} \\ & \quad +(4m_\mu^2 - 4m_\mu E_{1'})E_{2'}^2 - 4m_\mu E_{2'}^3] dE_{2'}.\end{aligned} \tag{9.26}$$

Evaluating the integral gives

$$\begin{aligned}&[(2m_\mu^2 E_{1'} - m_\mu^3)\tfrac{E_{2'}^2}{2} + (2m_\mu^2 - 4m_\mu E_{1'})\tfrac{E_{2'}^3}{3} - 4m_\mu \tfrac{E_{2'}^4}{4}]|_{m_\mu/2-E_{1'}}^{m_\mu/2} = \\ &(2m_\mu^2 E_{1'} - m_\mu^3)(m_\mu E_{1'} - E_{1'}^2)/2 \\ &+(4m_\mu^2 - 4m_\mu E_{1'})[3(m_\mu^2/4)E_{1'} - 3(m_\mu/2)E_{1'}^2 + E_{1'}^3]/3 \\ &-4m_\mu[4(m_\mu/2)^3 E_{1'} - 6(m_\mu/2)^2 E_{1'}^2 + 4(m_\mu/2)E_{1'}^3 - E_{1'}^4]/4.\end{aligned} \tag{9.27}$$

Now integrate over $E_{1'}$. The external factors of $E_{1'}$ cancel, leaving an integral over $x = E_{1'}/m_\mu$:

$$\begin{aligned}&m_\mu^5\left\{\left[-\tfrac{x}{2} + \tfrac{3x^2}{2} - x^3\right] + \left[x - 3x^2 + \tfrac{10x^3}{3} - \tfrac{4x^4}{3}\right] - \left[\tfrac{x}{2} - \tfrac{3x^2}{2} + 2x^3 - x^4\right]\right\} = \\ &m_\mu^5\left[\tfrac{x^3}{3} - \tfrac{x^4}{3}\right].\end{aligned} \tag{9.28}$$

Integrating x from 0 to 1/2 gives

$$\begin{aligned}\Delta\Gamma(p_1 \to p_2 p_{1'} p_{2'}) &= \frac{4\pi \times 2g^4 m_\mu^2/m_W^2}{32\pi E_1 m_W^4 (2\pi)^3} m_\mu^5 \left(\frac{m_\mu}{3\times 4\times 2^4} - \frac{m_\mu}{3\times 5\times 2^5}\right) \\ &= \frac{2g^4 m_\mu^2/m_W^2}{m_W^4(4\pi)^3} m_\mu^5 \left(\frac{5}{3\times 10\times 2^5} - \frac{2}{3\times 10\times 2^5}\right) \\ &= \frac{g^4 m_\mu^2/m_W^2}{160 m_W^4 (4\pi)^3} m_\mu^5.\end{aligned} \tag{9.29}$$

We can now compare this to the lifetime calculated without the correction,

$$\Gamma = \frac{g^4}{32m_W^4}\frac{m_\mu^5}{3(4\pi)^3}. \tag{9.30}$$

Taking the ratio gives

$$\Delta\Gamma/\Gamma = \frac{3m_\mu^2}{5m_W^2}. \tag{9.31}$$

The relative correction is thus $0.6 \times (0.105/80.4)^2$, or 1.02×10^{-6}.

9.3 Experimental measurements

Two ongoing measurements of the muon lifetime begin with a positive pion beam at the Paul Scherrer Institute in Switzerland. The pions are stopped in a target and have a predominant decay into an antimuon and a muon neutrino. Antimuons are preferred to muons, since muons can be captured by the electromagnetic field of a nucleus (thus affecting their lifetime). The FAST detector [17] reduces deadtime by simultaneously measuring the electrons from multiple muon decays. The MuLan Collaboration [18] uses a beam kicker to provide 5 μs pulses of muons with 22 μs spacing. MuLan has published its final measurement using 2 trillion candidates collected in 2006 and 2007, with a statistical precision of a part per million. Systematic uncertainties are roughly half of this. The part-per-million uncertainty on the lifetime corresponds to a 0.6 parts-per-million measurement of G_F ($= 1.1663788(7) \times 10^{-5}$ GeV^{-2}). The FAST experiment collected 420 billion candidates in 2008 and 2009, allowing a statistical precision of 1.5 parts per million. Analysis of these events is ongoing.

CHAPTER 10

Z BOSON PRODUCTION

The production or exchange of a single gauge boson is the simplest process to describe in perturbation theory. Such a process is therefore ideal for the determination of fundamental parameters of the theory. Measurements of the Z boson production rate in e^+e^- collisions as a function of mass have yielded a precise determination of the Z boson mass (m_Z). A dedicated run with the large electron-positron collider (LEP) from 1989-1995 resulted in 17 million measured events at a collision energy near m_Z [14]. The value of m_Z measured during this run is now used to fix a parameter of the Electroweak theory. In addition, measurements of the angular distributions of the final-state fermions were used to determine $\sin^2\theta_W$, allowing a high-precision test of Electroweak predictions. Measurements at the Stanford Linear Collider performed on a smaller sample of 0.6 million events yielded similar precision due to the polarization of the initial-state electrons.

10.1 Z boson measurements

In the Electroweak theory the Z boson mass is expressed in terms of gauge-coupling and scalar-potential parameters as

$$m_{Z^0} = \frac{\mu\sqrt{g^2 + g'^2}}{4\sqrt{\lambda}}. \tag{10.1}$$

The parameters in this equation determine all tree-level interactions between fermions and gauge bosons. In fact only three parameters are needed, as the scalar potential parameters always enter in the combination $\mu/\sqrt{2\lambda}$, i.e. the vacuum expectation value of the field.

The Z-boson mass is measured by determining the position of the resonance in the propagator using e^+e^- annihilation into Z/γ^*. Equation 10.1 relates the mass parameter of the Z-boson gauge field to other parameters in the Lagrangian. However, the effective mass of the propagator is different from this mass parameter due to loops in the propagator. When fixing Electroweak parameters and making predictions these effects need to be accounted for. Fortunately, the dominant contributions preserve the form of the equations, so the on-shell mass measured by LEP is a convenient input parameter. Other parameters measured using Z boson production include the width of the Z boson (Γ_Z), the weak mixing angle, and the number of light neutrinos.

10.1.1 Cross section for Z/γ^* production

Consider the simplest case of $\mu^+\mu^-$ production in e^+e^- collisions. The cross section for e^+e^- annihilation into a gauge boson, and subsequent "decay" to a muon-antimuon pair can be derived from the matrix element

$$\mathcal{M} = \mathcal{M}_Z + \mathcal{M}_\gamma, \tag{10.2}$$

with

$$\begin{aligned}\mathcal{M}_Z \;=\; & \int \frac{d^4k}{(2\pi)^4}\delta^4[p_1+p_2-k]\bar{v}_2\frac{ig\gamma_\mu}{2\cos\theta_W}\left[(g_V^e - g_A^e\gamma_5\right]u_1 \times \\ & \frac{-i}{k^2-m_Z^2+i\epsilon}\left[g^{\mu\nu}+(\alpha-1)\frac{k^\mu k^\nu}{k^2-\alpha m_Z^2}\right]\delta^4[k-(p_{1'}+p_{2'})] \times \\ & \bar{u}_{1'}\frac{ig\gamma_\nu}{2\cos\theta_W}\left[(g_V^\mu - g_A^\mu\gamma_5\right]v_{2'}\end{aligned} \tag{10.3}$$

and

$$\begin{aligned}\mathcal{M}_\gamma \;=\; & \int \frac{d^4k}{(2\pi)^4}\delta^4[p_1+p_2-k]\bar{v}_2(-ie\gamma_\mu)u_1\frac{-i}{k^2+i\epsilon}\left[g^{\mu\nu}+(\alpha-1)\frac{k^\mu k^\nu}{k^2}\right] \times \\ & \delta^4[k-(p_{1'}+p_{2'})]\bar{u}_{1'}(-ie\gamma_\nu)v_{2'}.\end{aligned} \tag{10.4}$$

For other final states, one factor of $-e$ or $g_V - g_A\gamma_5$ is modified to the appropriate weak and electromagnetic charges. For the e^+e^- and $\nu_e\bar{\nu}_e$ final states there are additional matrix elements corresponding to the t-channel exchange of a Z, γ, or W boson. The integrals over k simply enforce overall momentum conservation due to the delta functions enforcing momentum conservation at each vertex.

We now set the gauge to the 't Hooft-Feynman gauge ($\alpha = 1$). In this gauge there is an additional diagram for the longitudinal degree of freedom of the Z boson (the scalar field). This diagram is proportional to the product of the masses of the fermions divided by the square of the Z boson mass, and can thus be neglected. Alternatively one can use the unitary gauge, in which case the $k^\mu k^\nu$ term combines with the Dirac spinors to give mass terms via the Dirac equations. As expected, the results are equivalent to the 't Hooft-Feynman gauge, and these extra terms can be neglected.

Squaring the matrix element gives:

$$|\mathcal{M}|^2 = |\mathcal{M}_Z|^2 + |\mathcal{M}_\gamma|^2 + \mathcal{M}_Z^*\mathcal{M}_\gamma + \mathcal{M}_Z\mathcal{M}_\gamma^*. \tag{10.5}$$

For simplicity start with the $|\mathcal{M}_\gamma|^2$ calculation. Integrating over k replaces k with $\sqrt{s} = p_1 + p_2$:

$$|\mathcal{M}_\gamma|^2 = \frac{e^4}{s^2}[g^{\mu\nu}\bar{v}_2\gamma_\mu u_1\bar{u}_{1'}\gamma_\nu v_{2'}][g^{\alpha\beta}\bar{v}_{2'}\gamma_\alpha u_{1'}\bar{u}_1\gamma_\beta v_2]. \quad (10.6)$$

Multiplication of two spinor states with an intermediate gamma matrix produces a scalar number, so we can shift the $\bar{u}_1\gamma_\beta v_2$ factor from the end of the second bracket to the middle of the first bracket (after u_1). Then we move the spinor at the end of each bracket to the front and take the trace:

$$|\mathcal{M}_\gamma|^2 = \frac{e^4}{s^2}g^{\mu\nu}g^{\alpha\beta}\mathrm{Tr}[v_2\bar{v}_2\gamma_\mu u_1\bar{u}_1\gamma_\beta]\mathrm{Tr}[u_{1'}\bar{u}_{1'}\gamma_\nu v_{2'}\bar{v}_{2'}\gamma_\alpha]. \quad (10.7)$$

The spinor combinations can be simplified if we do not measure initial-state or final-state spins. In that case we average over initial spins and sum over final spins, and use $u\bar{u} = v\bar{v} = \gamma_\mu p^\mu$ for massless particles summed over spin states:

$$|\mathcal{M}_\gamma|^2 = \frac{e^4}{4s^2}g^{\mu\nu}g^{\alpha\beta}\mathrm{Tr}[\gamma_\lambda p_2^\lambda\gamma_\mu\gamma_\delta p_1^\delta\gamma_\beta]\mathrm{Tr}[\gamma_\epsilon p_{1'}^\epsilon\gamma_\nu\gamma_\phi p_{2'}^\phi\gamma_\alpha]. \quad (10.8)$$

Finally we can make use of the following identity:

$$\mathrm{Tr}(\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu) = 4[g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\nu}] \quad (10.9)$$

to obtain

$$\begin{aligned}
|\mathcal{M}_\gamma|^2 &= \frac{4e^4}{s^2}g^{\mu\nu}g^{\alpha\beta}p_2^\lambda p_1^\delta p_{1'}^\epsilon p_{2'}^\phi[g_{\lambda\beta}g_{\delta\mu} + g_{\lambda\mu}g_{\beta\delta} - g_{\lambda\delta}g_{\beta\mu}][g_{\epsilon\nu}g_{\phi\alpha} + g_{\epsilon\alpha}g_{\nu\phi} - g_{\epsilon\phi}g_{\nu\alpha}] \\
&= \frac{4e^4}{s^2}g^{\mu\nu}g^{\alpha\beta}[p_2^\beta p_1^\mu + p_2^\mu p_1^\beta - p_{1\delta}p_2^\delta g_{\beta\mu}][p_{1'}^\nu p_{2'}^\alpha + p_{1'}^\alpha p_{2'}^\nu - p_{1'\delta}p_{2'}^\delta g_{\nu\alpha}] \\
&= \frac{8e^4}{s^2}[(p_1p_{2'})(p_2p_{1'}) + (p_1p_{1'})(p_2p_{2'})]. \quad (10.10)
\end{aligned}$$

This is the Lorentz-invariant expression for the matrix element. Now we define a coordinate system: choose the z axis to be along the direction of the incoming electron (p_1), and θ the angle of the outgoing muon (p_1') with respect to the incoming electron. Then $p_1p_{1'} = p_2p_{2'} = s(1-\cos\theta)/4$ and $p_1p_{2'} = p_2p_{1'} = s(1+\cos\theta)/4$:

$$\begin{aligned}
|\mathcal{M}_\gamma|^2 &= \frac{8e^4}{s^2}\frac{s^2}{16}[(1+\cos\theta)^2 + (1-\cos\theta)^2] \\
&= e^4(1+\cos^2\theta). \quad (10.11)
\end{aligned}$$

Using similar methods we can calculate the remaining matrix element factors. The diagram with the Z-boson exchange yields:

$$\begin{aligned}
|\mathcal{M}_Z|^2 &= \frac{g^4}{16\cos^4\theta_W}\left[\frac{1}{s-m_Z^2+i\epsilon}\right]^2 g^{\mu\nu}\bar{v}_2\gamma_\mu\left[(g_V^e - g_A^e\gamma_5\right]u_1 \\
&\quad \bar{u}_{1'}\gamma_\nu\left[g_V^\mu - g_A^\mu\gamma_5\right]v_{2'}g^{\alpha\beta}\bar{v}_{2'}\gamma_\alpha\left[g_V^\mu - g_A^\mu\gamma_5\right]u_{1'} \\
&\quad \bar{u}_1\gamma_\beta\left[g_V^e - g_A^e\gamma_5\right]v_2 \\
&= \frac{g^4g^{\mu\nu}g^{\alpha\beta}}{16\cos^4\theta_W}\left[\frac{1}{s-m_Z^2+i\epsilon}\right]^2\mathrm{Tr}\{v_2\bar{v}_2\gamma_\mu\left[g_V^e - g_A^e\gamma_5\right] \\
&\quad u_1\bar{u}_1\gamma_\beta\left[g_V^e - g_A^e\gamma_5\right]\}\mathrm{Tr}\{u_{1'}\bar{u}_{1'}\gamma_\nu\left[g_V^\mu - g_A^\mu\gamma_5\right] \\
&\quad v_{2'}\bar{v}_{2'}\gamma_\alpha\left[g_V^\mu - g_A^\mu\gamma_5\right]\}. \quad (10.12)
\end{aligned}$$

To evaluate this matrix element we use the anticommutation of γ_5 with γ^μ to move it to the right end of the brackets. Then use $(\gamma_5)^2 = 1$ and average over initial spin states:

$$\begin{aligned}|\mathcal{M}_Z|^2 &= \frac{g^4 g^{\mu\nu} g^{\alpha\beta}}{64\cos^4\theta_W}\left[\frac{1}{s-m_Z^2+i\epsilon}\right]^2 \mathrm{Tr}\{\gamma_\delta p_2^\delta \gamma_\mu \gamma_\phi p_1^\phi \gamma_\beta [(g_V^e)^2 + (g_A^e)^2 - \\ &\quad 2g_V^e g_A^e \gamma_5]\}\mathrm{Tr}\{\gamma_\lambda p_{1'}^\lambda \gamma_\nu \gamma_\rho p_{2'}^\rho \gamma_\alpha \left[(g_V^\mu)^2 + (g_A^\mu)^2 - 2g_A^\mu g_V^\mu \gamma_5\right]\}. \quad (10.13)\end{aligned}$$

As in the case of muon decay, the traces can be evaluated using Eq. 9.9, giving:

$$\begin{aligned}|\mathcal{M}_Z|^2 &= \frac{g^4}{2\cos^4\theta_W}\left[\frac{1}{s-m_Z^2+im_Z\Gamma_Z}\right]^2 \{[(g_V^\mu)^2 + (g_A^\mu)^2][(g_V^e)^2 + (g_A^e)^2][(p_1 p_{2'}) \times \\ &\quad (p_2 p_{1'}) + (p_1 p_{1'})(p_2 p_{2'})] + [4g_A^\mu g_V^\mu g_V^e g_A^e][(p_1 p_{2'})(p_2 p_{1'}) - (p_1 p_{1'})(p_2 p_{2'})]\} \\ &= \frac{g^4}{16\cos^4\theta_W}\left[\frac{s}{s-m_Z^2+im_Z\Gamma_Z}\right]^2 \{[(g_V^\mu)^2 + (g_A^\mu)^2][(g_V^e)^2 + (g_A^e)^2] \times \\ &\quad [1+\cos^2\theta] + [4g_A^\mu g_V^\mu g_V^e g_A^e][2\cos\theta]\}, \quad (10.14)\end{aligned}$$

where we have introduced the Z-boson width Γ_Z. For the interference terms we calculate one of the terms and add its complex conjugate. We start with:

$$\begin{aligned}\mathcal{M}_Z\mathcal{M}_\gamma^* &= -\frac{g^2 e^2}{16\cos^2\theta_W s}\left[\frac{1}{s-m_Z^2+im_Z\Gamma_Z}\right]\{g^{\mu\nu}\bar{v}_2\gamma_\mu 2\left[g_V^e - g_A^e\gamma_5\right]u_1 \\ &\quad \bar{u}_{1'}\gamma_\nu 2\left[g_V^\mu - g_A^\mu\gamma_5\right]v_{2'}\}[g^{\alpha\beta}\bar{v}_{2'}\gamma_\alpha u_{1'}\bar{u}_1\gamma_\beta v_2]. \\ &= -\frac{g^2 e^2}{4\cos^2\theta_W s}\left[\frac{1}{s-m_Z^2+im_Z\Gamma_Z}\right] g^{\mu\nu} g^{\alpha\beta}\mathrm{Tr}\{v_2\bar{v}_2\gamma_\mu\left[g_V^e - g_A^e\gamma_5\right]u_1\bar{u}_1\gamma_\beta\} \\ &\quad \mathrm{Tr}\{u_{1'}\bar{u}_{1'}\gamma_\nu\left[g_V^\mu - g_A^\mu\gamma_5\right]v_{2'}\bar{v}_{2'}\gamma_\alpha\}. \quad (10.15)\end{aligned}$$

