

# *Nonlinear Integrable Optics as a Route to High Intensity Accelerators*

*Stephen D. Webb*

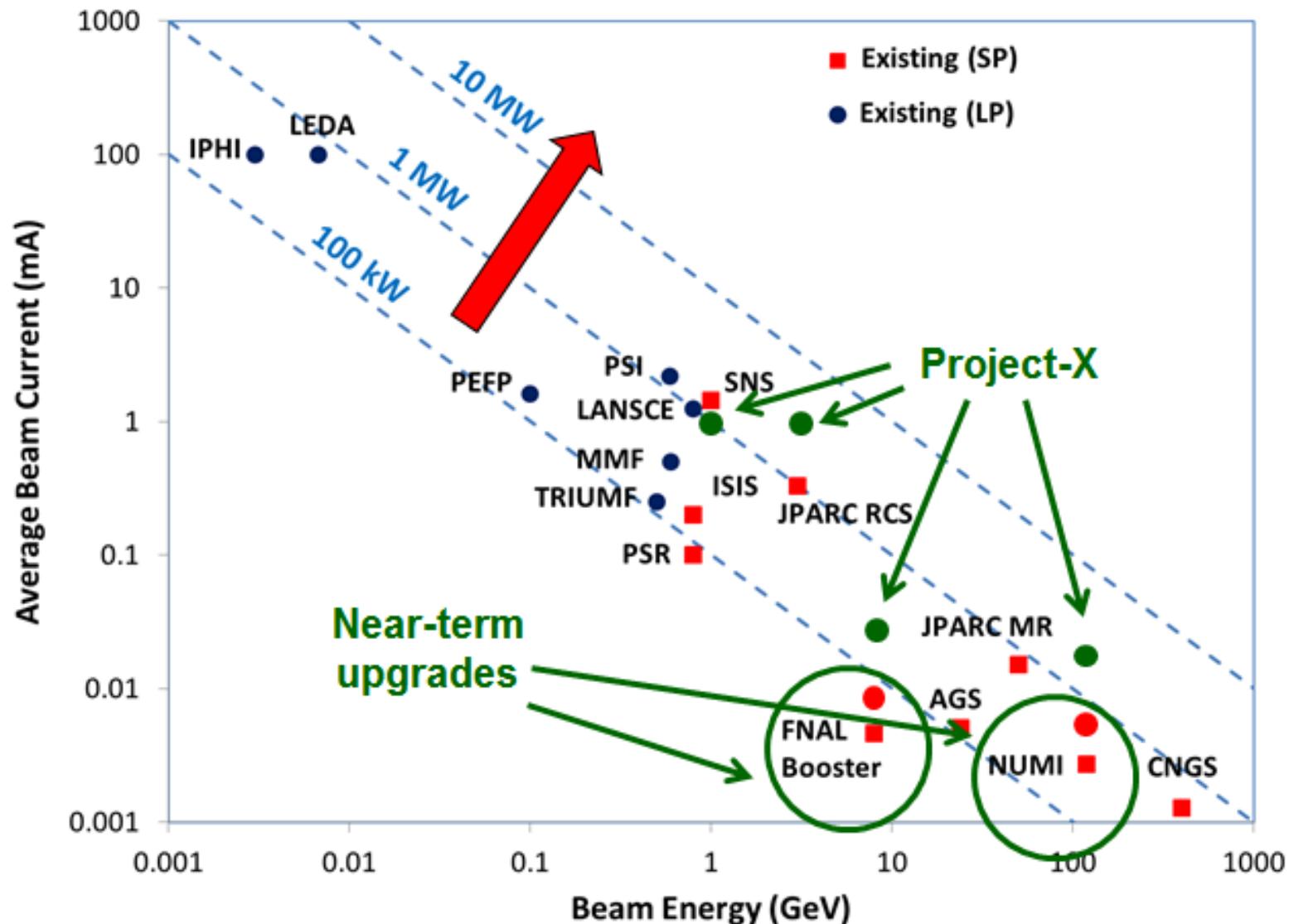
*RadiaSoft, LLC., Boulder, CO*

[swebb@radiasoft.net](mailto:swebb@radiasoft.net)

*7 September, 2015*

*Oxford University*

# *Hadron Accelerators at the Intensity Frontier*



from 2013 Snowmass WG3,  
"Issues and R&D Required for the Intensity Frontier Accelerators"  
arXiv:1305.6917v1 [physics.acc-ph]

