Strong Force, Quarkonium, and LIPS

- Why we believe there's a strong force.
- Why Colour?
  - Why not something with no inappropriate mental imagery
- Probing the Colour Force
  - The study of simple massive quark bound states

Strong Force, Quarkonium, and LIPS

- So Far…not speculated what holds the proton together.
- **Also Serious Mystery**: Ω^- (sss)
  - Ω^- is J = 3/2
    - So the spin wave function can be: $|\langle_1 \langle_2 \langle_3>$
    - χ(spin)ϕ(flavour) = $|s_1 s_2 s_3> | \langle_1 \langle_2 \langle_3>$

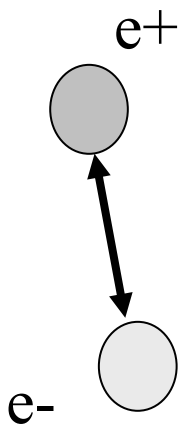
- **COMPLETELY SYMETRIC!!**

Strong Force And Quarkonium

- **Must have another part to wave function**
  - $\Psi = \chi(\text{spin})\phi(\text{flavour})\varphi(\text{space}) = \text{symmetric}$
  - **Not allowed for Fermions!**

- **Not only is colour needed, we know it must have an antisymmetric wave function and it must be in a singlet state with zero net colour.**

Positronium



Solve Hydrogen atom but with a reduced mass of $m_e/2$:

Get first bound state of -6.8eV.
Can do the same for the strong force.

$$V(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} + F_0 r$$

Use this potential for the quarks and fit to α_S and F_0.
Discover: $\alpha_S = 0.3$ and F_0 is 16 tons!

Discovery of J/Ψ

$$J/\Psi = \left(c\,\bar{c}\right)$$

One of the more interesting things is the 'width' in the mass of the J/ψ.

On a mass resonance:

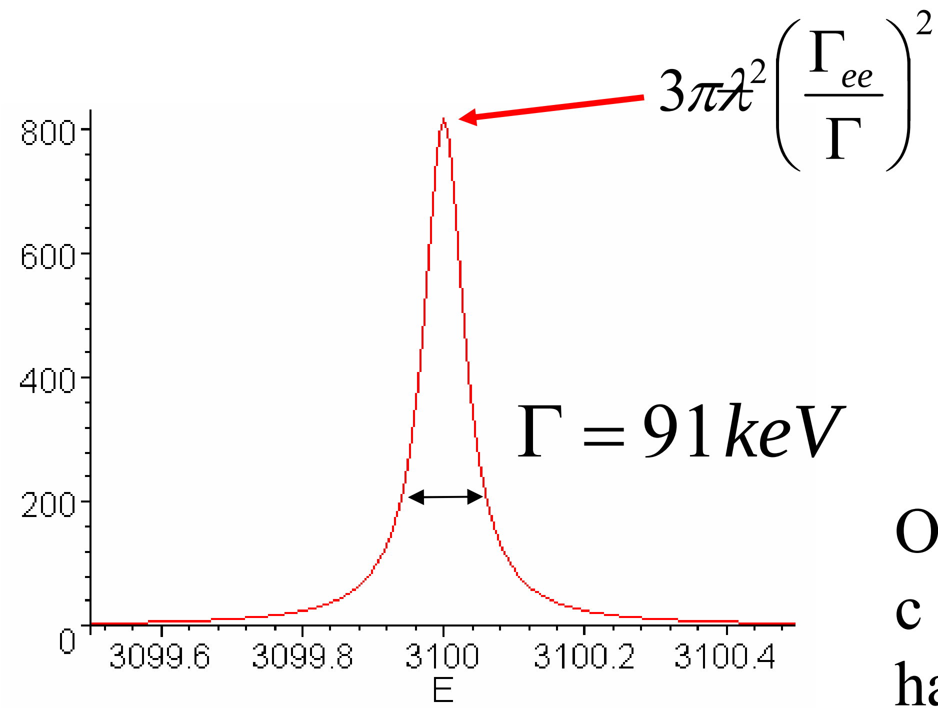
Breit-Wigner: $$\sigma(E) = \frac{4\pi\lambdabar^2 (2j+1) \quad \Gamma_{ee}^2 / 4}{(2S_1+1)(2S_2+1)\left\{(E-E_R)^2 + \Gamma^2/4\right\}}$$