We again move the γ_5 to the right, average over initial spin states, and use Eq. 9.9 to get:

$$\begin{aligned}\mathcal{M}_Z\mathcal{M}_\gamma^* &= -\frac{2g^2 e^2}{\cos^2\theta_W s}\left[\frac{1}{s-m_Z^2+im_Z\Gamma_Z}\right]\{[g_V^e g_V^\mu][(p_1 p_{2'})(p_2 p_{1'}) + \\ &\quad (p_1 p_{1'})(p_2 p_{2'})] + g_A^e g_A^\mu[(p_1 p_{2'})(p_2 p_{1'}) - (p_1 p_{1'})(p_2 p_{2'})]\} \quad (10.16) \\ &= -\frac{g^2 e^2}{4\cos^2\theta_W}\left[\frac{s}{s-m_Z^2+im_Z\Gamma_Z}\right][g_V^e g_V^\mu(1+\cos^2\theta) + 2g_A^e g_A^\mu\cos\theta].\end{aligned}$$

Adding the complex conjugate gives:

$$\begin{aligned}\mathcal{M}_Z\mathcal{M}_\gamma^* + \mathcal{M}_Z^*\mathcal{M}_\gamma &= -\frac{g^2 e^2}{2\cos^2\theta_W}\left[\frac{s(s-m_Z^2)}{(s-m_Z^2)^2 + m_Z^2\Gamma_Z^2}\right] \times \\ &\quad [g_V^e g_V^\mu(1+\cos^2\theta) + 2g_A^e g_A^\mu\cos\theta]. \quad (10.17)\end{aligned}$$

We can now obtain the differential cross section for Z-boson production. In the center of mass system, the cross section for distinguishable particles is

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2 p_{1'}}{64\pi^2 s p_1}. \quad (10.18)$$

Integrating over ϕ and inserting the matrix element gives:

$$\begin{aligned}\frac{d\sigma}{d\cos\theta} &= \frac{1}{32\pi s}\{16\pi^2\alpha^2 Q_e^2 Q_\mu^2(1+\cos^2\theta) - \frac{2\pi g^2\alpha Q_e Q_\mu}{\cos^2\theta_W}\left[\frac{s(s-m_Z^2)}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\right]\times \\ &[g_V^e g_V^\mu(1+\cos^2\theta)+2g_A^e g_A^\mu\cos\theta] + \frac{g^4}{16\cos^4\theta_W}\left[\frac{s^2}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\right]\times \\ &[((g_V^\mu)^2+(g_A^\mu)^2)((g_V^e)^2+(g_A^e)^2)(1+\cos^2\theta)+8g_A^\mu g_V^\mu g_V^e g_A^e\cos\theta]\}, \quad (10.19)\end{aligned}$$

where we have used $\alpha = e^2/(4\pi)$. The square of the weak coupling g^2 is frequently replaced with either $e^2/\sin^2\theta_W$ or $8G_F M_W^2/\sqrt{2}$.

The s dependence of the cross section includes a $1/s$ piece from the photon propagator, a small interference contribution that is negative below m_Z and positive above it, and a highly peaked contribution around m_Z from Z-boson production. By measuring the cross section precisely near m_Z, the LEP experiments obtained $m_Z = 91.1875 \pm 0.0021$ GeV. The measurement relies on a precise knowledge of the beam energy, which required among other things correcting for tidal effects from the moon and sun, leakage currents from Geneva trains, and geological deformations following heavy rainfall. The measurement also relies on an accurate calibration of the luminosity. This was performed with small-angle e^+e^- ("Bhabha") scattering, which is dominated by t-channel exchange of a photon.

The width measurement was performed inclusively using the s dependence of the cross section, and exclusively by calculating the total cross section for individual final states. From the combination of measurements, the number of neutrinos was measured to be 2.9840 ± 0.0082.

10.1.2 Longitudinal asymmetry

In 1993-1996 the Stanford Linear Collider produced half a million Z bosons on resonance ($\sqrt{s} = 91.2$ GeV) using a polarized electron beam colliding with an unpolarized positron beam. The effect of polarization on the cross section can be calculated by inserting $(1-\gamma_5)/2$ $[(1+\gamma_5)/2]$ for negatively (positively) polarized electrons in the matrix element for Z boson production:

$$\begin{aligned}\mathcal{M}_Z^L &= \int \frac{d^4k}{(2\pi)^4}\delta^4[p_1+p_2-k]\bar{v}_2\frac{ig\gamma_\mu}{8\cos\theta_W}\left[(-1+4\sin^2\theta_W)-\gamma_5\right]\times \\ &(1-\gamma_5)u_1\frac{-i}{k^2-m_Z^2+i\epsilon}\left[g^{\mu\nu}+(\alpha-1)\frac{k^\mu k^\nu}{k^2-\alpha m_Z^2}\right]\times \\ &\delta^4[k-(p_{1'}+p_{2'})]\bar{u}_{1'}\frac{ig\gamma_\nu}{4\cos\theta_W}\left[(-1+4\sin^2\theta_W)-\gamma_5\right]v_{2'}. \quad (10.20)\end{aligned}$$

We choose the 't Hooft-Feynman gauge ($\alpha = 1$) and again neglect the additional diagrams proportional to the product of fermion masses divided by the square of the Z boson mass. To perform the calculation, we again write $(-1+4\sin^2\theta_W)-\gamma_5 = 2(g_V - g_A\gamma_5)$ and square the matrix element to obtain

$$\begin{aligned}|\mathcal{M}_Z^L|^2 &= \frac{g^4 g^{\mu\nu}g^{\alpha\beta}}{128\cos^4\theta_W}\left[\frac{1}{s-m_Z^2+im_Z\Gamma_Z}\right]^2 \mathrm{Tr}\{v_2\bar{v}_2\gamma_\mu\,[g_V^e - g_A^e\gamma_5] \\ &(1-\gamma_5)u_1\bar{u}_1\gamma_\beta\,[g_V^e - g_A^e\gamma_5]\}\mathrm{Tr}\{u_{1'}\bar{u}_{1'}\gamma_\nu\,[g_V^\mu - g_A^\mu\gamma_5] \\ &v_{2'}\bar{v}_{2'}\gamma_\alpha\,[g_V^\mu - g_A^\mu\gamma_5]\}, \quad (10.21)\end{aligned}$$

where we have averaged over initial spin states and used $(1/2 - \gamma_5/2)^2 = 1/2 - \gamma_5/2$, since $\gamma_5^2 = 1$. We use $s = m_Z^2$ to simplify the propagator and move the γ_5 matrices to the right end of the brackets to obtain

$$\begin{aligned}|\mathcal{M}_Z^L|^2 &= \frac{g^4 g^{\mu\nu} g^{\alpha\beta}}{128\cos^4\theta_W}\left(\frac{1}{m_Z^2\Gamma_Z^2}\right)\mathrm{Tr}\{\gamma_\delta p_2^\delta \gamma_\mu \gamma_\phi p_1^\phi \gamma_\beta[(g_V^e + g_A^e)^2 \times \\ &(1-\gamma_5)]\}\mathrm{Tr}\{\gamma_\lambda p_{1'}^\lambda \gamma_\nu \gamma_\rho p_{2'}^\rho \gamma_\alpha[(g_v^\mu)^2 + (g_A^\mu)^2 - 2g_A^\mu g_V^\mu \gamma_5]\}. \end{aligned} \quad (10.22)$$

For the right-handed matrix element we replace $(g_V^e + g_A^e)^2(1-\gamma_5)$ with $(g_V^e - g_A^e)^2(1+\gamma_5)$. We now use Eq. 9.9 to obtain

$$\begin{aligned}|\mathcal{M}_Z^L|^2 &= \frac{g^4}{4\cos^4\theta_W}\frac{1}{m_Z^2\Gamma_Z}\{[(g_V^\mu)^2 + (g_A^\mu)^2][(g_V^e + g_A^e)^2][(p_1p_{2'}) \times \\ &(p_2p_{1'}) + (p_1p_{1'})(p_2p_{2'})] + [2g_A^\mu g_V^\mu(g_V^e + g_A^e)^2][(p_1p_{2'})(p_2p_{1'}) \\ &-(p_1p_{1'})(p_2p_{2'})]\} \\ &= \frac{g^4}{32\cos^4\theta_W}\frac{s^2}{m_Z^2\Gamma_Z^2}\{[(g_V^\mu)^2 + (g_A^\mu)^2][(g_V^e + g_A^e)^2][1+\cos^2\theta] \\ &+[2g_A^\mu g_V^\mu(g_V^e + g_A^e)^2][2\cos\theta]\}. \end{aligned} \quad (10.23)$$

Similarly,

$$\begin{aligned}|\mathcal{M}_Z^R|^2 &= \frac{g^4}{32\cos^4\theta_W}\frac{s^2}{m_Z^2\Gamma_Z^2}\{[(g_V^\mu)^2 + (g_A^\mu)^2][(g_V^e - g_A^e)^2] \times \\ &[1+\cos^2\theta] - [2g_A^\mu g_V^\mu(g_V^e - g_A^e)^2][2\cos\theta]\}. \end{aligned} \quad (10.24)$$

Finally, we insert the matrix elements into the differential cross section, $d\sigma/d\Omega = |\mathcal{M}|^2|p_{1'}|/(64\pi^2 s|p_1|)$, and integrate over ϕ:

$$\begin{aligned}\frac{d\sigma_L}{d\cos\theta} &= \frac{1}{32\pi m_Z^2}\{8\pi^2\alpha^2(1+\cos^2\theta) + \frac{g^4}{32\cos^4\theta_W}\frac{m_Z^2}{\Gamma_Z^2} \times \\ &[((g_V^\mu)^2 + (g_A^\mu)^2)(g_V^e + g_A^e)^2(1+\cos^2\theta) \\ &+4g_A^\mu g_V^\mu(g_V^e + g_A^e)^2\cos\theta]\} \end{aligned} \quad (10.25)$$

$$\begin{aligned}\frac{d\sigma_R}{d\cos\theta} &= \frac{1}{32\pi m_Z^2}\{8\pi^2\alpha^2(1+\cos^2\theta) + \frac{g^4}{32\cos^4\theta_W}\frac{m_Z^2}{\Gamma_Z^2} \times \\ &[((g_V^\mu)^2 + (g_A^\mu)^2)(g_V^e - g_A^e)^2(1+\cos^2\theta) \\ &-4g_A^\mu g_V^\mu(g_V^e - g_A^e)^2\cos\theta]\}. \end{aligned} \quad (10.26)$$

Integrating over $d\cos\theta$ and using $g^4 = 16\pi^2\alpha^2/\sin^4\theta_W$, we have

$$\begin{aligned}\sigma_L &= \frac{2\pi\alpha^2}{3m_Z^2}\left\{1 + \frac{m_Z^2[(g_V^\mu)^2 + (g_A^\mu)^2](g_V^e + g_A^e)^2}{16\Gamma_Z^2\sin^4\theta_W\cos^4\theta_W}\right\} \\ \sigma_R &= \frac{2\pi\alpha^2}{3m_Z^2}\left\{1 + \frac{m_Z^2[(g_V^\mu)^2 + (g_A^\mu)^2](g_V^e - g_A^e)^2}{16\Gamma_Z^2\sin^4\theta_W\cos^4\theta_W}\right\}. \end{aligned} \quad (10.27)$$

Now we can calculate the longitudinal asymmetry, defined as

$$\begin{aligned}A_L &= \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} \\ &\approx \frac{2g_V^e g_A^e}{(g_V^e)^2 + (g_A^e)^2}\left[1 - \frac{16\sin^4\theta_W\cos^4\theta_W\Gamma_Z^2}{m_Z^2[(g_V^e)^2 + (g_A^e)^2][(g_V^\mu)^2 + (g_A^\mu)^2]}\right], \end{aligned} \quad (10.28)$$

neglecting terms of order Γ_Z^4/m_Z^4.

Combining all final states, the SLD Collaboration measured $A_L = 0.1516 \pm 0.0021$. The uncertainty on the measurement is 1.4% and the $\frac{\Gamma_Z^2}{m_Z^2}$ correction is $< 1\%$, so we can neglect the correction term. Taking $g_V = (-1/2 + 2\sin^2\theta_W)$ and $g_A = 1/2$, the asymmetry is

$$\begin{aligned} A_L &= \frac{2(-1/2 + 2\sin^2\theta_W) \times 1/2}{1/4 - 2\sin^2\theta_W + 4\sin^4\theta_W + 1/4} \\ &= \frac{-1/2 + 2\sin^2\theta_W}{1/2 - 2\sin^2\theta_W + 4\sin^4\theta_W}. \end{aligned} \tag{10.29}$$

This can be rewritten as a quadratic in $s_W \equiv \sin^2\theta_W$:

$$4A_L s_W^2 - (2A_L - 2)s_w + (A_L/2 - 1/2) = 0 \tag{10.30}$$

with solution

$$s_W = \frac{2A_L - 2 \pm [(2A_L - 2)^2 - 16A_L(A_L/2 - 1/2)]^{1/2}}{8A_L} \tag{10.31}$$

The term under the square root is

$$\begin{aligned} (4A_L^2 - 8A_L + 4 - 8A_L^2 + 8A_L)^{1/2} &= 2(1 - A_L^2)^{1/2} \\ &\approx 2 - A_L^2. \end{aligned} \tag{10.32}$$

We can take other measurements (e.g. m_W and m_Z) to select the positive sign in front of the square root. Then

$$\begin{aligned} s_W &= \frac{A_L - 1 + (1 - A_L^2)^{1/2}}{4A_L} \\ &\approx \frac{1}{4} - \frac{A_L}{8}. \end{aligned} \tag{10.33}$$

Substituting $A_L = 0.1516$ gives $\sin^2\theta_W = 0.23105$ in the approximation and $\sin^2\theta_W = 0.23094$ from the analytic equation. The approximation can be used to calculate the relative uncertainty on s_W:

$$\sigma(s_W) = \frac{\sigma(A_L)}{8} = 0.00026. \tag{10.34}$$

Using $m_Z = 91.1876$ GeV and $m_W^2 = m_Z^2\cos^2\theta_W$, this translates into a tree-level W boson mass of

$$m_W = 79.968 \text{ GeV}. \tag{10.35}$$

10.1.3 Forward-backward asymmetry

The measurement of the weak charge of fermions, and the corresponding extraction of $\sin^2\theta_W$, is greatly simplified by integrating $\cos\theta$ above and below zero and taking the difference divided by the sum. At $\sqrt{s} = m_Z$ the interference term does not contribute and the symmetric $\cos\theta$ distribution in the photon term also removes its contribution. Then one has the simple relation

$$\frac{\int_0^1 d\cos\theta \frac{d\sigma}{d\cos\theta} - \int_{-1}^0 d\cos\theta \frac{d\sigma}{d\cos\theta}}{\int_0^1 d\cos\theta \frac{d\sigma}{d\cos\theta} + \int_{-1}^0 d\cos\theta \frac{d\sigma}{d\cos\theta}} = \frac{3g_A^\ell g_V^\ell g_V^q g_A^q}{[(g_V^\ell)^2 + (g_A^\ell)^2][(g_V^q)^2 + (g_A^q)^2]}. \tag{10.36}$$

Many pieces of the calculation have cancelled, and experimentally many of the uncertainties cancel in this ratio measurement. The determinations of $\sin^2\theta_W$ through forward-backward asymmetry measurements at LEP are among the most precise measurements of this parameter. The measurements are consistent with the predictions, with a moderate deviation in the case of the forward-backward asymmetry measured using decays to b quarks. The measurement $A_{FB}^{b,LEP} = 0.0992 \pm 0.0016$ is 2.5σ lower than the prediction, $A_{FB}^{b,SM} = 0.1034 \pm 0.0004$.

CHAPTER 11

W BOSON MASS

With the input parameter set m_Z, G_F, and α_{EM}, the mass of the W boson is the most straightforward quantity to extract from the Electroweak model. Event samples with large W-boson yields have been produced by the LEP, Tevatron, and LHC colliders, allowing precise measurements of the W boson mass and testing the Electroweak model at the level of a couple parts in ten thousand. The measurements are consistent with the prediction, providing a remarkable validation of the theory. The consistency hinged on the existence of a Higgs boson in a highly constrained mass range, and the recent discovery of the Higgs boson completes the model, allowing continued predictions to very high energy scales.