# Beam Halo

VOLUME 73, NUMBER 9

PHYSICAL REVIEW LETTERS

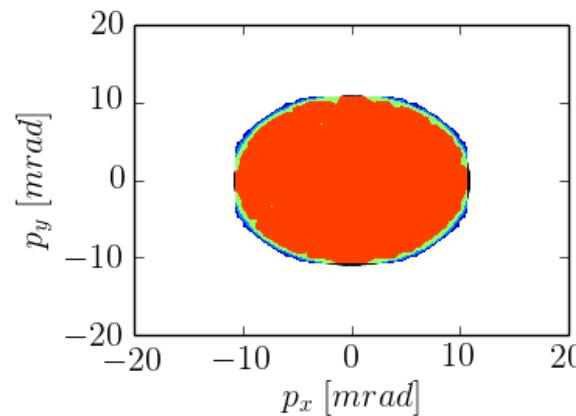
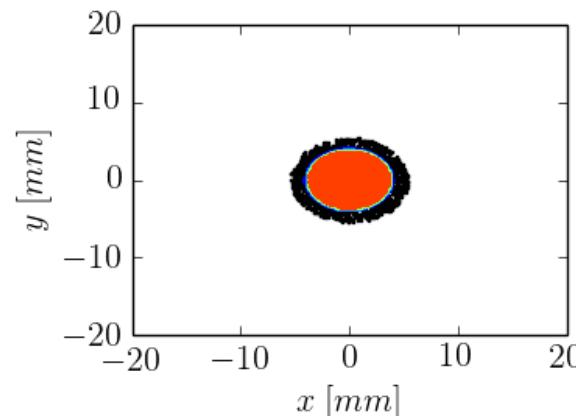
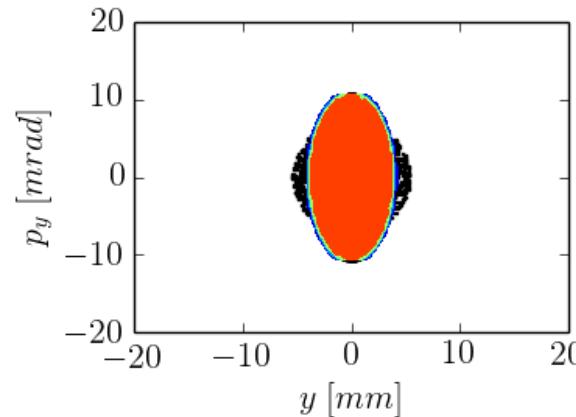
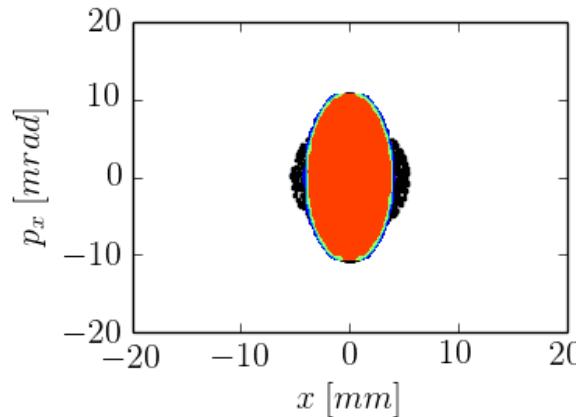
29 AUGUST 1994