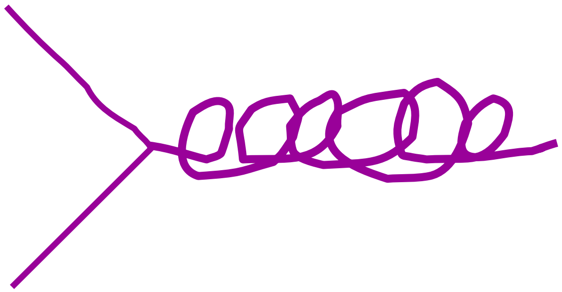
$$\lambdabar = \frac{\hbar c}{pc} = \frac{200\ MeV\ \ fm}{1500\ MeV} = 0.133\ \ fm$$

Why so narrow (long-lived)?

1). Gluons are spin 1 and massless like photons

2). Gluons have parity -1





One gluon forbidden:
c cbar is colour singlet; gluons have colour charge

Two gluons: P and C problem

Three gluons allowed but it is now suppressed by a factor of $\alpha_s^{\ 6}$

Spectroscopic notation:

$$n^{2s+1}L_J$$

$J/\psi \Rightarrow 1^3S_1$ $\chi_{c0} \Rightarrow 1^3P_0$ ***or*** 2^3P_0

L≦n is true only for 1/r potential

J/ψ has $J^{PC} = 1^{--}$ like the photon

What is the Isospin of the J/ψ?

We already know that $I_3=0$ because of the Quark composition.

Because $I_3 \le I$, we know that $I = 0,1,2,3\ldots$. Integer not ½ integer.

Look at the decays of the J/ψ to I-spin eigenstates:

$J/\psi \rightarrow \rho^+\pi^-, \rho^0\pi^0, \rho^-\pi^+$ in almost equal proportions

Both rho and pion have Isospin I = 1

So the J/ψ could have I = 0,1, or 2 only.

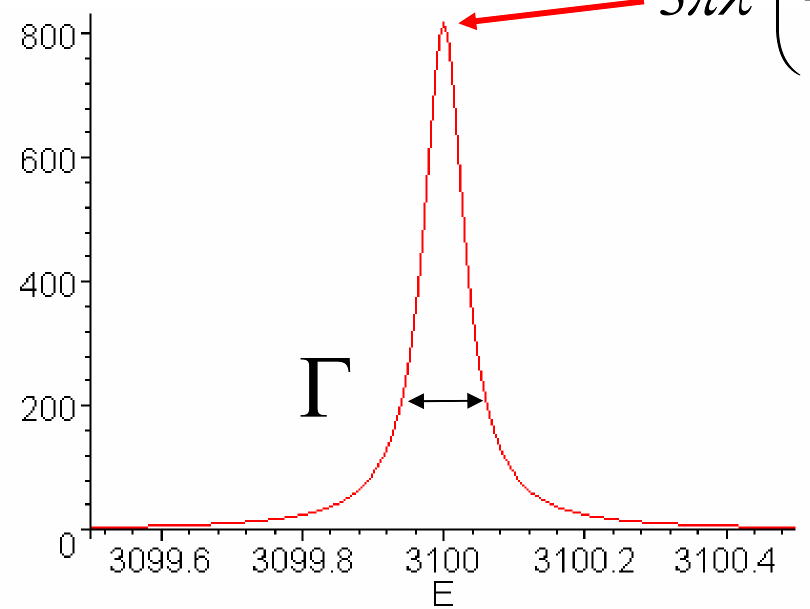
Use the 1x1 Clebsh-Gordon table to settle the matter.