11.1 Tree-level relations

At tree level, the W and Z boson mass terms in the Lagrangian are

$$\mathcal{L}_{m_V} = \frac{\mu^2}{2\lambda}\left[\frac{g^2}{2}W_\mu^+ W^{\mu-} + \frac{1}{4}(g' B_\mu - g W_\mu^3)^2\right], \qquad (11.1)$$

where $\mu^2/2$ and λ are respectively the coefficients of the scalar terms $\phi^\dagger\phi$ and $(\phi^\dagger\phi)^2$, g is the weak coupling, g' is the hypercharge coupling, and W_μ^i and B_μ are the weak and hypercharge vector fields, respectively. The vector fields W_μ^3 and B_μ are diagonal in the weak basis and can be expressed in terms of their mass eigenstates Z_μ and A_μ as

$$\begin{aligned} W_\mu^3 &= \cos\theta_W Z_\mu + \sin\theta_W A_\mu, \\ B_\mu &= \cos\theta_W A_\mu - \sin\theta_W Z_\mu, \end{aligned} \qquad (11.2)$$

where $\tan\theta_W = g'/g$. Defining the parameter $v \equiv \mu/\sqrt{\lambda}$ proportional to the vacuum expectation of the Higgs field, the masses of the W and Z bosons are

$$\begin{aligned} m_W &= \frac{gv}{2}, \\ m_Z &= \frac{v\sqrt{g^2+g'^2}}{2}. \end{aligned} \tag{11.3}$$

We can express m_W in terms of the input Electroweak parameters using $e = g\sin\theta_W$, $\alpha_{EM} = e^2/4\pi$, and $G_F = \sqrt{2}g^2/8m_W^2$:

$$\begin{aligned} m_W^2 &= \frac{\sqrt{2}e^2}{8G_F\sin^2\theta_W} \\ &= \frac{\pi\alpha_{EM}}{\sqrt{2}G_F(1-m_W^2/m_Z^2)} \\ &= m_Z^2\left[\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\pi\alpha_{EM}}{\sqrt{2}G_F m_Z^2}}\right]. \end{aligned} \tag{11.4}$$

This tree-level prediction can be calculated using measured quantities such as $m_Z = 91.1875 \pm 0.0021$ GeV [14] and $G_F = 1.16637 \pm 0.00001 \times 10^{-5}$ GeV^{-2} [24]. For the electromagnetic coupling α_{EM}, the most precise measurement is at $Q^2 \approx 0$, giving $1/\alpha_{EM}(Q^2 = 0) = 137.035999679$. However, for a prediction of the W boson mass, we need a measurement at $Q^2 = m_W^2$. The standard procedure is to calculate $\alpha_{EM}(Q^2 = m_W^2)$ by including higher order diagrams and measurements at intermediate Q^2 (to account for the hadronic contributions). Direct measurements are complicated by interference with the Z boson; a LEP measurement that used a global fit to account for the interference found $1/\alpha_{EM}(Q^2 = m_Z^2) = 128.937 \pm 0.047$ [14, 25]. Using this value of α_{EM}, and the other input parameters, gives

$$m_W^2 = 91.1875^2\left[\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\pi}{(128.937)\sqrt{2}(1.16637\times10^{-5})(91.1875^2)}}\right]. \tag{11.5}$$

The term in brackets is 0.76899; multiplying by m_Z^2 and taking the square root gives $m_W = 79.964$ GeV. This is 395 MeV below the current prediction of 80.359 GeV, which includes corrections to second order in the couplings. The precision of the input tree-level parameters predict m_W to better than 10 MeV, so the higher order corrections are tested with precise measurements of m_W.

11.2 Renormalization

In order to calculate higher order corrections to the W boson propagator, we need to define a renormalization procedure for the Lagrangian for the vector boson mass terms, Eq. (11.1). We define counterterms for the weak and hypercharge couplings g and g', as well as the scalar field parameter v^2 [26]. Including these counterterms, the Lagrangian of the mass eigenstates is

$$\begin{aligned} \mathcal{L}_{m_V} + \delta\mathcal{L}_{m_V} = {} & \tfrac{(v^2-\delta v^2)(g-\delta g)^2}{4} W_\mu^+ W^{\mu-} + \tfrac{(v^2-\delta v^2)}{8}[A_\mu(g'\cos\theta_W - g\sin\theta_W) \\ & -Z_\mu(g\cos\theta_W + g'\sin\theta_W) - \delta g'(\cos\theta_W A_\mu - \sin\theta_W Z_\mu) + \\ & \delta g(\cos\theta_W Z_\mu + \sin\theta_W A_\mu)]^2, \end{aligned} \tag{11.6}$$

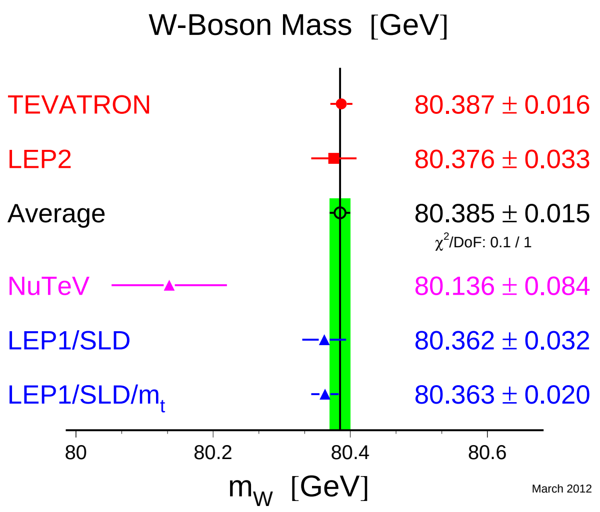


Figure 11.1 Combined direct and indirect measurements of m_W.

This can be rewritten in terms of mass counterterms,

$$\mathcal{L}_{m_V} + \delta\mathcal{L}_{m_V} = (m_W^2 - \delta m_W^2) W_\mu^+ W^{\mu-} + \frac{(m_Z^2 - \delta m_Z^2)}{2} Z_\mu Z^\mu + \delta m_{ZA}^2 Z_\mu A^\mu, \quad (11.7)$$

where

$$\begin{aligned} \delta m_W^2 &= \frac{v^2 \delta g^2 + g^2 \delta v^2}{4} \\ \delta m_Z^2 &= (g^2 + g'^2)\frac{\delta v^2}{4} + \frac{v^2}{4}\delta(g^2 + g'^2) \\ \delta m_{ZA}^2 &= \frac{m_Z^2}{(g^2 + g'^2)^{1/2}}(\cos\theta_W \delta g' - \sin\theta_W \delta g). \end{aligned} \quad (11.8)$$

We see that renormalizing the couplings produces a counterterm that mixes the Z_μ field with the A_μ field, so we expect loop diagrams to produce divergent terms coupling these fields, with convergent residual components. Thus, the tree-level mixing angle will in general be modified by higher order diagrams.

The mass counterterms can be determined by calculating each one-loop boson propagator, also known as the self-energy diagrams. In general the propagator has the form

$$\Pi_{VV'}^{\mu\nu}(q^2) = A_{VV'}(q^2) g^{\mu\nu} + B_{VV'}(q^2) q^\mu q^\nu, \quad (11.9)$$

where V and V' are the incoming and outgoing bosons respectively. There are four such self-energy propagators, Π_{WW}, Π_{ZZ}, $\Pi_{\gamma Z}$, and $\Pi_{\gamma\gamma}$. If we define the Lagrangian parameters m_W and m_Z to be the physical masses of the gauge bosons, the self-energy counterterms for the W and Z bosons are simply

$$\begin{aligned} \delta m_W^2 &= Re A_{WW}(m_W^2), \\ \delta m_Z^2 &= Re A_{ZZ}(m_Z^2). \end{aligned} \quad (11.10)$$

The cross term δm^2_{ZA} can be written in terms of δm^2_Z and δm^2_W as

$$\begin{aligned}\delta m^2_{ZA} &= \frac{m^2_W}{2\sin\theta_W\cos\theta_W}\left[\frac{\delta m^2_Z}{m^2_Z} - \frac{\delta m^2_W}{m^2_W}\right] \\ &= \frac{m^2_W}{2\sin\theta_W\cos\theta_W} Re\left[\frac{A_{ZZ}(m^2_Z)}{m^2_Z} - \frac{A_{WW}(m^2_W)}{m^2_W}\right] \qquad (11.11)\end{aligned}$$

so the divergences in $\Pi_{\gamma Z}$ are removed once Π_{WW} and Π_{ZZ} are renormalized. The relations between the mass and coupling counterterms also constrain two of the coupling counterterms. The third can be constrained with the $\Pi_{\gamma\gamma}$ propagator, which gives the vacuum polarization and does not have a mass counterterm. Rather it leads to a renormalization of the electric charge δe, which can be expressed in terms of the weak and hypercharge coupling counterterms as

$$\begin{aligned}\delta e &= \cos^3\theta_W\delta g' + \sin^3\theta_W\delta g \\ &= -e\Pi_{\gamma\gamma}/2, \qquad (11.12)\end{aligned}$$

where $\Pi_{\gamma\gamma}$ is defined by $A_{\gamma\gamma}(q^2) = -q^2\Pi_{\gamma\gamma}$; this form arises from the lack of a mass term for the photon propagator.

This prescription translates the fundamental Lagrangian coupling counterterms into more physically useful mass counterterms. The coupling counterterms can be expressed in terms of these mass counterterms as,

$$\begin{aligned}\delta g &= \frac{-e\Pi_{\gamma\gamma}}{2\sin\theta_W} - \frac{e\cos^2\theta_W}{2\sin^3\theta_W} Re\left[\frac{A_{ZZ}(m^2_Z)}{m^2_Z} - \frac{A_{WW}(m^2_W)}{m^2_W}\right], \\ \delta g' &= \frac{-e\Pi_{\gamma\gamma}}{2\cos\theta_W} + \frac{e}{2\cos\theta_W} Re\left[\frac{A_{ZZ}(m^2_Z)}{m^2_Z} - \frac{A_{WW}(m^2_W)}{m^2_W}\right]. \qquad (11.13)\end{aligned}$$

11.3 One-loop results

The Fermi coupling constant G_F is extracted from the measurement of the muon lifetime using the following formula (including the first-order photon correction):

$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 - \frac{8m_e^2}{m_\mu^2}\right)\left[1 + \frac{3}{5}\frac{m_\mu^2}{m_W^2} + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]. \qquad (11.14)$$

There are additional corrections not included in this equation, in particular the W boson self-energy diagrams, the electroweak vertex correction diagrams, and box diagrams corresponding to additional exchanges between the muon weak doublet line and the electron weak doublet line. These corrections add a term to the matrix element that can be expressed as $\Delta r\mathcal{M}_0$, where $\mathcal{M}_0$ is the tree-level matrix element. They would appear on the right-hand side of Eq. (11.14) as a multiplicative factor $(1+\Delta r)^2$, so G_F^{-1} extracted from the muon lifetime requires a $(1+\Delta r)$ correction to account for these contributions. Since $m_W \propto G_F^{-1/2}$, it receives a correction of $(1+\Delta r/2)$.

One component of the Δr correction arises from the bosonic contributions to the propagator, vertices, and box diagrams. These can be expressed as

$$\begin{aligned}\Delta r_b &= \frac{\alpha_{EM}}{4\pi s^2}\{F(\sin^2\theta_W) + \frac{1}{\sin^2\theta_W}\left[I\left(\frac{m_H^2}{m_Z^2}\right) - (1-2\sin^2\theta_W)I\left(\frac{m_H^2}{m_Z^2\cos^2\theta_W}\right)\right] \\ &\quad -\frac{3}{4}\left[\frac{\frac{m_H^2}{m_Z^2}\ln(\frac{m_H^2}{m_Z^2}) - \cos^2\theta_W\ln(\cos^2\theta_W)}{\frac{m_H^2}{m_Z^2} - \cos^2\theta_W}\right]\},\end{aligned} \tag{11.15}$$

where the expression $F(\sin^2\theta_W)$ has the value 2.68 for $\sin^2\theta_W = 0.23$ and the Feynman integral $I(y)$ is given by

$$I(y) = \int_0^1 dx\left[1 - \frac{1}{2}x^2 - \frac{1}{2}y(1-x)\right]\ln[x^2 + y(1-x)] + \frac{1}{4}y\left(\ln y - \frac{1}{2}\right). \tag{11.16}$$

The correction depends logarithmically on the Higgs boson mass. The contribution from light-fermion loops in the propagator is

$$\Delta r_f = \frac{2\alpha N_c}{3\pi}\left[\sum \ln\frac{m_Z}{m_i} - \frac{5}{2}a_q + \frac{3}{8}\frac{(2\sin^2\theta_W - 1)}{\sin^4\theta_W}\ln\cos\theta_W\right], \tag{11.17}$$

where quarks have $N_c = 3$ (the color factor) and $a_q = 5/9$. These contribute a relative correction of 3.3% to the W boson mass, with the largest uncertainty arising from the light quark masses. These can be incorporated into the running of the electromagnetic coupling, and currently contribute about 2 MeV uncertainty in the W boson mass prediction.

11.3.1 Top-bottom loop

The top-bottom loop is one of the largest contributors to the correction of the W-boson propagator. In the on-shell scheme the correction can be expressed as

$$\Delta r_{tb} = \frac{-3G_F m_W^2}{8\sqrt{2}\pi^2(m_Z^2 - m_W^2)}\left[m_t^2 + m_b^2 - \frac{2m_t^2 m_b^2}{m_t^2 - m_b^2}\ln\left(\frac{m_t^2}{m_b^2}\right)\right]. \tag{11.18}$$

The correction has two important features. First, it would be zero if the top and bottom quark masses were equal. This is a general feature of the corrections: in the limit where there is a symmetry between W and Z boson interactions their relative masses receive common corrections. Second, it increases with the square of the top mass and thus has the largest effect on the W boson mass in the on-shell scheme. The precise determination of the top-quark mass is therefore important for predicting the existence of other particles. Prior to its discovery, the top quark's mass was predicted to be about 170 GeV, in good agreement with its measured value. The current uncertainty on the mass of the top quark contributes about 4 MeV uncertainty on the W boson mass. Uncertainties from higher order corrections also contribute about 4 MeV. The combination of the world measurements of the W boson mass give an uncertainty of 15 MeV, with the precision dominated by the CDF measurement of $m_W = 80.387 \pm 0.019$ GeV. A measurement of equal precision has recently been performed by ATLAS, with the result $m_W = 80.370 \pm 0.019$ GeV. The measurements involve fitting the measured transverse mass and momentum distributions of W bosons produced via $q\bar{q}' \to W$ and decaying leptonically. An example distribution is shown in Fig. 11.2. For an input Higgs boson mass of $m_H = 125.14 \pm 0.24$ GeV, the

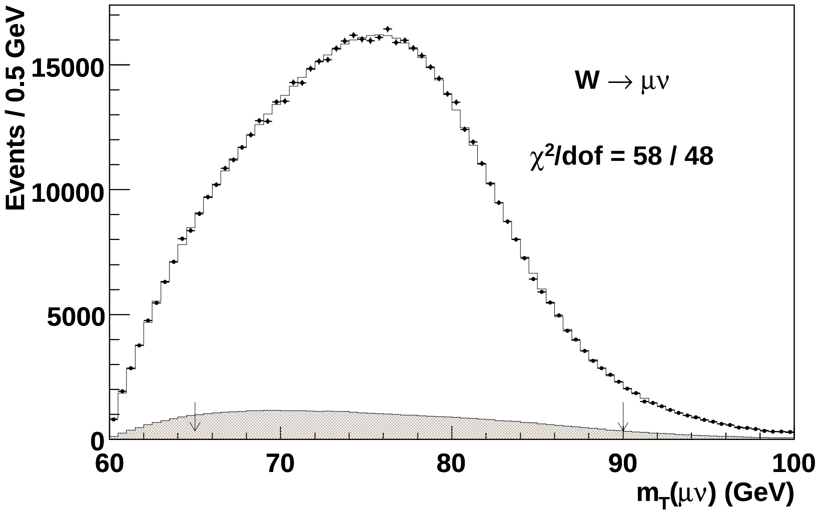


Figure 11.2 Fit for m_W using the transverse mass distribution.

predicted W boson mass is $m_W = 80.358 \pm 0.008$ GeV. Historically the measured value of the W boson mass was used to predict the mass of the Higgs boson; this prediction gives $m_H = 60^{+56}_{-19}$ GeV. A global fit to all W and Z measurements gives a predicted Higgs boson mass of $m_H = 93^{+25}_{-21}$ GeV.