## Analytic Model for Halo Formation in High Current Ion Linacs

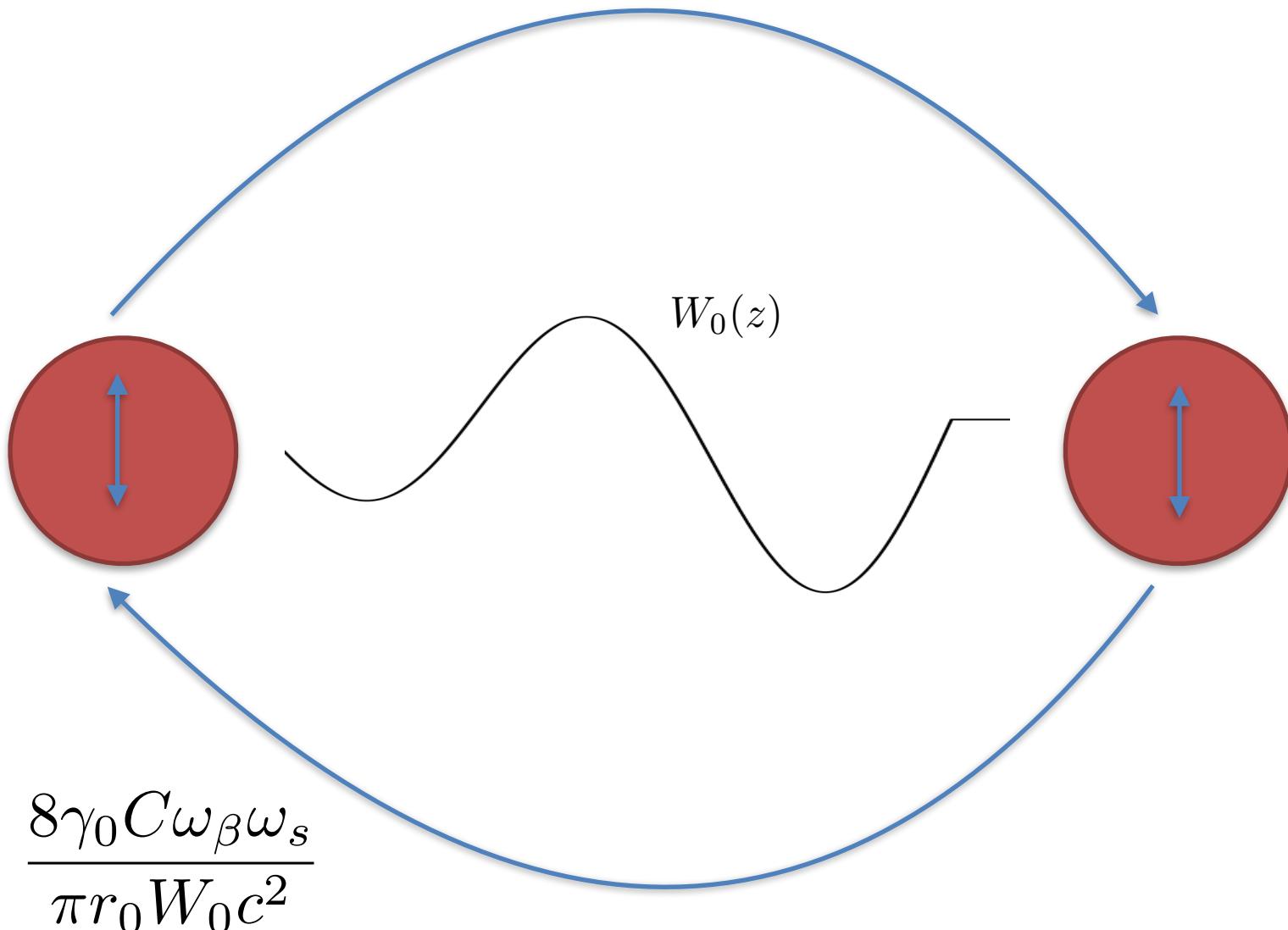
Robert L. Gluckstern

Physics Department, University of Maryland, College Park, Maryland 20742

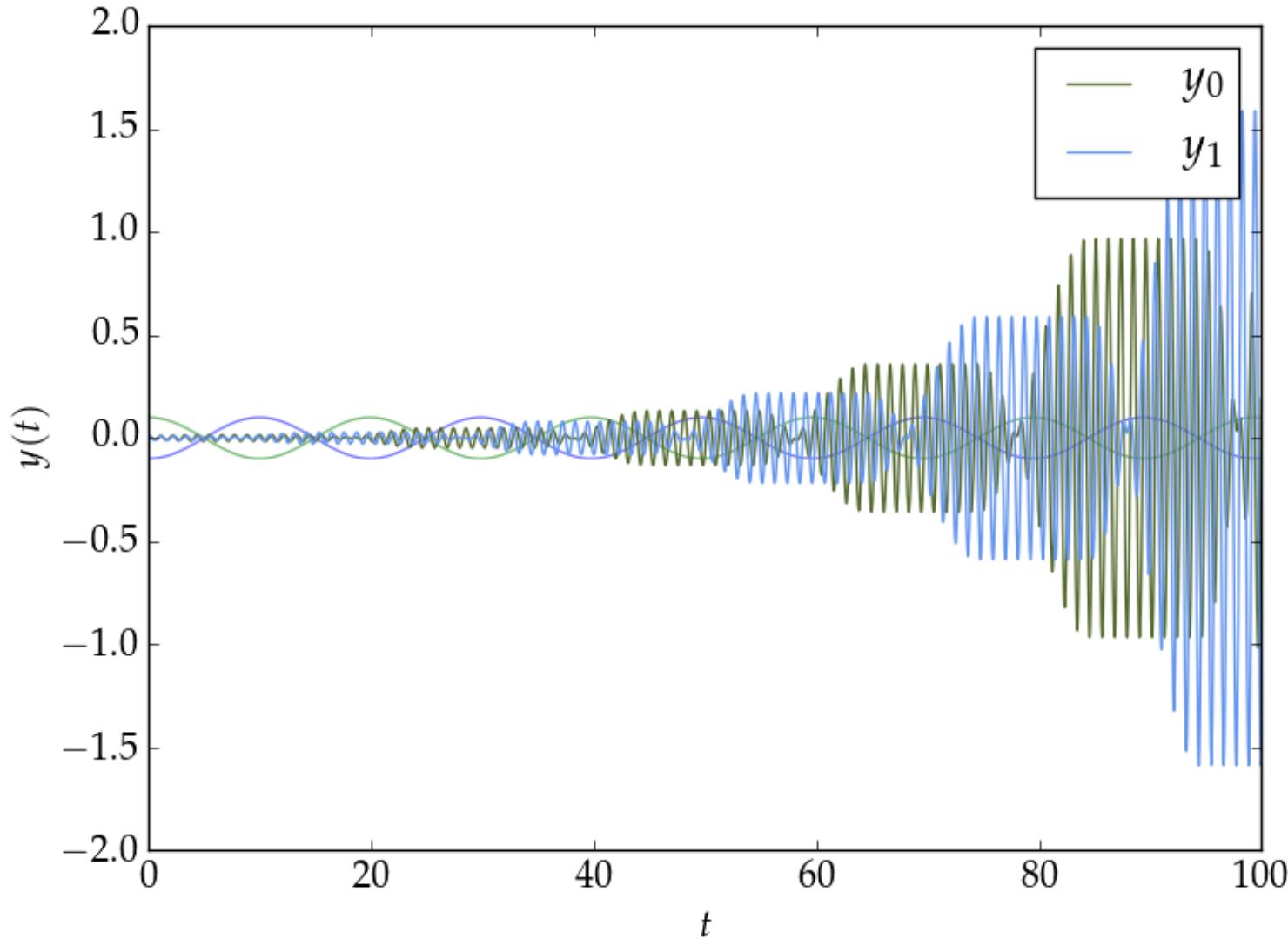
(Received 8 March 1994)



# *Strong Head-Tail Instability*



# *Strong Head-Tail Instability*



# **Parametric Resonances & Tune Spread**

**Nonlinear decoherence** — frequency spreads in ensembles which suppress the forcing terms in parametric resonances. This is logically distinct from Landau damping, but has a similar effect.

$$\ddot{r} + q^2 \left( r - \frac{L^2 a^4}{r^3} \right) = -\frac{\kappa}{a^2} r \left( 1 - \frac{a^2}{r^2} \right) \Theta(r - a) + \underbrace{2 \frac{\epsilon \kappa}{a^2} r \cos(pt) \Theta(a - r)}_{\text{parametric resonance}}$$

$$2 \frac{\epsilon \kappa}{a^2} r \cos(pt) \Theta(a - r) \mapsto 2 \frac{\epsilon \kappa}{a^2} r \Theta(a - r) \int_0^p dp' \varrho(p') \cos(p't)$$



$$\sim \frac{\sin(p_0 t)}{\sigma_p t}$$

# *Conventional Nonlinear Schemes Induce Chaos*

Perturbative single turn map

$$\mathcal{A}^{-1} e^{\mu_0 : J:} \mathcal{A} e^{S_4 : x^4:} \mapsto \mathcal{A}'^{-1} e^{\mu_i : J: + S'_4 : J^2: + \mathcal{O}(S_4^2)} \mathcal{A}'$$

$$\frac{n\mu_0}{2\pi} \neq \ell$$

Tune spread w/ amplitude

Near a resonance...