I,I3	ρ+π-	ρ0π0	ρ-π+	
$\|0,0\rangle =$	$\sqrt{\frac{1}{3}}\|1,1\rangle\|1,-1\rangle$	$-\sqrt{\frac{1}{3}}\|1,0\rangle\|1,0\rangle$	$+\sqrt{\frac{1}{3}}\|1,-1\rangle\|1,1\rangle$	yes
$\|1,0\rangle =$	$\sqrt{\frac{1}{2}}\|1,1\rangle\|1,-1\rangle$		$-\sqrt{\frac{1}{2}}\|1,-1\rangle\|1,1\rangle$	no
$\|2,0\rangle =$	$\sqrt{\frac{1}{6}}\|1,1\rangle\|1,-1\rangle$	$-\sqrt{\frac{2}{3}}\|1,0\rangle\|1,0\rangle$	$+\sqrt{\frac{1}{6}}\|1,-1\rangle\|1,1\rangle$	no

How the Beit-Wigner width is measured:

In Theory

$$3\pi\lambdabar^2\left(\frac{\Gamma_{ee}}{\Gamma}\right)^2$$

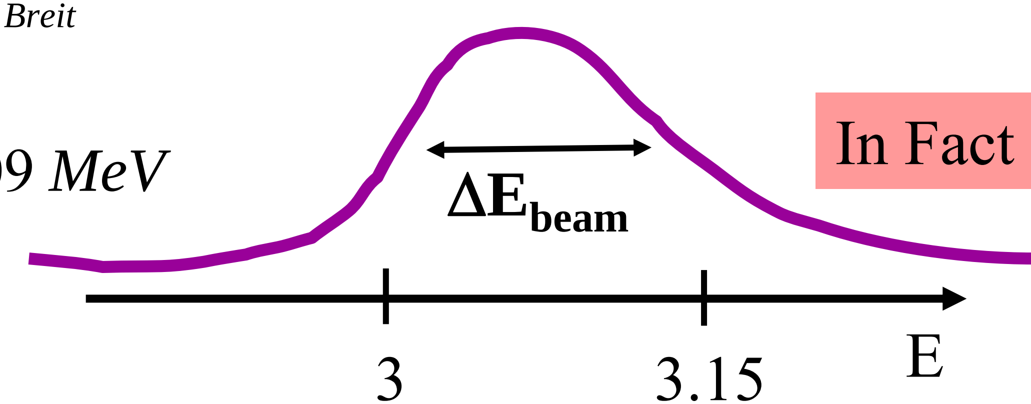


Conservation of Probability

$$\int_0^\infty \sigma(E_{exp})dE_{exp} = \int_0^\infty \sigma(E_{Breit})dE_{Breit}$$

$$= \frac{3}{2}\pi^2\lambdabar^2\left(\frac{\Gamma_{ee}}{\Gamma}\right)^2\Gamma \quad \Rightarrow \quad \Gamma = 0.09\ MeV$$

In Fact



#2 Fermi's Golden Rule

- Call it W
- Probability per unit time of a transition from initial state |i> to final state |f> is constant.

$$W = \frac{2\pi}{\hbar}\left|M_{fi}\right|^2 \rho(E)$$

Non-relativistic!

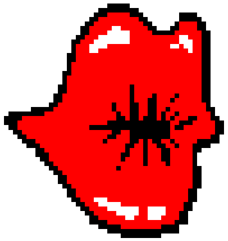
M_{fi} is the 'matrix element', in QM you recognize it as $\langle f|V|i\rangle$

$\rho(E)$ is the 'density of states available at energy E'.