Example calculation: The loop corrections require a consistent application of on-shell renormalization, so they must be calculated both for W boson and Z boson propagators. The form of each calculation is similar to that of the vacuum polarization in Sec. 7.3. As an example, the calculation of the top-bottom quark loop contribution to the W boson propagator is demonstrated here. Applying the Feynman rules to the Feynman diagram gives:

$$\frac{-3\mu^{4-d}g_0^2|V_{tb}|^2}{8}\int\frac{d^dp}{(2\pi)^d}\mathrm{Tr}\left[\gamma_\mu(1-\gamma_5)\frac{1}{\gamma_\alpha p^\alpha - m_t}\gamma_\nu(1-\gamma_5)\frac{1}{\gamma_\beta p^\beta - \gamma_\beta k^\beta - m_b}\right] \equiv i\Pi_{\mu\nu}^{WW}(k), \tag{11.19}$$

including factors of 3 for color and -1 for the fermion loop. To evaluate this integral, we move all gamma matrices to the numerator by multiplying the numerator and denominator by the same factor, and we separate the propagators with the Feynman integral:

$$i\Pi_{\mu\nu}^{WW} = \frac{-3g_0^2\mu^{4-d}|V_{tb}|^2}{8}\int_0^1 dz\int\frac{d^dp}{(2\pi)^d} \frac{\mathrm{Tr}[\gamma_\mu(1-\gamma_5)(\gamma_\alpha p^\alpha + m_t)\gamma_\nu(1-\gamma_5)(\gamma_\beta p^\beta - \gamma_\beta k^\beta + m_b)]}{\{(p^2-m_t^2)z + [(p-k)^2 - m_b^2](1-z)\}^2} \tag{11.20}$$

Redefining the integrand to $p' = p - k(1-z)$ and recalling that integrals over terms linear in p' give zero, the numerator becomes

$$2[p'^\alpha p'^\beta - k^\alpha k^\beta z(1-z)]\mathrm{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta(1+\gamma_5)]. \tag{11.21}$$

The traces can be evaluated in d dimensions using

$$\begin{aligned}\mathrm{Tr}(\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_5) &= di\epsilon_{\mu\alpha\nu\beta},\\ \mathrm{Tr}(\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta) &= f(d)(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\nu}g_{\alpha\beta}+g_{\mu\beta}g_{\nu\alpha}),\end{aligned} \tag{11.22}$$

where $f(d)$ is some function with the property $f(4)=4$. The antisymmetric tensor $\epsilon_{\mu\alpha\nu\beta}$ combined with the symmetric terms $p'^{\alpha}p'^{\beta}-k^\alpha k^\beta z(1-z)$ will give zero. The numerator becomes

$$2f(d)\{2p'_\mu p'_\nu - 2z(1-z)(k_\mu k_\nu - k^2 g_{\mu\nu}) - g_{\mu\nu}[p'^2+k^2z(1-z)]\}. \tag{11.23}$$

In terms of p', the denominator is

$$p'^2+k^2z(1-z)-m_t^2z-m_b^2(1-z). \tag{11.24}$$

Now the first and last terms of the numerator will cancel if we add $m_t^2z-m_b^2(1-z)$ to the last term (as in the case for the vacuum polarization of the photon). We are then left with the integral over the middle term,

$$\begin{aligned} i\Pi^{WW}_{\mu\nu} &= \frac{-3f(d)g_0^2\mu^{4-d}|V_{tb}|^2}{4}\int_0^1 dz\int\frac{d^dp'}{(2\pi)^d}\\ &\quad \frac{-2z(1-z)(k_\mu k_\nu-k^2g_{\mu\nu})-[m_t^2z+m_b^2(1-z)]g_{\mu\nu}}{\{p'^2+k^2z(1-z)-m_t^2z-m_b^2(1-z)\}^2}.\end{aligned} \tag{11.25}$$

The integral is calculated using the usual identity (Eq. 6.10) to give

$$\begin{aligned} i\Pi^{WW}_{\mu\nu} &= -3g_0^2\mu^{4-d}|V_{tb}|^2\int_0^1\frac{i(-\pi)^{d/2}\Gamma(\epsilon)dz}{(2\pi)^d\Gamma(2)[k^2z(1-z)-m_t^2z-m_b^2(1-z)]^\epsilon}\times\\ &\quad \{-2z(1-z)(k_\mu k_\nu-k^2g_{\mu\nu})-[m_t^2z+m_b^2(1-z)]g_{\mu\nu}\},\end{aligned} \tag{11.26}$$

where $f(d)$ has been set to 4 and ϵ has been defined to be $2-d/2$. Now we use $\Gamma(\epsilon)\approx\epsilon^{-1}-\gamma$, $\Gamma(2)=1$, and $a^\epsilon\approx 1+\epsilon\ln a$ to obtain

$$\begin{aligned} i\Pi^{WW}_{\mu\nu} &= -3g_0^2|V_{tb}|^2\int_0^1\frac{idz}{16\pi^2}\left\{1+\epsilon\ln\frac{4\pi\mu^2}{[-k^2z(1-z)+m_t^2z+m_b^2(1-z)]}\right\}\\ &\quad\times\left(\frac{1}{\epsilon}-\gamma\right)\{-2z(1-z)(k_\mu k_\nu-k^2g_{\mu\nu})-[m_t^2z+m_b^2(1-z)]g_{\mu\nu}\}\\ &= \frac{-3ig_0^2|V_{tb}|^2}{16\pi^2}\{\left[\frac{k^2-k^\mu k^\nu}{3}-\frac{m_t^2+m_b^2}{2}\right]\left(\frac{1}{\epsilon}-\gamma\right)+\\ &\quad\int_0^1 dz\left[z(1-z)(k^2g_{\mu\nu}-2k_\mu k_\nu)-\Delta g_{\mu\nu}\right]\ln\frac{4\pi\mu^2}{\Delta}\},\end{aligned} \tag{11.27}$$

where

$$\Delta = m_t^2z+m_b^2(1-z)-k^2z(1-z). \tag{11.28}$$

The correction has a divergent term, which is removed through renormalization, and a finite term proportional to the square of the top-quark mass.

CHAPTER 12

THE HIGGS BOSON

The Electroweak theory depends crucially on a mechanism to charge the vacuum and hide the underlying SU(2) $\times$ U(1) gauge symmetry. The Higgs boson was the last unobserved particle in the Standard Model and its discovery now provides direct access to the scalar sector of interactions. It fixes the μ parameter in the scalar potential, completing the input parameters of the minimal Standard Model (still undetermined are the masses and mixing phase of the neutrinos). Prior to the discovery, loop contributions of the Higgs boson to the W-boson mass gave a predicted Higgs-boson mass of 94^{+29}_{-24} GeV, with an upper bound of 152 GeV at 95% CL. Parts of this mass range were accessible to the LEP and Tevatron colliders, and the entire mass range (and up to a TeV or more) was accessible to the LHC.

The recent LHC discovery of a boson with a mass of about 125 GeV allows a variety of tests of the Standard Model predictions. In this mass range the Higgs boson can be measured using many production mechanisms and decays, individually testing the relationship between the mass of each fermion or gauge boson and its coupling to the Higgs boson.

12.1 Higgs boson mass

The renormalization procedure allows the calculation of physical quantities, given input measurements of the model parameters at a particular scale. The potential divergences are prevented by unknown contributions at a higher scale. If nothing else, we expect all quantities to be affected by gravitational interactions at the Planck scale. When parameter values at the measurement scale are highly sensitive to the values at the Planck scale, we say that the theory is fine-tuned. Most of the parameters of the theory grow logarithmically with energy, so no fine-tuning is required. However, the Higgs boson mass grows linearly with energy, so there is fine-tuning already near the TeV energy scale. This can be seen by explicitly calculating the one-loop contributions to the Higgs boson mass.

We have evaluated the one-loop contribution to a scalar propagator using dimensional regularization. In order to study fine-tuning as a function of the scale of new physics, it is more useful to use a cut-off in the integral. The vertex of the four-Higgs interaction can be expressed as $\frac{3ig^2m_H^2}{4m_W^2}$ and the vertex of the three-Higgs interaction can be expressed as $\frac{3igm_H^2}{2m_W^2}$, where $g = e\sin\theta_W$ is the weak coupling. The loop from the four-point vertex contributes the following term to the propagator:

$$\frac{-3ig^2m_H^2}{2\times 4m_W^2}\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m_H^2+i\epsilon} \equiv -i\Sigma^{loop\ H}(p^2), \tag{12.1}$$

where the factor of two in the denominator is a symmetry factor from the loop. The integral can be easily evaluated by the replacement $dE \rightarrow idE$ (a Wick rotation), so that the denominator is $-q^2 - m_H^2$. The integration factor d^4q can be expressed as $2\pi^2q^3dq$ after performing the angular integration. Changing variables to $u = q^2$, the integral becomes

$$\int_0^{\Lambda^2}\frac{-1}{8\pi^2}\frac{u}{2(u+m_H^2)}du = \frac{-1}{16\pi^2}[\Lambda^2 - m_H^2\ln(\Lambda^2/m_H^2+1)]. \tag{12.2}$$

Including the vertex factor and neglecting the logarithmic term, we get

$$\Sigma^{loop\ H}(p^2) = \frac{-3g^2m_H^2\Lambda^2}{32\pi^2\times 4m_W^2}. \tag{12.3}$$

The three-point function will contribute a logarithmic divergence, since it has two propagators in the denominator giving a factor of $1/q^4$. So we can neglect this contribution. The mass becomes

$$\begin{aligned}(m_H^{loop\ H})^2 &= m_H^2 - \Sigma^{loop\ H}(p^2)\\ &= m_H^2 - \frac{3g^2m_H^2\Lambda^2}{32\pi^2\times 4m_W^2}\\ &= m_H^2 - \frac{3\times 4\pi(128)^{-1}\times(1-80.385^2/91.188^2)^{-1}\times 125^2\Lambda^2}{32\pi^2\times 4\times 80.385^2}\\ &= m_H^2 - 2.53\times 10^{-3}\Lambda^2/\mathrm{GeV}^2.\end{aligned} \tag{12.4}$$

The dominant divergent fermion loop is the top loop, since its coupling is significantly greater than the other fermions. The vertex factor is $-igm_t/(2m_W)$, giving a loop contribution of

$$\frac{3g^2m_t^2}{4m_W^2}\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\gamma_\alpha p^\alpha - m_t}\frac{i}{\gamma_\beta p^\beta - \gamma_\beta k^\beta - m_t}\right] \equiv -i\Sigma^{loop\ t}(p^2), \tag{12.5}$$

where the factor of 3 accounts for the three top-quark colors and a -1 comes from the internal fermion loop. Moving all gamma matrices to the numerator by multiplying numerator and denominator by the same factor, and separating the propagators with the Feynman integral gives:

$$-i\Sigma^{loop\ t} = -\frac{3g^2m_t^2}{4m_W^2}\int_0^1 dz \int \frac{d^4p}{(2\pi)^4}\frac{\text{Tr}[(\gamma_\alpha p^\alpha + m_t)(\gamma_\beta p^\beta - \gamma_\beta k^\beta + m_t)]}{\{(p^2-m_t^2)z + [(p-k)^2 - m_t^2](1-z)\}^2}. \tag{12.6}$$

Redefining the integrand to $p' = p - kz$ and recalling that integrals over terms linear in p' give zero, the numerator becomes

$$[p'^\alpha p'^\beta - k^\alpha k^\beta z(1-z)]\text{Tr}(\gamma_\alpha\gamma_\beta) + 4m_t^2. \tag{12.7}$$

The trace can be evaluated using $\text{Tr}(\gamma_\alpha\gamma_\beta) = 4g_{\alpha\beta}$. The dominant divergence will come from the p^2 term, giving

$$-i\Sigma^{loop\ t} = -\frac{3g^2m_t^2}{4m_W^2}\int_0^1 dz \int \frac{d^4p}{(2\pi)^4}\frac{4p^2}{[p^2 - m_t^2 + k^2z(1-z)]^2} + ... \tag{12.8}$$

With a Wick rotation and defining $u = p^2 + m_t^2 - k^2z(1-z)$, keeping only the Λ^2 term:

$$\begin{aligned}\Sigma^{loop\ t} &= \frac{3g^2m_t^2}{16\pi^2m_W^2}\Lambda^2 \\ &= \frac{3\times 4\pi(128)^{-1}\times(1-80.385^2/91.188^2)^{-1}\times 173^2}{16\pi^2\times 80.385^2}\Lambda^2 \\ &= 3.88\times 10^{-2}\Lambda^2. \end{aligned}\tag{12.9}$$

Combining top-quark and Higgs-boson loops, the Higgs boson mass is

$$\begin{aligned}(m_H^{loop})^2 &= m_H^2 - 2.53\times 10^{-3}\Lambda^2/\text{GeV}^2 + 3.88\times 10^{-2}\Lambda^2/\text{GeV}^2 \\ &= m_H^2 + 3.63\times 10^{-2}\Lambda^2. \end{aligned}\tag{12.10}$$

Fine-tuning to a given percentage would mean that the difference on the right is that percentage times m_H^2. For 1% tuning, we get

$$125^2 = 0.01 \times 3.6\times 10^{-2}\Lambda^2, \tag{12.11}$$

or $\Lambda = 6.6$ TeV.

12.2 Higgs boson production at hadron colliders

The relative cross sections for Higgs boson production are dominated by the couplings of the Higgs boson to fermions and vector bosons. At the LHC the process with the highest cross section contains the $t\bar{t}H$ vertex, where the $t\bar{t}$ pair is produced by a t-channel interaction between two gluons. The diagram, known as gluon-gluon fusion, is a triangle top loop with two external gluons and an external Higgs boson. The next highest cross sections are the vector-boson fusion processes $WW \to H$ and $ZZ \to H$, where the W and Z bosons are radiated by the incoming quarks. Associated production follows: WH, ZH, $t\bar{t}H$, involving the same vertices but suppressed by the higher combined mass of the final-state system. Production via the $b\bar{b}H$ coupling is small by comparison, but can be relevant in other models. The Higgs boson was first observed via gluon-gluon fusion production, with several final states showing significances near 5σ. The vector-boson fusion signal reached $\approx 5\sigma$ with ATLAS and CMS 7-8 TeV data combined. The associated production channels should soon be observed in the ongoing 13 TeV collisions at the LHC.

12.2.1 Gluon fusion [27]

The matrix element for gluon-gluon fusion has two components corresponding to the two directions of the internal top-quark lines:

$$\begin{aligned}\mathcal{M} &= (-ig_s)^2\left(\frac{-igm_t}{2m_W}\right)i^3\mathrm{Tr}(t_at_b)(-1)\epsilon_\nu\epsilon_\mu\int\frac{d^dq}{(2\pi)^d}\\ &\left[\frac{i}{\gamma_\alpha q^\alpha+\gamma_\alpha k_2^\alpha-m_t}\gamma^\mu\frac{i}{\gamma_\beta q^\beta-m_t}\gamma^\nu\frac{i}{\gamma_\rho q^\rho-\gamma_\rho k_1^\rho-m_t}\right]+\\ &\left[\frac{i}{-\gamma_\alpha q^\alpha+\gamma_\alpha k_1^\alpha-m_t}\gamma^\nu\frac{i}{-\gamma_\beta q^\beta-m_t}\gamma^\mu\frac{i}{-\gamma_\rho q^\rho-\gamma_\rho k_2^\rho-m_t}\right],\end{aligned} \quad (12.12)$$

where the ϵ factors are the external gluon polarizations, t are the adjoint representations of the gluon color charges, g_s is the strong coupling constant, the integral is performed in d dimensions, and there is an overall factor of -1 because of the fermion loop. Rather than go through the extensive calculation of the matrix element squared, we consider its components and the various contributions to the cross section.

The cross section is given by

$$\begin{aligned}\sigma &= \frac{1}{2s}\int\frac{d^4p}{(2\pi)^4}2\pi\delta(p^2-m_H^2)(2\pi)^4\delta^4(k_1+k_2-p)\frac{1}{4}\frac{1}{64}\sum_{s,c}|\mathcal{M}|^2\\ &= \frac{\pi}{256s}\delta(s-m_H^2)\sum_{s,c}|\mathcal{M}|^2,\end{aligned} \quad (12.13)$$

where the factor of 1/4 comes from averaging over initial-state spins, and the factor of 1/64 comes from averaging over initial-state colors. We can additionally separate out the prefactors in the matrix element,

$$|\mathcal{M}|^2 \equiv g_s^4\frac{g^2m_t^2}{4m_W^2}\mathrm{Tr}(t_at_b)\mathrm{Tr}(t_at_b)|\mathcal{M}_{int}|^2. \quad (12.14)$$

We now use $\alpha_s = g_s^2/4\pi$, $G_F = g^2/(2\sqrt{2}m_W)$, and $\mathrm{Tr}(t_at_b) = \delta_{ab}/2$ and $\delta_{ab}\delta_{ab} = 8$ for a sum over color factors. The cross section becomes

$$\sigma = \frac{\pi^3\alpha_s^2\sqrt{2}G_Fm_t^2}{16m_H^4}\delta(s-m_H^2)\sum_{s,c}m_H^2|\mathcal{M}_{int}|^2, \quad (12.15)$$

The matrix element has the form

$$\mathcal{M}_{int} = a(\epsilon_1\cdot\epsilon_2) - \frac{2}{m_H^2}a(\epsilon_1\cdot k_2)(\epsilon_2\cdot k_1), \quad (12.16)$$

where

$$\begin{aligned}a &= \frac{8m_t}{1-\frac{1}{2}m_H^2(1-\tau)16\pi^2C_0},\\ \tau &= \frac{4m_t^2}{m_H^2},\\ C_0 &= \frac{-i\arcsin^2(\tau^{-1/2})}{8\pi^2s}\end{aligned} \quad (12.17)$$

for $m_H < m_t$. Summing over spin states gives $|\mathcal{M}_{int}|^2 = 2|a|^2$, so the total cross section is

$$\sigma = \frac{\alpha_s^2 G_F m_H^2}{288\sqrt{2}\pi} \left\{ \frac{3}{2}\tau \left[1 + (1-\tau)\arcsin^2(\tau^{-1/2}) \right] \right\}^2 \delta(s - m_H^2). \tag{12.18}$$

The cross section is approximately independent of the mass of the top quark since for $\tau \to \infty$ we have the approximation $\arcsin(\tau^{-1/2}) \approx 1/\tau$. Thus, the mass dependence from the propagator in the loop ($\propto 1/m_t^2$) cancels the Yukawa coupling at the ttH vertex ($\propto m_t^2/m_W^2$).

The leading-order term is at the order of α_s^2 and there are large corrections from higher orders. The calculation has been performed to NNNLO accuracy (α_s^5!), and there are signs of convergence (the uncertainty from missing higher orders is estimated to be 3%).