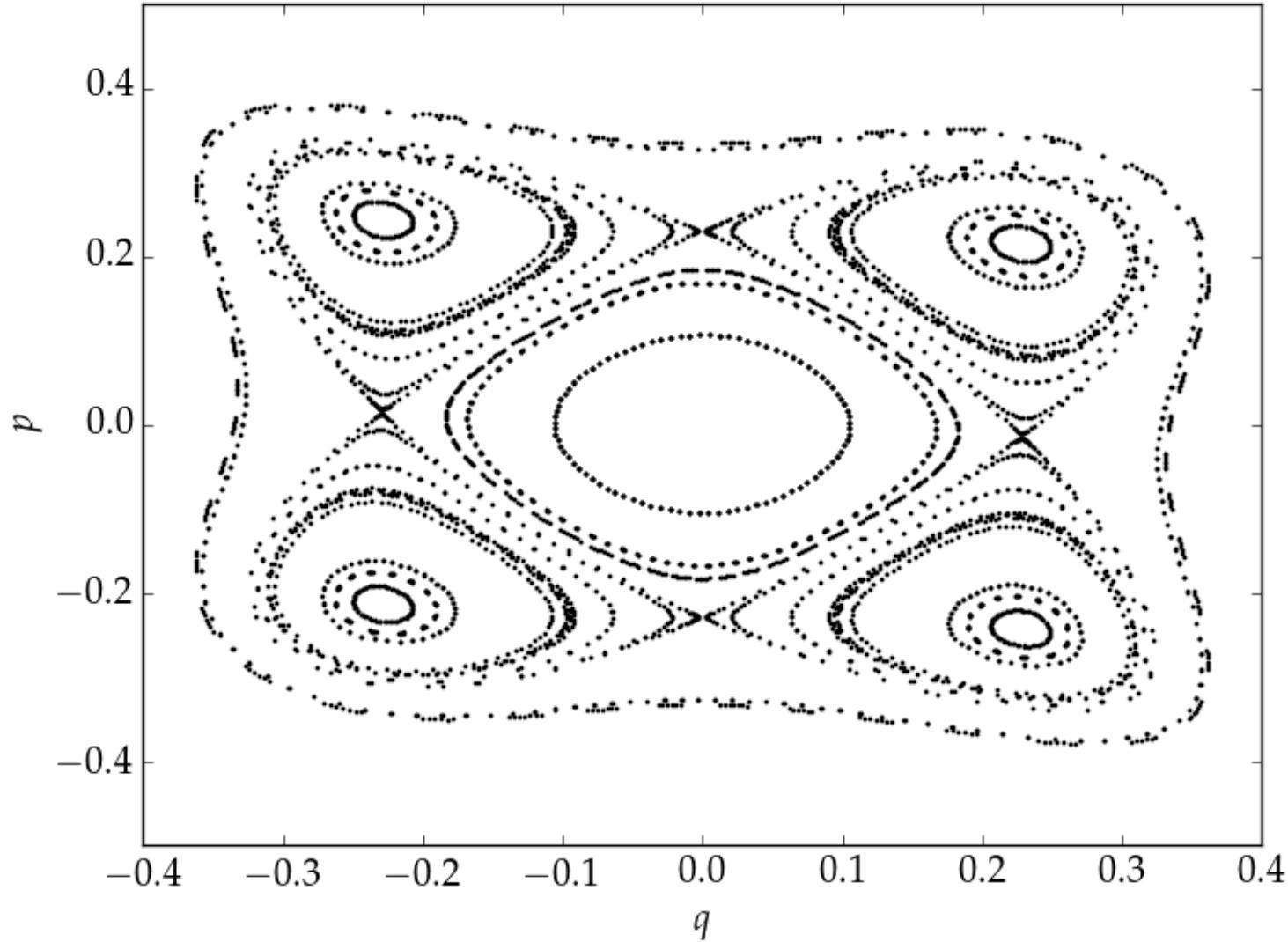
$$r_{\pm} = \sqrt{J} e^{\pm i\phi}$$

$$\mathcal{A}' = \exp \left( : \sum_{a+b=0}^4 C_{a,b} \frac{1}{1 - e^{i(a-b)\mu_0}} r_+^a r_-^b : \right) \mathcal{A}$$