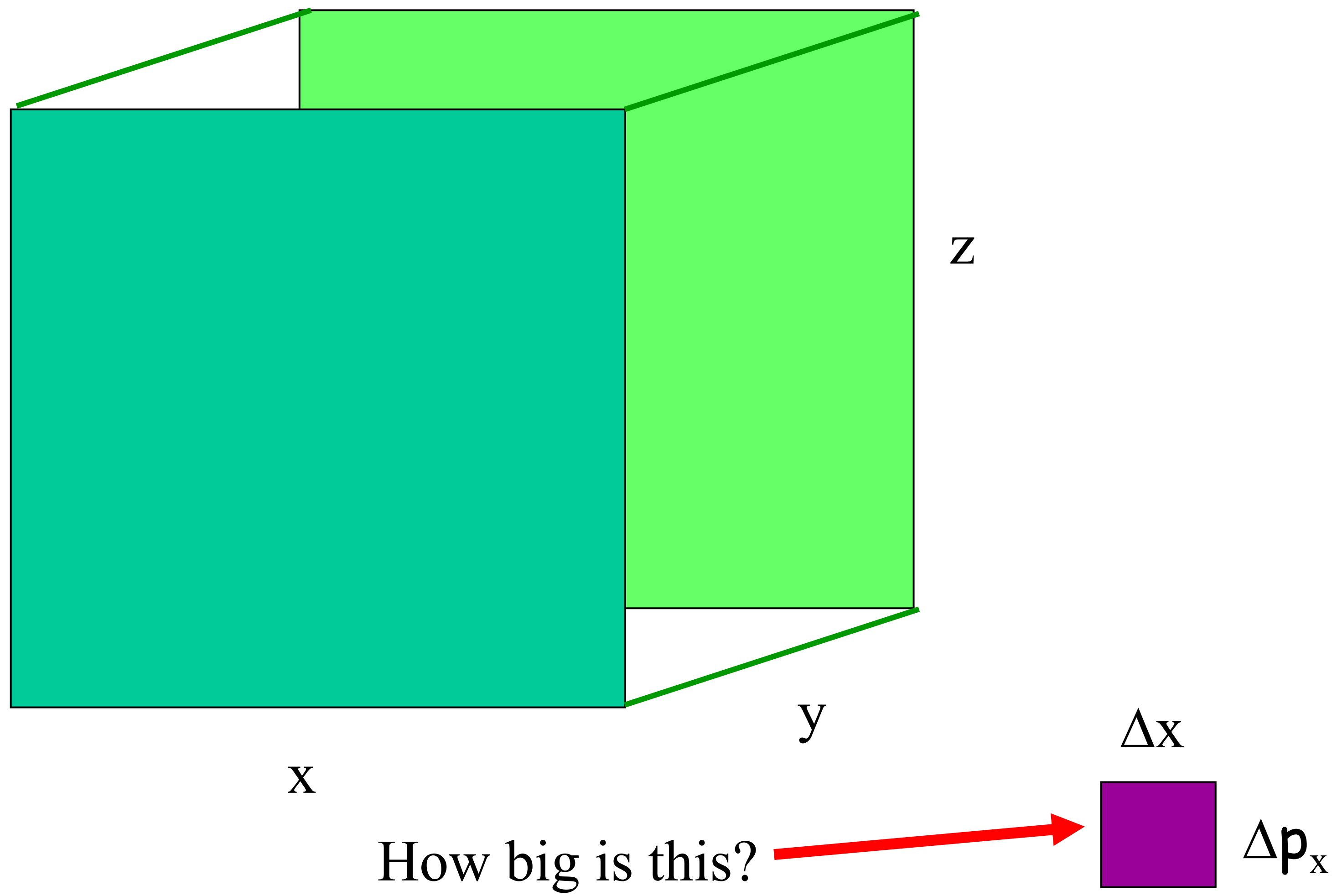
Understanding the:

Lorentz Invariant Phase Space Factor

dLIPS



How many QM ‘states’ are there in a volume ‘V’ up to a given momentum ‘P’?



Understanding dLIPS

6 – dimensions are needed in QM to describe a particle.

Three in space x_0, y_0, z_0;

and three in momentum p_{x0}, p_{y0}, p_{z0}

$$x_0 y_0 z_0 p_{x0} p_{y0} p_{z0} = Volume$$

6 dimensions!!

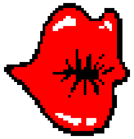
Want to increment these values by the ***smallest amount*** possible and be assured I have crossed to a new state:

$$(x_0 + \Delta x)(y_0 + \Delta y)(z_0 + \Delta z)(p_{x0} + \Delta p_x)(p_{y0} + \Delta p_y)(p_{z0} + \Delta p_z) =$$

$$x_0 y_0 z_0 p_{x0} p_{y0} p_{z0} + \underline{\Delta x \Delta p_x \Delta y \Delta p_y \Delta z \Delta p_z} + ManyCrossTerms$$

The ***smallest*** term in this group is the ΔxΔp term…

the uncertainty principle tells us its size!