12.2.2 Vector-boson fusion

Higgs boson production through vector-boson fusion was one of the original processes studied to show the problems that arise in the absence of a physical scalar field, and to constrain its mass. In particular, the $WW \to WW$ process (WW scattering) has a matrix element that grows quadratically with energy in the absence of a Higgs boson. Since the scattering of a massless gauge boson does not have this behavior, the unphysical growth has to come from the longitudinal degree of freedom (the Goldstone boson). From the perspective of the Electroweak theory, its origin is the consideration of only a subset of diagrams of a ϕ^4 theory; the physical Higgs boson is required for a complete renormalizable ϕ^4 theory.

The calculation of WW scattering involves matrix elements for a 4-point vertex and s- and t-channel $\gamma/Z/H$ exchange. The outgoing W-boson lines have associated polarization vectors, and we consider only the longitudinal polarizations since these are the polarizations that arise from the Higgs mechanism. The longitudinal polarization can be calculated by considering a boost of the polarization vector (0,0,0,1) along the z axis:

$$\begin{aligned} \epsilon_\mu &= (\gamma v, 0, 0, \gamma) \\ &= (p_1/m_W, 0, 0, E/m_W) \\ &\approx (E/m_W - m_W/2E, 0, 0, E/m_W). \end{aligned} \tag{12.19}$$

Projecting along the vector gives a factor $\vec{p}_1/|\vec{p}_1| \approx (1 + m_W^2/2E^2)\vec{p}_1/E$ for the spatial components of the polarization vector. Then the polarization tensor can be written:

$$\begin{aligned} \epsilon_\mu &= \frac{p_{1\mu}}{m_W} - \frac{m_W}{2E^2}(E, -\vec{p}_1) \\ &= \frac{p_{1\mu}}{m_W} - \frac{2m_W}{s} p_{2\mu}, \end{aligned} \tag{12.20}$$

where the last line is a general feature of $2 \to 2$ processes such as WW scattering.

Now we consider the 4-point WW vertex, whose matrix element is simply given by the Feynman rule for the coupling and the external polarizations $\epsilon_\mu(p_1)\epsilon_\nu(p_2) \to \epsilon_\rho(k_1)\epsilon_\lambda(k_2)$:

$$\begin{aligned}
\mathcal{M}_4 &= ig^2\{2[\epsilon(p_2)\epsilon(k_1)][\epsilon(p_1)\epsilon(k_2)] - [\epsilon(p_1)\epsilon(p_2)][\epsilon(k_1)\epsilon(k_2)] - [\epsilon(p_1)\epsilon(k_1)][\epsilon(p_2)\epsilon(k_2)]\} \\
&\approx i\frac{g^2}{m_W^4}\{[2(p_2k_1)(p_1k_2) - (p_1p_2)(k_1k_2) - (p_1k_1)(p_2k_2)] + \frac{2m_W^2}{s} \times \\
&\quad [2(p_1k_1 + p_2k_2)(p_1k_2 + p_2k_1) - 2m_W^2(p_1p_2 + k_1k_2) - (p_1k_2 + p_2k_1)(p_2k_2 + p_1k_1)]\} \\
&\approx i\frac{g^2}{4m_W^4}\left[2(s+t-2m_W^2)^2 - (s-2m_W^2)^2 - (t-2m_W^2)^2 - \frac{8m_W^2}{s}tu\right] \\
&\approx i\frac{g^2}{4m_W^4}\left[s^2 + t^2 + 4st - 4(s+t)m_W^2 - \frac{8m_W^2}{s}tu\right], \qquad (12.21)
\end{aligned}$$

where we have used $p_1p_2 = k_1k_2 = s/2 - m_W^2$, $p_1k_1 = p_2k_2 = -t/2 + m_W^2$, $p_1k_2 = p_2k_1 = -u/2 + m_W^2$, and $s + t + u = 4m_W^2$.

We next consider s-channel γ/Z exchange:

$$\begin{aligned}
\mathcal{M}_{s(\gamma/Z)} &= -ig^2\left(\frac{\sin^2\theta_W}{s} + \frac{\cos^2\theta_W}{s - m_Z^2}\right)\{(p_1 - p_2)_\mu[\epsilon(p_1)\epsilon(p_2)] + \\
&\quad [(-2p_1 - p_2)\epsilon(p_2)]\epsilon_\mu(p_1) + [(p_1 + 2p_2)\epsilon(p_1)]\epsilon_\mu(p_2)\}\{(k_2 - k_1)^\mu[\epsilon(k_1)\epsilon(k_2)] \\
&\quad +[(-2k_2 - k_1)\epsilon(k_1)]\epsilon^\mu(k_2) + [(2k_1 + k_2)\epsilon(k_2)]\epsilon^\mu(k_1)\} \qquad (12.22)
\end{aligned}$$

The polarization vectors satisfy $p_\mu\epsilon^\mu = 0$, so the matrix element becomes

$$\begin{aligned}
\mathcal{M}_{s(\gamma/Z)} &= -i\frac{g^2}{m_W^4}\left(\frac{\sin^2\theta_W}{s} + \frac{\cos^2\theta_W}{s - m_Z^2}\right)[(p_1k_2 + p_2k_1 - p_1k_1 - p_2k_2)(p_1p_2)(k_1k_2) + \\
&\quad (p_1k_2 - p_2k_2 - 2p_1k_1m_W^2/s + 2p_2k_1m_W^2/s)(p_1p_2)(-2k_2k_1) + \\
&\quad (p_1k_1 - p_2k_1 - 2p_1k_2m_W^2/s + 2p_2k_2m_W^2/s)(p_1p_2)(2k_1k_2) + \\
&\quad (-2p_1p_2)(p_1k_2 - p_1k_1 - 2p_2k_2m_W^2/s + 2p_2k_1m_W^2/s)(k_1k_2) + \\
&\quad (-2p_1p_2)(p_1k_2 - 2m_W^2p_1k_1/s - 2m_W^2p_2k_2/s)(-2k_2k_1) + \\
&\quad (-2p_1p_2)(p_1k_1 - 2m_W^2p_2k_1/s - 2m_W^2p_1k_2/s)(2k_1k_2) + \\
&\quad (2p_2p_1)(k_1k_2)(p_2k_2 - p_2k_1 - 2m_W^2p_1k_2/s + 2m_W^2p_1k_1/s) + \\
&\quad (2p_2p_1)(p_2k_2 - 2m_W^2p_1k_2/s - 2m_W^2p_2k_1/s)(-2k_2k_1) + \\
&\quad (2p_2p_1)(p_2k_1 - 2m_W^2p_1k_1/s - 2m_W^2p_2k_2/s)(2k_1k_2)]. \qquad (12.23)
\end{aligned}$$

Again using the Mandelstam variables and assuming $m_W^2/s \ll 1$, we get:

$$\begin{aligned}
\mathcal{M}_{s(\gamma/Z)} &\approx i\frac{g^2}{m_W^4 s}(1 + m_W^2/s)\{(u-t)s^2/4 - (u-t)s^2/4 + (t-u)s^2/4 - \\
&\quad (u-t)s^2/4 + s^2u/2 - s^2t/2 + s^2(t-u)/4 - s^2t/2 + s^2u/2 + m_W^2 \times \\
&\quad [(t-u)s + (u-t)s/4 + (t-u)s/2 + 3(u-t)s/4 + (t-u)s/2 + \\
&\quad (u-t)s/4 + (t-u)s/2 - 2us - 2ts + 2ts + 2us + 3(u-t)s/4 + \\
&\quad (t-u)s/2 + 2ts + 2us - 2us - 2ts]\} \\
&\approx -i\frac{g^2}{4m_W^4}\left[s(t-u) - 3m_W^2(t-u)\right]. \qquad (12.24)
\end{aligned}$$

The t-channel Z/γ exchange can be obtained by simply exchanging p_2 with $-k_1$:

$$\begin{aligned}\mathcal{M}_{t(\gamma/Z)} \quad = \quad & -ig^2\left(\frac{\sin^2\theta_W}{t}+\frac{\cos^2\theta_W}{t-m_Z^2}\right)\{(p_1+k_1)_\mu[\epsilon(p_1)\epsilon(k_1)]-[2p_1\epsilon(k_1)]\epsilon(p_1)_\mu \\ & -[2k_1\epsilon(p_1)]\epsilon(k_1)_\mu\}\{(k_2+p_2)^\mu[\epsilon(p_2)\epsilon(k_2)]-[2k_2\epsilon(p_2)]\epsilon(k_2)^\mu- \\ & [2p_2\epsilon(k_2)]\epsilon(p_2)^\mu\}. \qquad (12.25)\end{aligned}$$

Multiplying the terms gives

$$\begin{aligned}\mathcal{M}_{t(\gamma/Z)} \quad = \quad & -i\frac{g^2}{m_W^4}\left(\frac{\sin^2\theta_W}{t}+\frac{\cos^2\theta_W}{t-m_Z^2}\right)[(p_1k_2+p_1p_2+k_1k_2+k_1p_2)\times \\ & (p_1k_1-2p_1k_2m_W^2/s-2p_2k_1m_W^2/s)\times \\ & (p_2k_2-2p_1k_2m_W^2/s-2p_2k_1m_W^2/s)+ \\ & (p_1k_2+k_1k_2-2p_1k_1m_W^2/s)(p_1k_1-2p_1k_2m_W^2/s-2p_2k_1m_W^2/s)\times \\ & (-2k_2p_2+4k_2p_1m_W^2/s)+(p_1p_2+k_1p_2-2k_1p_1m_W^2/s)\times \\ & (p_1k_1-2p_1k_2m_W^2/s-2p_2k_1m_W^2/s)(-2p_2k_2+4p_2k_1m_W^2/s)+ \\ & (-2p_1k_1+4p_1k_2m_W^2/s)(p_1k_2+p_1p_2-2p_2k_2m_W^2/s)\times \\ & (p_2k_2-2p_1k_2m_W^2/s-2p_2k_1m_W^2/s)+ \\ & (-2p_1k_1+4p_1k_2m_W^2/s)(p_1k_2-2m_W^2p_1k_1/s-2m_W^2p_2k_2/s)\times \\ & (-2k_2p_2+4k_2p_1m_W^2/s)+ \\ & (-2p_1k_1+4p_1k_2m_W^2/s)(p_1p_2)(-2p_2k_2+4p_2k_1m_W^2/s)+ \\ & (-2k_1p_1+4k_1p_2m_W^2/s)(k_1k_2+k_1p_2-2k_2p_2m_W^2/s)\times \\ & (p_2k_2-2p_1k_2m_W^2/s-2p_2k_1m_W^2/s)+ \\ & (-2k_1p_1+4k_1p_2m_W^2/s)(k_1k_2)(-2k_2p_2+4k_2p_1m_W^2/s) \\ & (-2k_1p_1+4k_1p_2m_W^2/s)(k_1p_2-2k_2p_2m_W^2/s-2k_1p_1m_W^2/s)\times \\ & (-2p_2k_2+4p_2k_1m_W^2/s). \qquad (12.26)\end{aligned}$$

In terms of Mandelstam variables this is

$$\mathcal{M}_{t(\gamma/Z)} \quad \approx \quad -i\frac{g^2}{4m_W^4}\left[t(s-u)-3m_W^2(s-u)+8m_W^2u^2/s\right]. \qquad (12.27)$$

Combining the gauge interactions gives

$$\begin{aligned}& \mathcal{M}_4+\mathcal{M}_{s(\gamma/Z)}+\mathcal{M}_{t(\gamma/Z)}\approx i\tfrac{g^2}{4m_W^4} \\ & [s^2+t^2+4st-4(s+t)m_W^2-8m_W^2tu/s-s(t-u)+3m_W^2(t-u)-t(s-u)+ \\ & 3m_W^2(s-u)-8m_W^2u^2/s]. \qquad (12.28)\end{aligned}$$

The term quadratic in the Mandelstam variables is

$$\begin{aligned}i\frac{g^2}{4m_W^4}[s^2+t^2+4st-st+su-ts+tu] \quad & = \quad i\frac{g^2}{4m_W^4}[(s+t)^2+(s+t)(-s-t+4m_W^2)] \\ & = \quad -i\frac{g^2}{m_W^2}(u-4m_W^2). \qquad (12.29)\end{aligned}$$

The term linear in the Mandelstam variables is

$$i\tfrac{g^2}{4m_W^4}[-4(s+t)m_W^2 - 8m_W^2 tu/s + 3m_W^2(t-u) + 3m_W^2(s-u) - 8m_W^2 u^2/s] =$$
$$i\tfrac{g^2}{4m_W^2}[4(u-4m_W^2) - 8(t+u)u/s + 3(s+t-2u)] =$$
$$i\tfrac{g^2}{4m_W^2}[4u - 16m_W^2 - 8(-s+4m_W^2)u/s + 3(-3u+4m_W^2)] =$$
$$i\tfrac{g^2}{4m_W^2}[3u - 4m_W^2 - 32m_W^2 u/s]. \quad (12.30)$$

Combining the terms gives

$$\begin{aligned}\mathcal{M}_4 + \mathcal{M}_{s(\gamma/Z)} + \mathcal{M}_{t(\gamma/Z)} &\approx -i\frac{g^2}{4m_W^2}(4u - 16m_W^2 - 3u + 4m_W^2 + 32m_W^2 u/s) \\ &\approx -i\frac{g^2}{4m_W^2}(u - 12m_W^2 + 32m_W^2 u/s). \quad (12.31)\end{aligned}$$

The matrix elements for Higgs exchange are more straightforward. For the s-channel we have

$$\begin{aligned}\mathcal{M}_{s(H)} &= -ig^2\frac{m_W^2}{s-m_H^2}[\epsilon(p_1)\epsilon(p_2)][\epsilon(k_1)\epsilon(k_2)] \\ &\approx ig^2\frac{1}{m_W^2 s}(1+m_H^2/s)(s/2-m_W^2)(s/2-m_W^2) \\ &\approx ig^2\frac{1}{m_W^2 s}(s^2/4 - m_H^2 s/4 - m_W^2 s), \quad (12.32)\end{aligned}$$

while the t-channel expression is

$$\begin{aligned}\mathcal{M}_{t(H)} &= -ig^2\frac{m_W^2}{t-m_H^2}[\epsilon(p_1)\epsilon(k_1)][\epsilon(p_2)\epsilon(k_2)] \\ &\approx ig^2\frac{1}{m_W^2 t}(1+m_H^2/t)[(t/2-m_W^2)(t/2-m_W^2) - 2tum_W^2/s] \\ &\approx ig^2\frac{1}{m_W^2 t}(t^2/4 - m_H^2 t/4 - m_W^2 t - 2tum_W^2/s). \quad (12.33)\end{aligned}$$

Combining the terms linear in the Mandelstam variables gives

$$\begin{aligned}\mathcal{M}_{s(H)} + \mathcal{M}_{t(H)} &= -i\frac{g^2}{4m_W^2}(s+t) \\ &= -i\frac{g^2}{4m_W^2}(-u+4m_W^2) \quad (12.34)\end{aligned}$$

We see that the linear term cancels once the Higgs propagator is included in the matrix element.

A full analysis, including longitudinal Z production, shows that the cross section for boson-boson scattering violates unitarity for $m_H \gtrsim 1$ TeV. This motivates the energy reach of the LHC, which was designed to definitively determine whether the Higgs boson exists.

We have focused on WW pairs in the initial state and only considered the divergent terms. At high enough energy, the initial-state W bosons can be considered as components of the original protons as is done with quarks and gluons. At the energy of the LHC one needs to consider the $qq \to qqWW$ process, which has 96 diagrams and is thus significantly more complicated.

12.3 Higgs boson production at lepton colliders

An intensive search for Higgs boson production at LEP via the $e^+e^- \to ZH$ process, whose matrix element is given in Eq. 5.24, took place between 1996 and 2001. No evidence for Higgs boson production was observed, and a lower limit on the Higgs-boson mass was set at 114.4 GeV. Future proposed e^+e^- colliders could produce a large sample of Higgs bosons with less additional energy in the detector than a hadron collider. Studies of the possible precision of measurements of such a sample are ongoing.