$$e^{\frac{\mu_0}{2} : \beta p^2 + 2\alpha p x + \gamma x^2:}$$

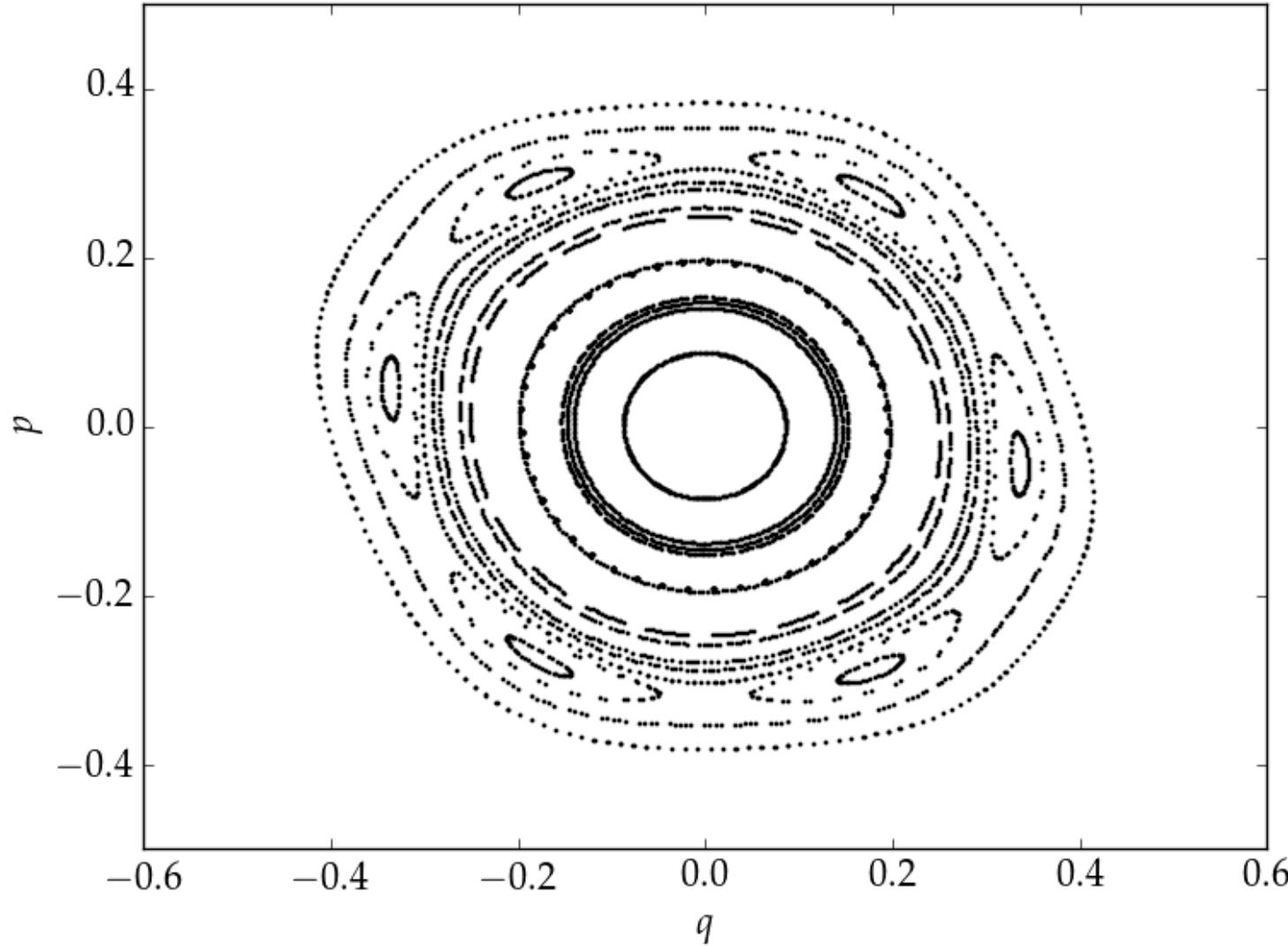
Famous “small denominators”

# *Conventional Nonlinear Schemes Induce Chaos*



**Resonance Condition**     $n \times \nu_x + m \times \nu_y = \ell$

# *Conventional Nonlinear Schemes Induce Chaos*



**Resonance Condition**     $n \times \nu_x + m \times \nu_y = \ell$

# *Nonlinear Integrable Optics*

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS 13, 084002 (2010)