Understanding dLIPS

Smallest distinguishable volume ➔ one state!

$$\Delta x \Delta p_x \Delta y \Delta p_y \Delta z \Delta p_z = (2\pi\hbar)^3$$

$$\frac{\Delta x \Delta p_x \Delta y \Delta p_y \Delta z \Delta p_z}{(2\pi\hbar)^3} = \Delta N$$

$$\Delta x \Delta y \Delta z \frac{\Delta p_x \Delta p_y \Delta p_z}{(2\pi\hbar)^3} = \Delta N$$

$$dV \frac{d^3 \vec{p}}{(2\pi\hbar)^3} = dN$$

Understanding dLIPS

Fine for the i^{th} particle. →

$$dV_i \frac{d^3\vec{p}_i}{(2\pi\hbar)^3} = dN_i$$

But suppose we have ***N*** total particles in the final state (and we only need to worry about the final state because in any experiment we take great pain to put the initial particles into a single, well-defined state).

$N_i + N_j$ = wrong!! These states are more like dice!
Or calculation of specific heat of a crystal.

How many possible combinations of two distinguishable pairs are possible?

Understanding dLIPS

$$dN_T = dN_1 dN_2 dN_3 \ldots dN_n = \prod_{i=1}^{n} dV_i \frac{d^3\vec{p}_i}{(2\pi\hbar)^3}$$

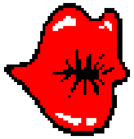
One Problem …This isn't Lorentz invariant.

Another Problem … $V_1 V_2 \ldots V_n$ is annoying

Turns out the Matrix Element has Volumes that will cancel with these volume elements.

$$dN_T = \prod_{i=1}^{n} \frac{d^3\vec{p}_i}{2E_i(2\pi\hbar)^3}$$

Is Lorentz Invariant!!!



Understanding dLIPS

Now we need to add up all of our dN's to get the total.

$$N_T = [V]^n \int \prod_{i=1}^{n} \frac{d^3 \vec{p}_i}{2E(2\pi\hbar)^3}$$

We expect to get a $1/V^n$ factor from the matrix element.

We will ignore that for now….so then

Integration is over all possible values of the i^{th} momentum.

$$\rho(E) \equiv \frac{dN_T}{dE} = \frac{d}{dE} \int \prod_{i=1}^{n} \frac{d^3 \vec{p}_i}{2E_i(2\pi\hbar)^3}$$

But we do not have independent momenta, if all but one of the momenta is known, the last one is also known!

Understanding dLIPS

$$\rho(E) = \frac{d}{dE} \int \prod_{i=1}^{n-1} \left(\frac{d^3 \vec{p}_i}{2E_i (2\pi\hbar)^3} \right) \frac{1}{2E_n}$$

Needed to keep Lorentz inv.

And this is perfectly fine to use…just remember to apply energy and momentum conservation at the end.

But using properties of the Dirac delta Function we can re-cast this equation in the following form (and explicitly include energy and momentum conservation).

Understanding dLIPS

We do have Energy and Momentum conservation.
So, for example, in CM frame with total Energy W.

$$(2\pi\hbar)^4\,\delta\left(W-\sum_{i=1}E_i\right)\delta^3\left(\sum_{i=1}^{n}\vec{p}_i\right)$$

Note the additional factors

Final Phase-space factor

$$R=\int_{all}\prod_{i=1}^{n}\frac{d^3\vec{p}_i}{\left(2E_i(2\pi\hbar)^3\right)}(2\pi\hbar)^4\,\delta\left(W-\sum_{i=1}E_i\right)\delta^3\left(\sum_{i=1}^{n}\vec{p}_i\right)$$

dLIPS: A Cool Recursion Formula!

