# Exercises: Electroweak gauge parameters

1. Consider the contribution of an axion (a pseudoscalar) to the anomalous magnetic moment of the electron, with Lagrangian term $\lambda\gamma_5\phi\bar{\psi}\psi$. Take the axion mass to be negligible, since limits from supernova 1987a are $m_a < 10^{-5}$ eV.

   (a) Use the Feynman rules to write down an expression for the correction to the $ee\gamma$ vertex from axion exchange.

   (b) Exploit the similarity to the photon-exchange vertex correction to combine and simplify the denominator using Feynman integrals and a shift of the integrated momentum.

   (c) The numerator has a divergent piece, which is the same as that of a scalar since $\{\gamma^5, \gamma^\mu\} = 0$. Again use the similarity with the vertex correction with photon exchange to write down the contribution to the counterterm in the minimum subtraction scheme.

   (d) Now consider the convergent part of the integral over axion momentum. Use the anticommutation of $\gamma^5$ and $\gamma^\mu$ and the Dirac equation to combine terms in the numerator. Perform the integrals and use the Gordon identity to find the contribution to the anomalous magnetic moment.

   (e) Fixing the electromagnetic coupling using an experiment based on Bloch oscillations, the measurement of the anomalous magnetic moment is consistent with prediction to one part in a trillion at 90% confidence level (C.L.). What is the inferred 90% C.L. upper limit on the axion-electron coupling coefficient $\lambda$?

2. Peskin and Takeuchi defined parameters $(S, T, U)$ for the loop corrections to the electroweak propagators that separate the effect of each fermion generation from the mass splitting within each generation. These are all the parameters needed to describe interactions between fermions whose masses can be neglected. The procedure can be seen as follows.

   (a) We have seen that the vacuum polarization of the photon propagator gives a charge renormalization in the exchange between two particles via

   $$\Delta_F^{AA} = \frac{QQ'e^2}{k^2(1 - \Pi'^{AA})},$$

   where $\Pi^{AA}(k^2) \approx k^2 d\Pi^{AA}/dk^2|_{k^2=0} \equiv k^2\Pi'^{AA}$ is the contribution from the vacuum polarization. Similarly, we have seen that for massive particles the vacuum polarization leads to a shift in mass, which for the gauge bosons can be written

   $$\Delta_F^{WW} = \frac{e^2 I_+ I_-}{2\sin^2\theta_W(k^2 - m_W^2 - \Pi^{WW})}$$

and

$$\Delta_F^{ZZ} = \frac{e^2(I_3 - \sin^2\theta_W Q)(I_3' - \sin^2\theta_W Q')}{\sin^2\theta_W \cos^2\theta_W (k^2 - m_Z^2 - \Pi^{ZZ})}.$$

There is also a vacuum polarization mixing term

$$\Delta_F^{AZ} = \frac{e\Pi^{AZ}[Q(I_3' - \sin^2\theta_W Q') + (I_3 - \sin^2\theta_W Q)Q']}{\sin\theta_W \cos\theta_W k^2(1-\Pi^{AA})(k^2 - m_Z^2 - \Pi^{ZZ})},$$

with $\Pi^{AZ} \approx k^2 d\Pi^{AZ}/dk^2|_{k^2=0} \equiv k^2\Pi'^{AZ}$. Show that the mixing term causes a correction to the effective $\sin^2\theta_W$ coupling of $\sin^2\theta_W' = \sin^2\theta_W - \sin\theta_W\cos\theta_W\Pi'^{ZA}/(1-\Pi'^{AA})$.

(b) We can further factor out the leading couplings that appear in the loop, i.e. $\Pi^{AA} = e^2\Pi^{QQ}$ and $\Pi^{WW} = e^2\Pi^{11}/sin^2\theta_W$ (with $\Pi^{11} = \Pi^{22}$). Write down $\Pi^{ZA}$ and $\Pi^{ZZ}$ in terms of $\Pi^{QQ}$, $\Pi^{3Q}$, and $\Pi^{33}$.

(c) Express the running couplings $e^2(k^2)$ and $sin^2\theta_W(k^2)$ as additive corrections to $e^2(0)$ and $\sin^2\theta_W(0)$ to first order in $\Pi^{QQ}$ and $\Pi^{3Q}$.

(d) Expanding the propagators as e.g. $\Pi^{33}(k^2) = \Pi^{33}(0) + k^2\Pi'^{33}(0)$, there are six propagator factors to this order in the expansion (since $\Pi^{3Q}(0) = \Pi^{QQ}(0) = 0$). Three of these are fixed with input parameters $\alpha_{EM}$, $G_F$ and $m_Z$. The other three can be predicted for any known loop correction, and are parameterized by Peskin and Takeuchi as

$$\begin{aligned}
\alpha_{EM}S &\equiv 4e^2[\Pi'^{33}(0) - \Pi'^{3Q}(0)],\\
\alpha_{EM}T &\equiv \frac{e^2}{\sin^2\theta_W\cos^2\theta_W m_Z^2}[\Pi^{11}(0) - \Pi^{33}(0)],\\
\alpha_{EM}U &\equiv 4e^2[\Pi'^{11}(0) - \Pi'^{33}(0)].
\end{aligned}$$

The effect of loops on the $W$ boson mass can be written as

$$\delta m_W^2 = \frac{\alpha_{EM}\cos^2\theta_W}{\cos^2\theta_W - \sin^2\theta_W} m_Z^2 \left[-\frac{1}{2}S + \cos^2\theta_W T + \frac{\cos^2\theta_W - \sin^2\theta_W}{4\sin^2\theta_W}U\right].$$

A fourth lepton generation with $m_N = m_L$ will give a correction $S = 1/(6\pi)$. Calculate the corresponding correction to $m_W$ using $\alpha_{EM} = 1/137$, $\sin^2\theta_W = 0.23146$, and $m_Z = 91.188$ GeV.

(e) Changing the top-quark mass predominantly affects $T$, which is large when there is a large mass splitting between up and down fermions. Taking the correction to be

$$T = \frac{3}{16\pi\sin^2\theta_W\cos^2\theta_W}\left[\frac{m_t^2 - 173^2}{m_Z^2}\right], \tag{-1}$$

where $m_t$ and $m_Z$ are in GeV, find the impact on the $W$ boson mass from a 1 GeV shift in the measured top-quark mass.