An interesting long-term possibility is the construction of a muon collider to produce Higgs bosons on resonance. To get an idea of the potential rates, we can calculate the production cross section for $\tau^+\tau^-$ in $\mu^+\mu^-$ collisions. This can occur through s-channel production of a photon, a Z boson or a Higgs boson. The matrix element can be written as

$$\mathcal{M} = \mathcal{M}_Z + \mathcal{M}_\gamma + \mathcal{M}_H, \tag{12.35}$$

with

$$\begin{aligned}\mathcal{M}_Z &= \int \frac{d^4k}{(2\pi)^4}\delta^4(p_1+p_2-k)\bar{v}_2\frac{ig\gamma_\mu}{4\cos\theta_W}\,(g_V^\mu - g_A^\mu\gamma_5)\,u_1\frac{-ig^{\mu\nu}}{k^2-m_Z^2+i\Gamma_Z}\times \\ &\quad \delta^4[k-(p_{1'}+p_{2'})]\bar{u}_{1'}\frac{ig\gamma_\nu}{4\cos\theta_W}\,(g_V^\tau - g_A^\tau\gamma_5)\,v_{2'},\end{aligned} \tag{12.36}$$

$$\begin{aligned}\mathcal{M}_\gamma &= \int \frac{d^4k}{(2\pi)^4}\delta^4[p_1+p_2-k]\bar{v}_2(-ie\gamma_\mu)u_1\frac{-ig^{\mu\nu}}{k^2+i\epsilon}\delta^4[k-(p_{1'}+p_{2'})]\times \\ &\quad \bar{u}_{1'}(-ie\gamma_\nu)v_{2'},\end{aligned} \tag{12.37}$$

and

$$\begin{aligned}\mathcal{M}_H &= \int \frac{d^4k}{(2\pi)^4}\delta^4(p_1+p_2-k)\bar{v}_2\frac{-igm_\mu}{2m_W}u_1\frac{i}{k^2-m_H^2+i\Gamma_H}\delta^4(k-p_{1'}-p_{2'}) \\ &\quad \times\bar{u}_{1'}\frac{-igm_\tau}{2m_W}v_{2'}\end{aligned} \tag{12.38}$$

in the Feynman gauge. We have already evaluated the contributions of the Z and γ propagators for $e^+e^- \to \mu^+\mu^-$; the contributions are the same for $\mu^+\mu^- \to \tau^+\tau^-$ at $\sqrt{s} = 125$ GeV since we can neglect lepton masses. The additional contributions from the Higgs-propagator matrix element are:

$$\Delta|\mathcal{M}|^2 = |\mathcal{M}_H|^2 + \mathcal{M}_H^*\mathcal{M}_\gamma + \mathcal{M}_\gamma^*\mathcal{M}_H + \mathcal{M}_H^*\mathcal{M}_Z + \mathcal{M}_Z^*\mathcal{M}_H^*. \tag{12.39}$$

The square of the Higgs propagator can be calculated by integrating over k:

$$\begin{aligned}|\mathcal{M}_H|^2 &= \frac{g^4m_\mu^2m_\tau^2}{16m_W^4[(s-m_H^2)^2+m_H^2\Gamma_H^2]}[\bar{v}_2u_1\bar{u}_{1'}v_{2'}][\bar{v}_{2'}u_{1'}\bar{u}_1v_2] \\ &= \frac{g^4m_\mu^2m_\tau^2}{16m_W^4m_H^2\Gamma_H^2}\mathrm{Tr}[\bar{u}_1v_2\bar{v}_2u_1\bar{u}_{1'}v_{2'}\bar{v}_{2'}u_{1'}] \\ &= \frac{g^4m_\mu^2m_\tau^2}{16m_W^4m_H^2\Gamma_H^2}\mathrm{Tr}[u_1\bar{u}_1v_2\bar{v}_2]\mathrm{Tr}[u_{1'}\bar{u}_{1'}v_{2'}\bar{v}_{2'}] \\ &= \frac{g^4m_\mu^2m_\tau^2}{64m_W^4m_H^2\Gamma_H^2}[4(p_2p_1)4(p_{1'}p_{2'})]. \\ &= \frac{g^4m_\mu^2m_\tau^2m_H^2}{16m_W^4\Gamma_H^2},\end{aligned} \tag{12.40}$$

where we have averaged over initial spin states and used $\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4g_{\mu\nu}$ and $p_1 p_2 = p_{1'} p_{2'} = s/2$. As expected, there is no angular dependence since the Higgs boson is a scalar particle.

Turning to the interference terms, we first consider interference between the Higgs boson and the photon. Again integrating over the propagators' momenta, we get:

$$\begin{aligned}
\mathcal{M}_H \mathcal{M}_\gamma^* &= \bar{v}_2 \frac{-igm_e}{2m_W} u_1 \frac{i}{s - m_H^2 + im_H\Gamma_H} \bar{u}_{1'} \frac{-igm_\mu}{2m_W} v_{2'} \frac{g^{\alpha\beta}}{s} [\bar{v}_{2'} \gamma_\alpha u_{1'} \bar{u}_1 \gamma_\beta v_2] \\
&= \frac{e^2 g^2 m_e m_\mu}{4m_W^2} \frac{g^{\alpha\beta}}{s(s - m_H^2 + im_H\Gamma_H)} \bar{v}_2 u_1 \bar{u}_{1'} v_{2'} \bar{v}_{2'} \gamma_\alpha u_{1'} \bar{u}_1 \gamma_\beta v_2 \\
&= \frac{e^2 g^2 m_e m_\mu}{4m_W^2} \frac{g^{\alpha\beta}}{s(s - m_H^2 + im_H\Gamma_H)} [\bar{u}_1 \gamma_\beta v_2 \bar{v}_2 u_1][\bar{u}_{1'} v_{2'} \bar{v}_{2'} \gamma_\alpha u_{1'}] \\
&= \frac{e^2 g^2 m_e m_\mu}{4m_W^2} \frac{g^{\alpha\beta}}{s(s - m_H^2 + im_H\Gamma_H)} \mathrm{Tr}[u_1 \bar{u}_1 \gamma_\beta v_2 \bar{v}_2] \mathrm{Tr}[u_{1'} \bar{u}_{1'} v_{2'} \bar{v}_{2'} \gamma_\alpha] \\
&= 0, \qquad (12.41)
\end{aligned}$$

where in the last line we have used the fact that the trace of an odd number of gamma matrices is zero. This is to be expected, since the Higgs boson is a spin-0 state and the photon only has spin states of ± 1, so there is no interference.

Finally we consider interference between the Higgs boson and the Z boson. In the Feynman gauge the spin-1 degrees of freedom are explicitly separated from the spin-0 degree of freedom. The matrix element for the spin-1 degrees of freedom will have a similar $g_{\mu\nu}$ factor to that of the photon and give a trace over an odd number of gamma matrices. This contribution will be zero, again as expected for an interference term between spin-1 and spin-0 propagators. The spin-0 degree of freedom of the Z boson, represented by the ϕ_2 vertex and propagator, has the following matrix element:

$$\begin{aligned}
\mathcal{M}_H \mathcal{M}_Z^* &= \bar{v}_2 \frac{-igm_e}{2m_W} u_1 \frac{i}{s - m_H^2 + im_H\Gamma_H} \bar{u}_{1'} \frac{-igm_\mu}{2m_W} v_{2'} \times \\
&\quad \bar{v}_{2'} \frac{gm_e\gamma_5}{2m_W} u_{1'} \frac{-i}{s - m_Z^2 + im_Z\Gamma_Z} \bar{u}_1 \frac{gm_\mu\gamma_5}{2m_W} v_2 \\
&= -\frac{g^4 m_e^2 m_\mu^2}{16m_W^4} \frac{\bar{v}_2 u_1 \bar{u}_{1'} v_{2'} \bar{v}_{2'} \gamma_5 u_{1'} \bar{u}_1 \gamma_5 v_2}{(s - m_H^2 + im_H\Gamma_H)(s - m_Z^2 + im_Z\Gamma_Z)} \\
&= \frac{\mathrm{Tr}(u_1 \bar{u}_1 \gamma_5 v_2 \bar{v}_2) \mathrm{Tr}(u_{1'} \bar{u}_{1'} v_{2'} \bar{v}_{2'} \gamma_5)}{(s - m_H^2 + im_H\Gamma_H)(s - m_Z^2 + im_Z\Gamma_Z)} \\
&= 0, \qquad (12.42)
\end{aligned}$$

where in the last line we have used $\mathrm{Tr}(\gamma_5) = 0$. Once again there is no interference. In this case the interference term disappears because there is no overlap between the production of a scalar (the Higgs boson) and a pseudoscalar (the ϕ_2 degree of freedom). Pseudoscalar interactions are identified by the presence of a γ_5 matrix.

Now we combine the terms to calculate the cross section. Since the Higgs term has no angular dependence, its contribution to the cross section will be $|\mathcal{M}_H|^2/(16\pi m_H^2)$. The cross section is

$$\sigma = \frac{4\pi\alpha^2}{3m_H^2}\{1 + \frac{m_H^4[(g_V^\mu)^2 + (g_A^\mu)^2][(g_V^\tau)^2 + (g_A^\tau)^2]}{16[(m_H^2 - m_Z^2)^2 + \Gamma_Z^2 m_Z^2]\sin^4\theta_W\cos^4\theta_W} - \frac{m_H^2(m_H^2 - m_Z^2)g_V^\mu g_V^\tau}{2[(m_H^2 - m_Z^2)^2 + \Gamma_Z^2 m_Z^2]\sin^2\theta_W\cos^2\theta_W} + \frac{3m_\mu^2 m_\tau^2 m_H^2}{64\sin^4\theta_W m_W^4 \Gamma_H^2}\}. \quad (12.43)$$

For a Higgs boson with a mass of 125 GeV, the width is about 4 MeV. Thus, the last term dominates and we have

$$\sigma \approx \frac{\pi\alpha^2 m_\mu^2 m_\tau^2}{16\sin^4\theta_W m_W^4 \Gamma_H^2}. \quad (12.44)$$

Using $\alpha \approx 128^{-1}$, $m_\mu = 0.1057$ GeV, $m_\tau = 1.777$ GeV, $\sin_W^\theta = (1 - m_W^2/m_Z^2) = 0.2229$, $m_W = 80.385$ GeV, and $\Gamma_H = 4.07$ MeV, we get $\sigma \approx 1.2 \times 10^{-8}$ GeV^{-2}. Multiplying by 0.389 GeV^2 mbarn gives 4.7 pb. Thus, a muon collider giving 1 fb^{-1} of integrated luminosity on the Higgs boson resonance would produce about 4700 Higgs bosons decaying to $\tau^+\tau^-$.

12.4 Higgs boson decay

The relation between fermion mass and its coupling to the Higgs boson gives detailed predictions for the decays of the Higgs boson. The Higgs branching ratios as a function of mass are shown in Fig. 12.1. Its partial width to fermions is given by

$$\Gamma(H \to f\bar{f}) = \frac{G_F m_f^2 m_H}{4\sqrt{2}\pi}\left(1 - \frac{4m_f^2}{m_H^2}\right)^{3/2}. \quad (12.45)$$

A Higgs boson with mass above the threshold for decay to on-shell weak bosons would have partial widths of

$$\Gamma(H \to WW) = \frac{G_F m_W^2 m_H}{8\sqrt{2}\pi}\frac{(1 - x_W)^{1/2}}{x_W}(4 - 4x_W + 3x_W^2),$$
$$\Gamma(H \to ZZ) = \frac{G_F m_Z^2 m_H}{16\sqrt{2}\pi}\frac{(1 - x_Z)^{1/2}}{x_Z}(4 - 4x_Z + 3x_Z^2), \quad (12.46)$$

where $x_{W,Z} = m_{W,Z}^2/m_H^2$. However, since at least one of the weak bosons must be off its mass shell in the decay of the observed Higgs boson, there is a suppression from the propagator that requires a full integration over the weak-boson phase space.

There is no direct coupling between the Higgs boson and the massless photon, but the Higgs boson can decay to photons through loops. For the decay through a fermion loop there are two diagrams distinguished by the direction of fermion flow in the loop. Choosing one of the orientations with external momenta $p + k$ for the Higgs and $p,\ k$ for the photons (with respective polarizations $\mu,\ \nu$), we can write the matrix element of the dominant top-

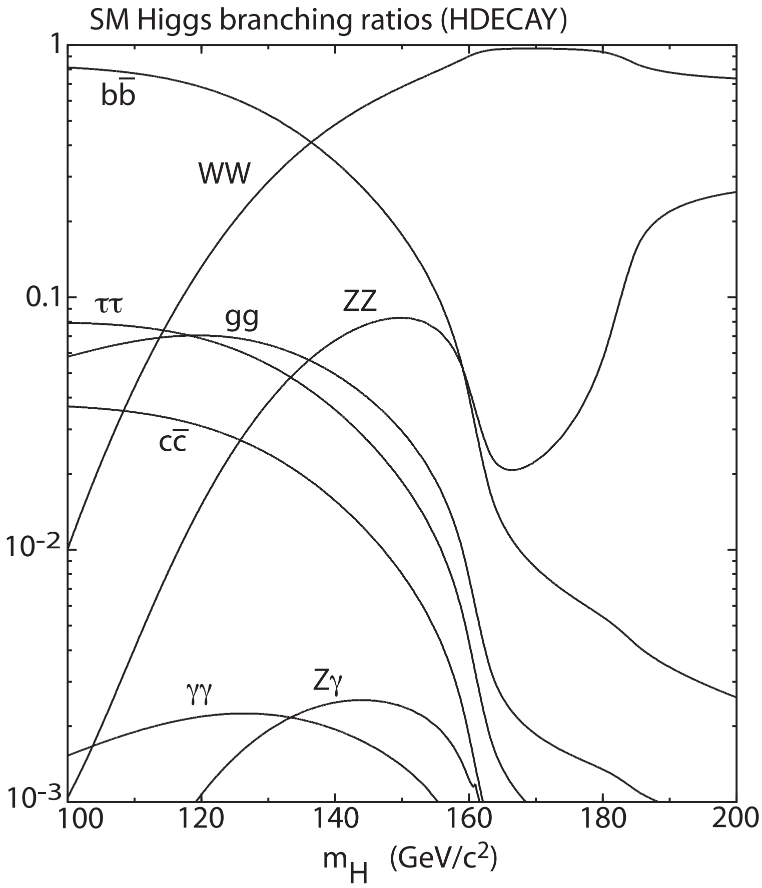


Figure 12.1 Higgs decay branching ratios.

quark loop as

$$\begin{aligned}\mathcal{M} &= (-1)\frac{-igm_t}{2m_W}\int\frac{d^4l}{(2\pi)^4}\mathrm{Tr}[\frac{i}{\gamma_\alpha(l^\alpha-k^\alpha)-m_t}\frac{2ie\gamma_\nu}{3}\frac{i}{\gamma_\beta l^\beta-m_t}\frac{2ie\gamma_\mu}{3}\times \\ &\quad \frac{i}{\gamma_\delta(l^\delta+p^\delta)-m_t}] \\ &= \frac{-2gm_te^2}{9m_W}\int\frac{d^4l}{(2\pi)^4}\times \\ &\quad \frac{\mathrm{Tr}[(\gamma_\alpha l^\alpha-\gamma_\alpha k^\alpha+m_t)\gamma_\nu(\gamma_\beta l^\beta+m_t)\gamma_\mu(\gamma_\delta l^\delta+\gamma_\delta p^\delta+m_t)]}{[(l-k)^2-m_t^2](l^2-m_t^2)[(l+p)^2-m_t^2]}. \end{aligned} \tag{12.47}$$

We combine the denominator using the two-parameter Feynman integral,

$$\frac{1}{abc} = 2\int_0^1 dx\int_0^{1-x} dy\frac{1}{[a(1-x-y)+bx+cy]^3}, \tag{12.48}$$

to obtain

$$\begin{aligned}\mathcal{M} &= \frac{-4gm_t e^2}{9m_W}\int_0^1 dx\int_0^{1-x} dy \int \frac{d^4l}{(2\pi)^4}\times \\ &\frac{\mathrm{Tr}[(\gamma_\alpha l^\alpha - \gamma_\alpha k^\alpha + m_t)\gamma_\nu(\gamma_\beta l^\beta + m_t)\gamma_\mu(\gamma_\delta l^\delta + \gamma_\delta p^\delta + m_t)]}{\{(l^2-m_t^2)(1-x-y)+[(l-k)^2-m_t^2]x+[(l+p)^2-m_t^2]y\}^3}\end{aligned} \quad (12.49)$$

Expanding the denominator,

$$l^2 - 2klx + 2ply + k^2x + p^2y - m_t^2 = (l-kx+py)^2 + 2pkxy - m_t^2, \quad (12.50)$$

we see that it can be simplified by defining $l' = l - kx + py$, and using $m_H^2 = (p+k)^2 = 2pk$. The numerator is

$$\begin{aligned}&\mathrm{Tr}[(\gamma_\alpha l^\alpha - \gamma_\alpha k^\alpha + m_t)\gamma_\nu(\gamma_\beta l^\beta + m_t)\gamma_\mu(\gamma_\delta l^\delta + \gamma_\delta p^\delta + m_t)] = \\ &\mathrm{Tr}\{[\gamma_\alpha(l^\alpha - k^\alpha)\gamma_\nu\gamma_\beta l^\beta\gamma_\mu + \gamma_\alpha(l^\alpha-k^\alpha)\gamma_\nu\gamma_\mu\gamma_\delta(l^\delta+p^\delta)+ \\ &\gamma_\nu\gamma_\beta l^\beta\gamma_\mu\gamma_\delta(l^\delta+p^\delta)]m_t + \gamma_\nu\gamma_\mu m_t^3\},\end{aligned} \quad (12.51)$$