Given the phase space factor for a certain reference frame and the case of *N-1* final state particles; we can find the phase space factor for *N* particles.

$$R = \int_{all} \prod_{i=1}^{n} \frac{d^3 \vec{p}_i}{\left(2E_i (2\pi\hbar)^3\right)} (2\pi\hbar)^4 \delta\left(W - \sum_{i=1}^{n} E_i \right) \delta^3\left(\sum_{i=1}^{n} \vec{p}_i \right)$$

Rewrite this, pulling out the n^{th} particle.

$$R_n(0,W) = \int \frac{d^3 \vec{p}_n}{2E_n (2\pi\hbar)^3} \int_{all} \prod_{i=1}^{n-1} \frac{d^3 \vec{p}_i}{\left(2E_i (2\pi\hbar)^3\right)} (2\pi\hbar)^4 \delta\left(\sum_{i=1}^{n-1} E_i - (W - E_n) \right) \delta^3\left(\sum_{i=1}^{n-1} \vec{p}_i - (-\vec{p}_n) \right)$$

The second integral is the phase space factor for n-1 particles with total momentum $-p_n$ and total energy $(E-E_n)$!

dLIPS: A Cool Recursion Formula!

Remember! This is Lorentz invariant!
This means good things: R_{n-1} must also be the same in a system with **zero** total momentum where the total energy is:

$$\varepsilon^2 = \left(E - E_n\right)^2 - \left(-\vec{p}_n\right)^2$$

So that:

$$R_{n-1}\left[\left(-\vec{p}_n\right),\left(E - E_n\right)\right] = R_{n-1}\left(0,\varepsilon\right).$$

So we can find the 3-body phase space based on what we know of the 2-body phase space!

dLIPS: 2-body to 3-body

The ubiquitous exercise for the student….but I will get you started on it!

$$R_3(0,W) = \int \frac{d^3 p_3}{2E_3(2\pi\hbar)^3} R_2(0,\varepsilon)$$

Where:

$$\varepsilon^2 = (W - E_3)^2 - p_3^2$$

Using our 2-body example:

$$R_3(0,E) = \int \frac{d^3 p_3}{2E_3(2\pi\hbar)^3} \left(\frac{\alpha p'}{E'} \right) \qquad \alpha = \frac{\pi}{(2\pi\hbar)^3}$$

p' is C.o.M. 2-body momentum with C.o.M. energy E'

dLIPS: 3-body answer

I'm not as cruel as I should be.
Here is the 3-body result.

$$R_3(0,W) = \int \frac{4\pi p_3^2\, dp_3}{2E_3(2\pi\hbar)^3} \cdot \alpha \times$$

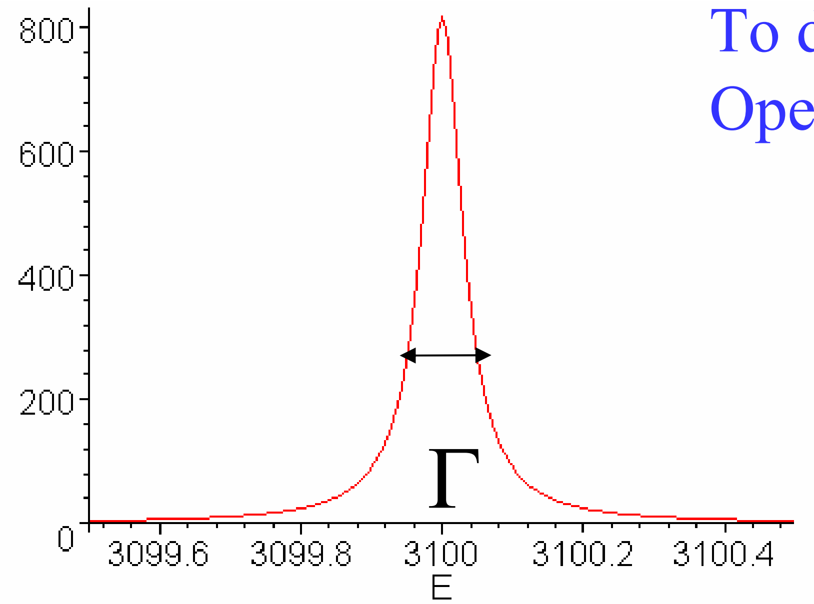
$$\frac{\left\{\left[W^2 + m_3^2 - 2WE_3 - (m_1 - m_2)^2\right]\left[W^2 + m_3^2 - 2WE_3 - (m_1 + m_2)^2\right]\right\}^{1/2}}{2\left(W^2 + m_3^2 - 2WE_3\right)}$$

The limits of integration must run from the minimum allowed momentum of the third particle to the maximum allowed momentum of the third particle
….now THAT is a true exercise for the student!