---

## **Nonlinear accelerator lattices with one and two analytic invariants**

V. Danilov

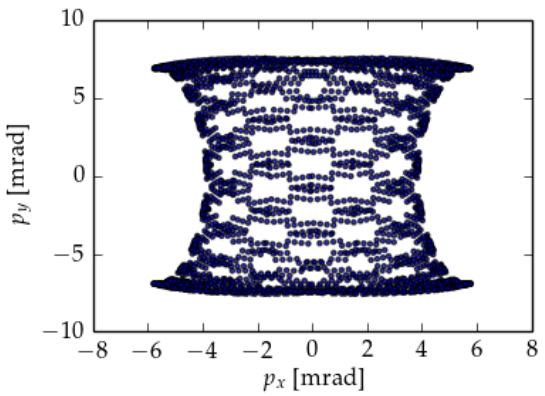
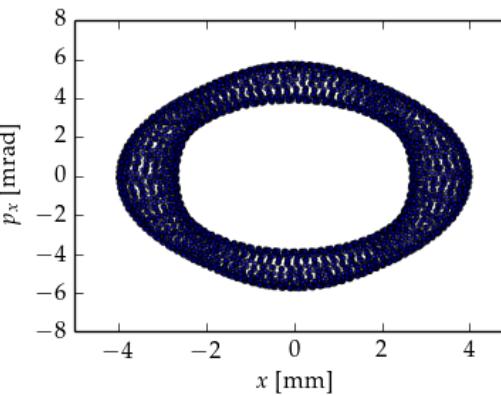
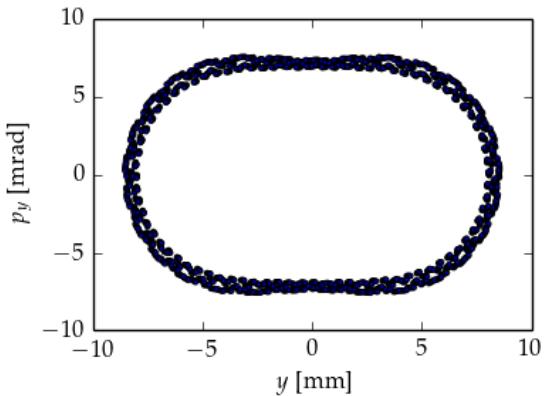
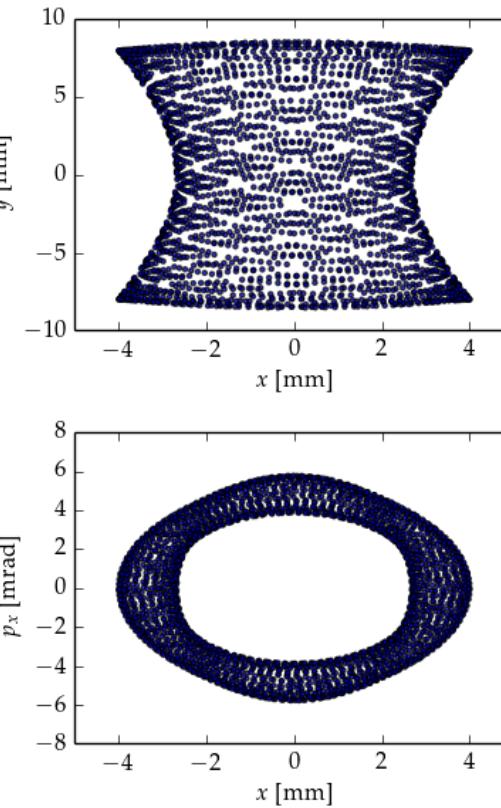
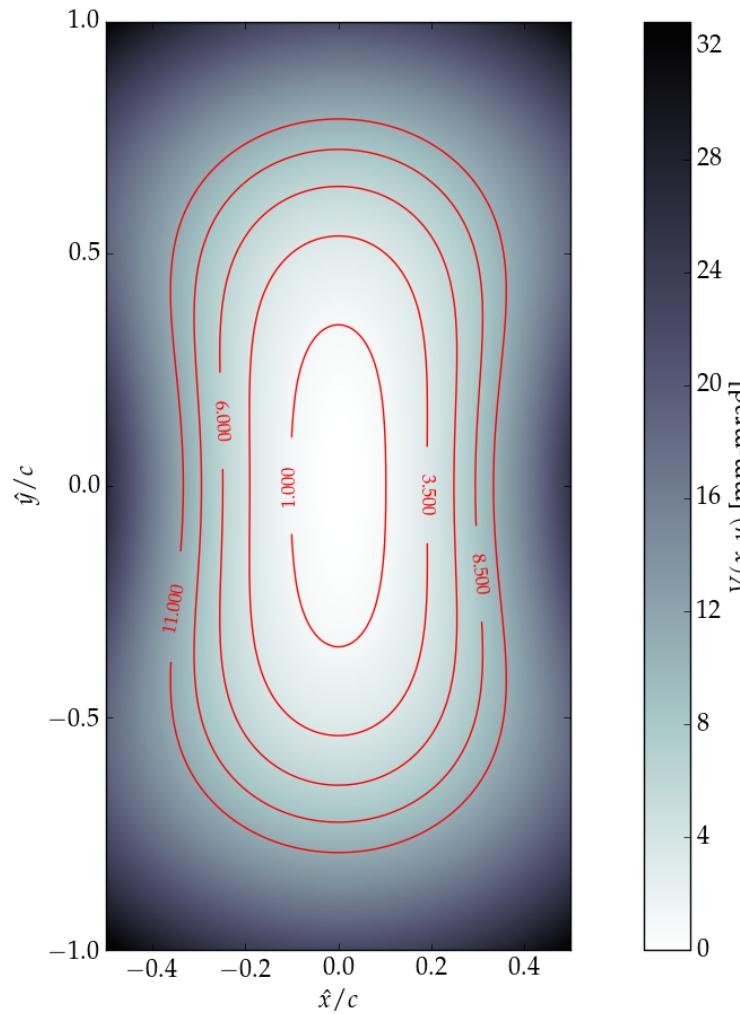
*Spallation Neutron Source Project, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830, USA*