neglecting terms with odd multiples of γ. In terms of l' this is

$$\begin{aligned}&\mathrm{Tr}\{[\gamma_\alpha(l'^\alpha - k^\alpha(1-x) - p^\alpha y)\gamma_\nu\gamma_\beta(l'^\beta + k^\beta x - p^\beta y)\gamma_\mu + \\ &\gamma_\alpha(l'^\alpha - k^\alpha(1-x) - p^\alpha y)\gamma_\nu\gamma_\mu\gamma_\delta(l'^\delta + p^\delta(1-y) + k^\delta x)+ \\ &\gamma_\nu\gamma_\beta(l'^\beta + k^\beta x - p^\beta y)\gamma_\mu\gamma_\delta(l'^\delta + p^\delta(1-y) + k^\delta x)]m_t + \gamma_\nu\gamma_\mu m_t^3\} \\ &= 4m_t[(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\nu\mu} + g_{\alpha\mu}g_{\nu\beta})(l'^\alpha - k^\alpha(1-x) - p^\alpha y)(l'^\beta + k^\beta x - p^\beta y)+ \\ &(g_{\alpha\nu}g_{\mu\delta} - g_{\alpha\mu}g_{\nu\delta} + g_{\alpha\delta}g_{\nu\mu})(l'^\alpha - k^\alpha(1-x) - p^\alpha y)(l'^\delta + p^\delta(1-y) + k^\delta x)+ \\ &(g_{\nu\beta}g_{\mu\delta} - g_{\nu\mu}g_{\beta\delta} + g_{\nu\delta}g_{\beta\mu})(l'^\beta + k^\beta x - p^\beta y)(l'^\delta + p^\delta(1-y) + k^\delta x) + g_{\nu\mu}m_t^2] \\ &= 4m_t\{(l'_\nu - k_\nu(1-x) - p_\nu y)(l'_\mu + k_\mu x - p_\mu y) - g_{\nu\mu}[l'^2 + kp(1-2x)y]+ \\ &(l'_\mu - k_\mu(1-x) - p_\mu y)(l'_\nu + k_\nu x - p_\nu y)+ \\ &(l'_\nu - k_\nu(1-x) - p_\nu y)(l'_\mu + p_\mu(1-y) + k_\mu x)- \\ &(l'_\mu - k_\mu(1-x) - p_\mu y)(l'_\nu + p_\nu(1-y) + k_\nu x)+ \\ &g_{\nu\mu}[l'^2 - kp(1-x-y+2xy)] + (l'_\nu + k_\nu x - p_\nu y)(l'_\mu + p_\mu(1-y) + k_\mu x)- \\ &g_{\nu\mu}[l'^2 + kp(x-2xy)] + (l'_\mu + k_\mu x - p_\mu y)(l'_\nu + p_\nu(1-y) + k_\nu x) + g_{\nu\mu}m_t^2]\} \\ &= 4m_t\{4l'_\nu l'_\mu - 2k_\nu k_\mu x(1-x) + 2p_\nu p_\mu y^2 + k_\nu p_\mu y(1-x) - 3p_\nu k_\mu xy- \\ &g_{\nu\mu}[l'^2 + kp(1-2x)y] - 2p_\mu p_\nu y(1-y) + k_\mu p_\nu(1-x)y- \\ &k_\nu p_\mu(1-x)(1-y) + k_\mu p_\nu(1-x)(1-y) + g_{\nu\mu}[l'^2 - kp(1-x-y+2xy)]+ \\ &2k_\mu k_\nu x^2 + k_\nu p_\mu x(1-y) + k_\mu p_\nu x(1-y) - k_{nu}p_\mu xy- \\ &g_{\nu\mu}[l'^2 + kp(x-2xy)] + g_{\nu\mu}m_t^2\} \\ &= 4m_t[4l'_\nu l'_\mu - g_{\nu\mu}l'^2 + (4y^2-2y)p_\nu p_\mu + (4x^2-2x)k_\nu k_\mu + \\ &(2x+2y-1-4xy)k_\nu p_\mu + (1-4xy)k_\mu p_\nu + g_{\nu\mu}(2xy-1)kp + g_{\nu\mu}m_t^2],\end{aligned} \quad (12.52)$$

where we have used $k^2 = p^2 = 0$ and neglected terms linear in l'. The terms with k_ν and p_μ will give zero when multiplied by the photon polarization vector. The terms quadratic in l' are potentially divergent, so we evaluate the integrals in $d = 4 - 2\epsilon$ dimensions. The quadratic terms can be evaluated using the generic integral

$$\int d^dp \frac{p_\mu p_\nu}{(p^2-\Delta)^\alpha} = (-1)^{d/2}\frac{i\pi^{d/2}}{\Gamma(\alpha)}(-\Delta)^{d/2-\alpha}\left[\frac{-\Delta g_{\mu\nu}}{2}\Gamma\left(\alpha - 1 - \frac{d}{2}\right)\right]. \quad (12.53)$$

In our case $\Delta = m_t^2 - xym_H^2$ so the quadratic terms give

$$
\begin{aligned}
&(-1)^{d/2}\tfrac{4im_t\pi^{d/2}}{\Gamma(3)(2\pi)^d}(-m_t^2+xym_H^2)^{d/2-3}\left[\tfrac{(-m_t^2+xym_H^2)g_{\mu\nu}(4-d)}{2}\Gamma(3-1-\tfrac{d}{2})\right]\\
&\quad=(-1)^{2-\epsilon}\tfrac{4im_t}{2(4\pi)^{2-\epsilon}}(-m_t^2+xym_H^2)^{-\epsilon}\left[\tfrac{g_{\mu\nu}(2\epsilon)}{2}\Gamma(\epsilon)\right]\\
&\quad=\left(\tfrac{-1}{4\pi(-m_t^2+xym_H^2)}\right)^{\epsilon}\tfrac{ig_{\mu\nu}}{32\pi^2}=\tfrac{im_tg_{\mu\nu}}{8\pi^2},
\end{aligned}
\tag{12.54}
$$

where $g_{\mu\nu}g^{\mu\nu} = d$. We see that this term is not divergent; indeed, this must be the case because there is no tree-level $H\gamma\gamma$ term in the Lagrangian to renormalize the vertex. The other terms can be evaluated using the general integral relation

$$
\int \frac{d^dp}{(p^2-\Delta)^\alpha} = (-1)^{d/2}\frac{i\pi^{d/2}\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}(-\Delta)^{d/2-\alpha}. \tag{12.55}
$$

For $\alpha = 3$ this is convergent so we obtain

$$
\int \frac{d^dp}{(2\pi)^4(p^2-\Delta)^3} = \frac{-i\pi^2}{2\Delta(2\pi)^4}\left\{(1-4xy)k_\mu p_\nu + g_{\mu\nu}\left[m_H^2\left(xy-\frac{1}{2}\right)+m_t^2\right]\right\}. \tag{12.56}
$$

Combining the terms gives

$$
\begin{aligned}
&\tfrac{-16gm_t^2e^2}{9m_W}\int_0^1 dx\int_0^{1-x} dy\tfrac{i}{32\pi^2}\left\{\tfrac{g_{\mu\nu}\left[\Delta-m_H^2(xy-\frac{1}{2})-m_t^2\right]-(1-4xy)k_\mu p_\nu}{\Delta}\right\}=\\
&\quad\tfrac{-igm_t^2e^2}{18\pi^2m_W}\int_0^1 dx\int_0^{1-x} dy\left\{\tfrac{g_{\mu\nu}\left[-m_H^2(2xy-\frac{1}{2})\right]-(1-4xy)k_\mu p_\nu}{\Delta}\right\}=\\
&\quad\tfrac{-ige^2}{18\pi^2m_W}\left(\tfrac{g_{\mu\nu}m_H^2}{2}-k_\mu p_\nu\right)\int_0^1 dx\int_0^{1-x} dy\tfrac{1-4xy}{1-xym_H^2/m_t^2}.
\end{aligned}
\tag{12.57}
$$

The diagram with the fermion loop going in the opposite direction will give an equal contribution. Summing the diagrams and squaring gives:

$$
\begin{aligned}
4|\mathcal{M}|^2 &= \frac{g^2e^4}{81\pi^4m_W^2}\left(\frac{g_{\mu\nu}g^{\mu\nu}m_H^4}{4}-2k_\mu p_\nu g^{\mu\nu}\frac{m_H^2}{2}\right)\left(\int_0^1 dx\int_0^{1-x}\frac{dy(1-4xy)}{1-xy\frac{m_H^2}{m_t^2}}\right)^2\\
&= \frac{g^2e^4m_H^4}{162\pi^4m_W^2}\left[\int_0^1 dx\int_0^{1-x} dy\frac{1-4xy}{1-xy\frac{m_H^2}{m_t^2}}\right]^2.
\end{aligned}
\tag{12.58}
$$

There will be another factor of two due to the two possible polarizations of the outgoing photons. The integral over dy can be performed using

$$
\int \frac{1-ay}{1-by}dy = \frac{a}{b}y + \frac{(a-b)\ln|by-1|}{b^2}. \tag{12.59}
$$

We can now insert the matrix element into the general formula for the width of a particle with mass m decaying into two massless particles,

$$
\Gamma = \frac{S}{16\pi m}|\mathcal{M}|^2 \tag{12.60}
$$

where S is the symmetry factor; for identical final-state particles this factor is 1/2. We thus obtain

$$
\begin{aligned}
\Gamma &= \frac{g^2\alpha^2 m_H^3}{162\pi^3 m_W^2}\left[\int_0^1 dx \int_0^{1-x} dy \frac{1-4xy}{1-xy\frac{m_H^2}{m_t^2}}\right]^2 \qquad (12.61)\\
&= \frac{2\sqrt{2}G_F\alpha^2 m_H^3}{81\pi^3}\left\{\frac{2m_t^2}{m_H^2}+\frac{m_t^2}{m_H^2}\left(4\frac{m_t^2}{m_H^2}-1\right)\int_0^1 dx \frac{\ln\left[1-\frac{m_H^2}{m_t^2}x(1-x)\right]}{x}\right\}^2 .
\end{aligned}
$$

The width is proportional to $\alpha_{EM}^2 m_H^3$, and as in the case of gluon fusion is approximately independent of the top quark mass.

CHAPTER 13

MESON MIXING

The off-diagonal Yukawa couplings in the fundamental Lagrangian lead to four physical parameters to describe flavor-changing charged-current interactions. A given flavor-changing interaction can be calculated using these parameters in the CKM matrix, which translates the mass basis to the weak basis. Neutral mesons with different quark flavors ($K^0 = \bar{s}d$, $D^0 = c\bar{u}$, $B^0 = \bar{b}d$, and $B_s^0 = \bar{b}s$) can oscillate into their antiparticles via intermediate states of different flavor. The measurements of these oscillations precisely determine the parameters of the CKM matrix, with uncertainties dominated by the non-perturbative QCD description of the bound states.

13.1 CP Violation

A key feature of the CKM matrix is the presence of a complex phase, which necessarily leads to CP violation. In the fermion Lagrangian with weak eigenstate spinors,

$$\mathcal{L}_{\text{fermion}} = i\bar{\psi}_L\gamma^\mu D_\mu\psi_L + i\bar{\psi}_R\gamma^\mu D_\mu\psi_R - (y_{ij}^d\bar{\psi}_{iL}\phi\psi_{jR}^d + y_{ij}^u\bar{\psi}_{iL}\tilde{\phi}\psi_{jR}^u + h.c.). \quad (13.1)$$

The measured fermions are mass eigenstates, related to the weak eigenstates by the CKM matrix:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}. \quad (13.2)$$

This matrix appears explicitly in the charged current Lagrangian with mass eigenstates,

$$\mathcal{L}_{\rm cc} = ig\bar{u}^j_L\gamma^\mu W^+_\mu V_{ij} d^i_L + ig^*\bar{d}^i_L\gamma^\mu W^-_\mu V^*_{ji} u^j_L. \tag{13.3}$$

Under a CP transformation, $u^j_L \to e^{i\theta^j_u}\bar{u}^j_L$, $d^i_L \to e^{i\theta^i_d}\bar{d}^i_L$, $W^+_\mu \to -W^{-\mu}$, and $\bar{q}'\gamma_\mu q \to -\bar{q}\gamma^\mu q'$ (where the phases have been absorbed in the quark fields). The charged current then becomes

$$\mathcal{L}^{\rm CP}_{\rm cc} = ig^*\bar{u}^j_L\gamma^\mu W^+_\mu V^*_{ij} d^i_L + ig\bar{d}^i_L\gamma^\mu W^-_\mu V_{ji} u^j_L. \tag{13.4}$$

The phase in the CKM matrix causes a difference between the charged-current Lagrangian and its CP transformation. The charged-current Lagrangian thus does not conserve CP; this is inherently due to the existence of three generations, which in general gives a matrix V different to V^*.

13.2 Mixing overview

The neutral mesons with weak decays oscillate between the particle and antiparticle states at a measurable rate before they decay. Because these states have a small mass difference, oscillations into other states are negligible. A 2×2 matrix can be constructed for the transitions $\langle M^0_{ij}|\bar{M}^0_{ij}\rangle$, $\langle \bar{M}^0_{ij}|\bar{M}^0_{ij}\rangle$, and the transitions with $M^0_{ij} \leftrightarrow \bar{M}^0_{ij}$ (M^0_{ij} is the meson composed of $q_i,\ q_j$ on-shell valence quarks):

$$H = \begin{pmatrix} m_0 + \delta E & W_{12} + \delta E_{12} \\ W^*_{12} + \delta E^*_{12} & m_0 + \delta E \end{pmatrix} + i \begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma^*_{12} & \Gamma \end{pmatrix}, \tag{13.5}$$

where m_0 corresponds to the “tree-level” QCD bound state, δE is the correction to the meson or anti-meson propagator due to weak interactions, W_{12} is the perturbative weak off-shell meson-antimeson transition, δE_{12} is the off-shell transition through intermediate states, and Γ_{12} is the transition through on-shell states. The diagonal elements are equal, due to CPT invariance.

Diagonalizing the Hamiltonian gives eigenvectors for the physical masses and widths of the observed states M^1_{ij} and M^2_{ij}, where

$$\begin{aligned} |M^1_{ij}\rangle &= p|M^0_{ij}\rangle - q|\bar{M}^0_{ij}\rangle, \\ |M^2_{ij}\rangle &= p|M^0_{ij}\rangle + q|\bar{M}^0_{ij}\rangle, \end{aligned} \tag{13.6}$$

with

$$\frac{q}{p} = \sqrt{\frac{m^*_{12} - \frac{1}{2}\Gamma^*_{12}}{m_{12} - \frac{1}{2}\Gamma_{12}}}, \tag{13.7}$$

where $m_{12} = W_{12} + \delta E_{12}$. One can rewrite the eigenstates to more clearly delineate the CP-violating contribution, e.g.

$$|M^2_{ij}\rangle = \frac{p+q}{2}\left[\left((|M^0_{ij}\rangle + |\bar{M}^0_{ij}\rangle) + \frac{1-q/p}{1+q/p}(|M^0_{ij}\rangle - |\bar{M}^0_{ij}\rangle\right)\right], \tag{13.8}$$

where $\epsilon = (1-q/p)/(1+q/p)$ provides a measure of the CP violation.

The eigenvalues are

$$\lambda_{1,2} = m_{1,2} - \frac{i}{2}\Gamma_{1,2}, \tag{13.9}$$

where

$$m_{1,2} = m \pm Re\sqrt{|m_{12}|^2 - \frac{|\Gamma_{12}|^2}{4} - iRe(m_{12}\Gamma_{12}^*)} \equiv m \pm \Delta m/2, \tag{13.10}$$

$$\Gamma_{1,2} = \Gamma \pm 2Im\sqrt{|m_{12}|^2 - \frac{|\Gamma_{12}|^2}{4} - iRe(m_{12}\Gamma_{12}^*)} \equiv \Gamma \pm \Delta\Gamma/2 \tag{13.11}$$

and $m = m_0 + \delta E$. Given these eigenvalues, one can calculate the time evolution of e.g. M_{ij}^0:

$$\begin{aligned} |M_{ij}^0\rangle &= \frac{1}{2}e^{-imt}e^{-\Gamma t/2}[(e^{\Delta\Gamma t/4}e^{i\Delta mt/2} + e^{-\Delta\Gamma t/4}e^{-i\Delta mt/2})|M_{ij}^0\rangle + \\ &\quad \frac{q}{p}(e^{\Delta\Gamma t/4}e^{i\Delta mt/2} - e^{-\Delta\Gamma t/4}e^{-i\Delta mt/2})|\bar{M}_{ij}^0\rangle]. \end{aligned} \tag{13.12}$$

The time evolution of the anti-meson $\bar{M}_{ij}^0$ has the same form, with a factor of p/q for the meson state M_{ij}^0 and 1 for the anti-meson state.

13.3 Mixing at leading order

The leading-order perturbative contribution to meson-antimeson oscillation occurs via the s or t-channel exchange of two W bosons. In the Feynman-'t Hooft gauge these box diagrams are manifestly convergent, though one needs to include diagrams with both the W gauge boson and the charged Goldstone boson. There are four diagrams each in the s and t channels. The matrix element is the same for the two channels so only one needs to be evaluated.

The matrix element with two W-boson propagators and internal quarks j and k is

$$\begin{aligned} i\mathcal{M}_{WW}^{jk} &= \langle M_{il}^0|\frac{g^4}{64}\xi_j\xi_k\int\frac{d^4k}{(2\pi)^4}\bar{q}_l(p_3)\gamma^\mu(1-\gamma_5)\frac{1}{\gamma_\alpha k^\alpha - m_{q_k}}\gamma^\nu(1-\gamma_5)q_i(p_1)\bar{q}_l(p_2) \\ &\quad \times\gamma_\nu(1-\gamma_5)\frac{1}{\gamma_\beta(k-p_1-p_2)^\beta - m_{q_j}}\gamma_\mu(1-\gamma_5)q_i(p_4)\frac{-i}{(k-p_1)^2 - m_W^2}\times \\ &\quad \frac{-i}{(k-p_3)^2 - m_W^2}|\bar{M}_{il}^0\rangle, \end{aligned} \tag{13.13}$$

where $\xi_j = V_{ij}V_{jl}^*$, $\xi_k = V_{lk}V_{ki}^*$, and M_{il}^0 is a meson composed of quarks i and l. In the Feynman-'t Hooft gauge, the diagrams with a scalar line replacing a $W_{\mu\nu}$ line will remove $\gamma^\mu\gamma_\mu$ or $\gamma^\nu\gamma_\nu$ and replace it with quark mass terms. The diagram with two scalar lines will remove both pairs of γ matrices.

To simplify the calculation we take the external momenta p_i to be negligible compared to m_W and the quark masses in the internal propagators. These external momenta are the individual quark momenta in the bound hadron and should be small compared to the quark masses. The mixing is dominated by the internal quark line with the highest mass, since the effects of mixing disappear as quark masses go to zero.