Back up slides

Why This Shape?

Would like to motivate this with a relativistic wave equation.
To do this try using $E^2=p^2c^2+m^2c^4$
Operator equivalents of E and P:



$$E \Rightarrow i\hbar \frac{\partial}{\partial t}$$

$$\vec{p} \Rightarrow -i\hbar \nabla$$

So the QM equivalent of Energy and momentum conservation is:

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \Rightarrow$$

$$-\hbar^2 \frac{\partial^2 \phi(x,t)}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \phi(x,t) + m^2 c^4 \phi(x,t)$$

This is called the Klein-Gordon equation.

Solutions to K-G equation:

$$\phi(\vec{x},t) = Ne^{-i\frac{E}{\hbar}t + i\frac{\vec{p}\bullet\vec{x}}{\hbar}}$$

With: $E = \pm\left(c^2 p^2 + m^2 c^4\right)^{\frac{1}{2}}$

For now, ignore negative E solutions. Throw in a potential (But can we stay fully relativistic??)

$$-\hbar^2 \frac{\partial^2 \phi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \phi + m^2 c^4 \phi + V\phi$$

Naïve Solution Becomes: $\phi(\vec{x},t) = N'e^{-i\frac{E'}{\hbar}t + i\frac{\vec{p}\bullet\vec{x}}{\hbar}}$

$$E' = \pm\left(c^2 p^2 + m^2 c^4 + V\right)^{\frac{1}{2}}$$

If we assume V is a real function you can multiply the K-G equation by ϕ* and multiply the complex conjugate of the K-G equation by ϕ. The potential, V, drops out.
You then get a continuity equation that you can interpret as some sort of conservation of probability

$$\frac{i\partial}{\partial t}\left[\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right] + \vec{\nabla} \cdot \left[\frac{\vec{p}}{i}\left(\phi^* \vec{\nabla}\phi - \phi \vec{\nabla}\phi^*\right)\right] = 0$$

$$\frac{\partial}{\partial t}\left[\frac{2E\left|N\right|^2}{\hbar}\right] + \vec{\nabla} \cdot \left[\vec{p}\frac{\left|N\right|^2}{\hbar}\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

Suppose that V is a purely imaginary constant

V = iΓ

$$\phi(\vec{x},t) = Ne^{-i\frac{t}{\hbar}\left(p^2c^2+m^2c^4+i\Gamma\right)^{1/2}+i\frac{\vec{p}\bullet\vec{x}}{\hbar}}$$

$$= Ne^{i\frac{\vec{p}\bullet\vec{x}}{\hbar}} e^{-i\frac{t}{\hbar}\left(p^2c^2+m^2c^4\right)^{1/2}\left(1+\frac{i\Gamma}{\left(p^2c^2+m^2c^4\right)}\right)^{1/2}}$$

Assume Γ is << ($p^2c^2+m^2c^4$) and expand the second term:

$$\phi(\vec{x},t) = Ne^{i\frac{\vec{p}\bullet\vec{x}}{\hbar}} e^{-i\frac{t}{\hbar}\left(p^2c^2+m^2c^4\right)^{1/2}-t\frac{\Gamma}{2\hbar}}$$

Note that we no longer have Probability conservation!!!

In the Rest Frame of the particle **p**=0 and we are left with:

$$\phi(t) = N e^{-i\frac{t}{\hbar}\left(mc^2\right) - t\frac{\Gamma}{2\hbar}}$$

Transform into 'Energy' space:

$$\phi(E) = \int e^{i\frac{E}{\hbar}t}\phi(t)dt$$

$$\propto \int_0^\infty e^{i\left(E - mc^2 + i\frac{\Gamma}{2}\right)\frac{t}{\hbar}}dt = \frac{N}{\hbar\sqrt{2\pi}}\frac{-i}{(E - mc^2) + i\frac{\Gamma}{2}}$$

$$\phi^*(E)\phi(E) \propto \left|\frac{-i}{(E - mc^2) + i\frac{\Gamma}{2}}\right|^2 = \frac{1}{(E - mc^2)^2 + \frac{\Gamma^2}{4}}$$

And The Breit-Wigner form emerges.