Setting $p_i = 0$ and moving gamma matrices to the numerator gives

$$i\mathcal{M}_{WW}^{jk} = \langle M_{il}^0|\frac{g^4}{64}\xi_j\xi_k \int \frac{d^4k}{(2\pi)^4}\bar{q}_l\gamma^\mu(1-\gamma_5)\left(\frac{\gamma_\alpha k^\alpha + m_{q_k}}{k^2 - m_{q_k}^2}\right)\gamma^\nu(1-\gamma_5)q_i \times \bar{q}_l\gamma_\nu(1-\gamma_5)\left(\frac{\gamma_\beta k^\beta + m_{q_j}}{k^2 - m_{q_j}^2}\right)\gamma_\mu(1-\gamma_5)q_i\left(\frac{-1}{k^2 - m_W^2}\right)^2|\bar{M}_{il}^0\rangle \quad (13.14)$$

First, consider the $m_{q_k}m_{q_j}$ term. Moving the γ_5 matrix on the left across the γ^ν will flip its sign, giving $(1+\gamma_5)(1-\gamma_5) = 0$. The term linear in k will also be zero, since it is an odd function of k. We are left with the $k^\alpha k^\beta$ term. The integral is simplified using a Feynman integral,

$$\frac{1}{(k^2 - m_{q_k}^2)(k^2 - m_{q_j}^2)(k^2 - m_W^2)^2} = 6\int \frac{dxdydzdw\delta(1-x-y-z-w)}{(k^2 - (x+y)m_W^2 - zm_{q_k}^2 - wm_{q_j}^2)^4}, \quad (13.15)$$

where the integrations over $x,\ y,\ z,\ w$ go from 0 to 1. We integrate over k using

$$\int d^dp\frac{p_\mu p_\nu}{(p^2 + 2pq - m^2)^\alpha} = (-1)^{d/2}\frac{i\pi^{d/2}}{\Gamma(\alpha)}\frac{1}{(-q^2 - m^2)^{\alpha - d/2}}[q_\mu q_\nu\Gamma(\alpha - d/2) + \frac{1}{2}g_{\mu\nu}(-q^2 - m^2)\Gamma(\alpha - 1 - d/2)]. \quad (13.16)$$

The integral becomes

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\frac{k^\alpha k^\beta}{(k^2 - m_{q_k}^2)(k^2 - m_{q_j}^2)(k^2 - m_W^2)^2} &= \frac{-6i\pi^2}{6\times(2\pi)^4}\int dxdydzdw \times \\ &\quad \delta(1-x-y-z-w) \times \\ &\quad [(x+y)m_W^2 + zm_{q_k}^2 + wm_{q_j}^2]^{-2} \times \\ &\quad \frac{1}{2}[(x+y)m_W^2 + zm_{q_k}^2 + wm_{q_j}^2]g^{\alpha\beta} \\ &= \int \frac{dxdydzdw\delta(1-x-y-z-w)g^{\alpha\beta}}{32\pi^2[(x+y)m_W^2 + zm_{q_k}^2 + wm_{q_j}^2]}.\end{aligned}$$

Integrating over w removes the delta function; integrating over z gives

$$\begin{aligned}&\int \frac{dxdydz(-ig^{\alpha\beta})}{32\pi^2[(x+y)m_W^2 + (1-x-y)m_{q_j}^2 + z(m_{q_k}^2 - m_{q_j}^2)]} = \\ &\int \frac{dxdy(-ig^{\alpha\beta})}{32\pi^2(m_{q_k}^2 - m_{q_j}^2)}\{\ln[(m_{q_k}^2 - m_{q_j}^2)(1-x-y) + \\ &(x+y)m_W^2 + (1-x-y)m_{q_j}^2] - \ln[(x+y)m_W^2 + (1-x-y)m_{q_j}^2]\} = \\ &\int \frac{dxdy(-ig^{\alpha\beta})}{32\pi^2(m_{q_k}^2 - m_{q_j}^2)}\{\ln[m_{q_k}^2 + (x+y)(m_W^2 - m_{q_k}^2)] - \ln[m_{q_j}^2 + (x+y)(m_W^2 - m_{q_j}^2)]\}.\end{aligned}$$

Next we integrate over y to obtain

$$\begin{aligned}&\int \frac{dx(-ig^{\alpha\beta})}{32\pi^2(m_{q_k}^2 - m_{q_j}^2)}\{\left[\frac{m_W^2}{m_W^2 - m_{q_k}^2}\ln(m_W^2) - \frac{m_W^2}{m_W^2 - m_{q_j}^2}\ln(m_W^2)\right] - \\ &\frac{[m_{q_k}^2 + x(m_W^2 - m_{q_k}^2)]}{m_W^2 - m_{q_k}^2}\ln[m_{q_k}^2 + x(m_W^2 - m_{q_k}^2)] + \\ &\frac{[m_{q_j}^2 + x(m_W^2 - m_{q_j}^2)]}{m_W^2 - m_{q_j}^2}\ln[m_{q_j}^2 + x(m_W^2 - m_{q_j}^2)]\}. \quad (13.17)\end{aligned}$$

Finally, the integration over x gives

$$\frac{(-ig^{\alpha\beta})}{32\pi^2(m_{q_k}^2-m_{q_j}^2)}\{\left[\frac{m_W^2}{m_W^2-m_{q_k}^2}\ln(m_W^2)-\frac{m_W^2}{m_W^2-m_{q_j}^2}\ln(m_W^2)\right]-$$
$$\frac{1}{4(m_W^2-m_{q_k}^2)^2}[2m_W^4\ln(m_W^2)-(m_W^2-m_{q_k}^2)(m_W^2+m_{q_k}^2)-2m_{q_k}^4\ln(m_{q_k}^2)]+$$
$$\frac{1}{4(m_W^2-m_{q_j}^2)^2}[2m_W^4\ln(m_W^2)-(m_W^2-m_{q_j}^2)(m_W^2+m_{q_j}^2)-2m_{q_j}^4\ln(m_{q_j}^2)]\}=$$
$$\frac{(-ig^{\alpha\beta})}{32\pi^2(m_{q_k}^2-m_{q_j}^2)}\{\frac{1}{4(m_W^2-m_{q_k}^2)^2}[(4m_W^4-4m_W^2m_{q_k}^2-2m_W^4)\ln(m_W^2)+$$
$$(m_W^2-m_{q_k}^2)(m_W^2-m_{q_k}^2+2m_{q_k}^2)+2m_{q_k}^4\ln(m_{q_k}^2)]-$$
$$\frac{1}{4(m_W^2-m_{q_j}^2)^2}[(4m_W^4-4m_W^2m_{q_j}^2-2m_W^4)\ln(m_W^2)+$$
$$(m_W^2-m_{q_j}^2)(m_W^2-m_{q_j}^2+2m_{q_j}^2)+2m_{q_j}^4\ln(m_{q_j}^2)]\}=$$
$$\frac{(-ig^{\alpha\beta})}{32\pi^2(m_{q_k}^2-m_{q_j}^2)}\{\frac{1}{4(m_W^2-m_{q_k}^2)^2}[(2(m_W^2-m_{q_k}^2)^2-2m_{q_k}^4)\ln(m_W^2)+$$
$$(m_W^2-m_{q_k}^2)^2+(m_W^2-m_{q_k})2m_{q_k}^2+2m_{q_k}^4\ln(m_{q_k}^2)]-$$
$$\frac{1}{4(m_W^2-m_{q_j}^2)^2}[(2(m_W^2-m_{q_j}^2)^2-2m_{q_j}^4)\ln(m_W^2)+$$
$$(m_W^2-m_{q_j}^2)^2+(m_W^2-m_{q_j})2m_{q_j}^2+2m_{q_j}^4\ln(m_{q_j}^2)]\}=$$
$$\frac{(-ig^{\alpha\beta})}{32\pi^2(m_{q_k}^2-m_{q_j}^2)}\{\frac{1}{4(m_W^2-m_{q_k}^2)^2}[(m_W^2-m_{q_k})2m_{q_k}^2+2m_{q_k}^4\ln(m_{q_k}^2/m_W^2)]-$$
$$\frac{1}{4(m_W^2-m_{q_j}^2)^2}[(m_W^2-m_{q_j})2m_{q_j}^2+2m_{q_j}^4\ln(m_{q_j}^2/m_W^2)]\} \qquad (13.18)$$

If we now write $x_{j,k}\equiv m_{q_{j,k}}^2/m_W^2$, the integral becomes

$$\frac{(-ig^{\alpha\beta})}{64\pi^2m_W^2(x_k-x_j)}\left(\frac{x_k}{1-x_k}+\frac{x_k^2\ln x_k}{(1-x_k)^2}-\frac{x_j}{1-x_j}-\frac{x_j^2\ln x_j}{(1-x_j)^2}\right). \qquad (13.19)$$

To calculate the remainder of the $k^\alpha k^\beta$ term we need to multiply a set of gamma matrices, which can be simplified using

$$\gamma^\mu\gamma^\alpha\gamma^\nu=g^{\mu\alpha}\gamma^\nu+g^{\nu\alpha}\gamma^\mu-g^{\mu\nu}\gamma^\alpha-i\epsilon^{\mu\alpha\nu\beta}\gamma_5\gamma_\beta. \qquad (13.20)$$

The $(1-\gamma_5)$ terms can be moved to the right to allow the multiplication of $[\gamma^\mu\gamma^\alpha\gamma^\nu][\gamma_\nu\gamma_\alpha\gamma_\mu]$. There are three contractions of $g^{\delta\epsilon}g_{\delta\epsilon}$ with an independent gamma matrix to give $12\gamma^\alpha\gamma_\alpha$, one pair of cross terms giving $2\gamma^\alpha\gamma_\alpha$ and another two pairs of cross terms giving $-4\gamma^\alpha\gamma_\alpha$. Finally the contraction of the ϵ pair give $-3!\gamma^\alpha\gamma_\alpha$, leaving a total of $4\gamma^\alpha\gamma_\alpha$ after all terms are summed. Moving the γ_5 back between the two γ factors gives

$$[\gamma^\mu\gamma^\alpha\gamma^\nu(1-\gamma_5)]q_i\bar{q}_l[\gamma_\nu\gamma_\alpha\gamma_\mu(1-\gamma_5)]=4[\gamma^\alpha(1-\gamma_5)]q_i\bar{q}_l[\gamma_\alpha(1-\gamma_5)]. \qquad (13.21)$$

We now return to the matrix element, using $g^2=8G_Fm_W^2/\sqrt{2}$ and $g^2=4\pi\alpha/\sin^2\theta_W$ to write $g^4/64=\pi\alpha G_Fm_W^2/(2\sqrt{2}\sin^2\theta_W)$:

$$i\mathcal{M}_{WW}^{jk} = \frac{-i\pi\alpha G_F\xi_j\xi_k}{2\sqrt{2}\sin^2\theta_W\pi^2(x_k-x_j)}\left(\frac{x_k}{1-x_k}+\frac{x_k^2\ln x_k}{(1-x_k)^2}-\frac{x_j}{1-x_j}-\frac{x_j^2\ln x_j}{(1-x_j)^2}\right)$$
$$\langle M_{il}^0|\left(\bar{q}_l\frac{1-\gamma_5}{2}\gamma^\alpha\frac{1-\gamma_5}{2}q_i\bar{q}_l\frac{1-\gamma_5}{2}\gamma_\alpha\frac{1-\gamma_5}{2}q_i\right)|\bar{M}_{il}^0\rangle. \qquad (13.22)$$

There are two channels, and the matrix element will be summed over internal quark lines, so the total matrix element for WW exchange is $2\sum_{jk}\mathcal{M}^{jk}_{WW}$. The calculation of the matrix element with $W\phi$ exchange proceeds along similar lines, except there is e.g. no $\gamma^\mu\gamma_\mu$ so moving $(1-\gamma_5)$ to the right removes the $k^\alpha k^\beta$ term rather than the $m_{q_k}m_{q_j}$ term. The integral can be calculated using equation 6.10; the result is $\mathcal{M}^{jk}_{W\phi} = \mathcal{M}^{jk}_{\phi W} = -x_j x_k \mathcal{M}^{jk}_{WW}$. The diagrams with $\phi\phi$ exchange have no $\gamma^\mu\gamma^\nu$ matrices, so once again only the $m_{q_k}m_{q_j}$ term contributes, giving $\mathcal{M}^{jk}_{\phi\phi} = x_j x_k \mathcal{M}^{jk}_{WW}/4$. Summing all the terms gives

$$i\mathcal{M}^{jk}_{WW} = \frac{-i\pi\alpha G_F\xi_j\xi_k}{2\sqrt{2}\sin^2\theta_W\pi^2}[\frac{x_j x_k}{x_k - x_j}\left(\ln\frac{x_k}{x_j} - \frac{3}{4}\frac{x_k^2\ln x_k}{(1-x_k)^2} + \frac{3}{4}\frac{x_j^2\ln x_j}{(1-x_j)^2}\right) - \frac{3}{4}\frac{x_j x_k}{(1-x_k)(1-x_j)}]\langle M^0_{il}|\,(\bar{q}_{lL}\gamma^\alpha q_{iL}\bar{q}_{lL}\gamma_\alpha q_{iL})\,|\bar{M}^0_{il}\rangle. \tag{13.23}$$

Table 13.1 shows the numerical values of the $x_k,\ x_j$ expression (known as the Inami-Lim factor) for the dominant charm and top quark combinations for $B^0_d,\ B^0_s$, and K_0 mixing, along with the corresponding CKM factors. Because of the large contribution from internal top-quark lines, the B_s system has the largest mixing matrix element.

Table 13.1 The numerical factors entering the leading-order mixing matrix element for B and K meson mixing [28].

Quarks	Inami-Lim factor	B^0_d CKM factor	B^0_s CKM factor	K^0 CKM factor
c, c	3.5×10^{-4}	$A^2\lambda^6$	$A^2\lambda^4$	λ^2
		(7.4×10^{-5})	(1.4×10^{-3})	(2.7×10^{-2})
c, t	3.0×10^{-3}	$A^2\lambda^6\|1-\rho-i\eta\|$	$A^2\lambda^4$	$A^2\lambda^6\|1-\rho-i\eta\|$
		(7.3×10^{-5})	(1.5×10^{-3})	(8.8×10^{-6})
t, t	2.5	$A^2\lambda^6\|1-\rho-i\eta\|$	$A^2\lambda^4$	$A^4\lambda^{10}\|1-\rho-i\eta\|^2$
		(7.2×10^{-5})	(1.5×10^{-3})	(1.1×10^{-7})

The matrix elements for the valence quarks in the meson bound state can be approximated by a leading-order vacuum insertion, with an empirical "bag" factor B_M to account for the neglected intermediate states. The meson transition to the vacuum can then be expressed in terms of the meson decay constant f_M^2 as $8f_M^2 m_M^2/(3\times 2m_M)$, including symmetry (4), color (2/3), and normalization $[1/(2m_M)]$ factors. Contributions from gluons between any pairs of quark lines are included as a multiplicative factor η to the matrix element; for B mesons this factor is 0.55 at next-to-leading order.

Given the mixing matrix element, we can calculate the mass and lifetime differences between the physical states. Typically $\Gamma_{12} \ll M_{12}$, so the mass difference is $\Delta m \approx 2|M_{12}|$ and the width difference is $\Delta\Gamma \approx 2Re(M_{12}\Gamma^*_{12})/|M_{12}|$. Because of the empirical bag and decay-constant factors, it is more theoretically precise to consider ratios of mass differences for similar systems. For example, the ratio of B^0_d to B^0_s mass differences is

$$\frac{\Delta m_s}{\Delta m_d} = \frac{m_{B_s} f^2_{B_s} B_{B_s}}{m_{B_d} f^2_{B_d} B_{B_d}}|\frac{V_{ts}}{V_{td}}|^2. \tag{13.24}$$

The decay constants can be measured in B-meson decays, so the mass-difference ratio allows the extraction of the CKM ratio $|\frac{V_{ts}}{V_{td}}|^2$. The measured mass differences are $\Delta m_d =$

0.507 ± 0.004 ps^{-1} and $\Delta m_s = (17.719 \pm 0.043)$ ps^{-1}, and the combinations of decay-constant and bag factors are $f_{B_d}\sqrt{B_{B_d}} = (211 \pm 12)$ MeV and $f_{B_s}\sqrt{B_{B_s}} = (248 \pm 15)$ MeV. The CKM matrix elements and the ratio of matrix elements are thus

$$|V_{td}| = (8.4 \pm 0.6) \times 10^{-3}, \tag{13.25}$$

$$|V_{ts}| = (42.9 \pm 2.6) \times 10^{-3}, \tag{13.26}$$

$$|V_{td}/V_{ts}| = 0.211 \pm 0.001 \pm 0.006. \tag{13.27}$$

13.4 CP violation in mixing

The phases of the CKM elements can be accessed in mixing measurements of CP violation. The phases are usually presented as a triangle, corresponding to one of the off-diagonal elements in the $VV^\dagger = 1$ matrix. The most commonly studied triangle, shown in Fig. **??**, corresponds to the element

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0. \tag{13.28}$$

The sides have lengths 1, $|V_{ud}V_{ub}^*/(V_{cd}V_{cb}^*)|$ and $|V_{td}V_{tb}^*/(V_{cd}V_{cb}^*)|$, with angles given by

$$\beta = arg(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}), \tag{13.29}$$

$$\alpha = arg(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}), \tag{13.30}$$

$$\gamma = arg(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}). \tag{13.31}$$

In the Wolfenstein parametrization, the triangle has axes at (0,0), (1,0), and $\bar{\rho}$. Kaon mixing provides hyperbolic constraints in the $\bar{\rho} - \bar{\eta}$ plane, and the B-meson mass differences give circular constraints in this plane.

CHAPTER 14

NEUTRINO MASSES

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