S. Nagaitsev

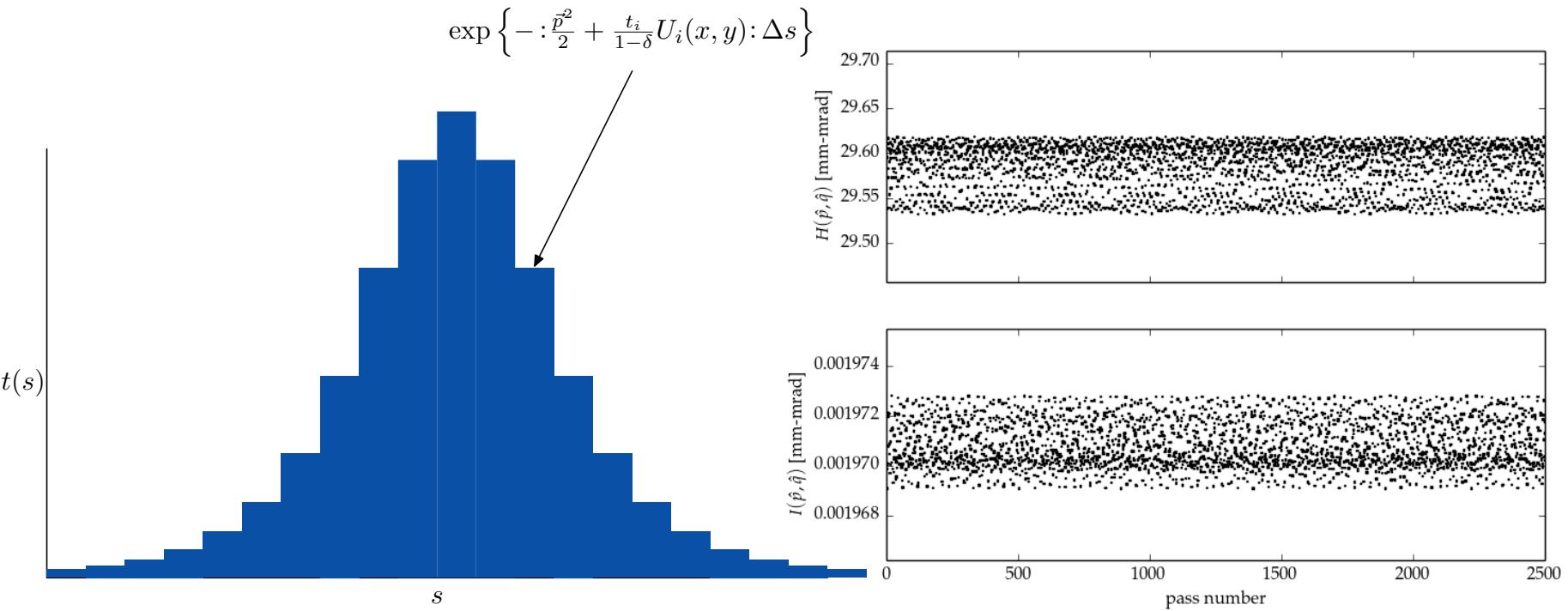
*Fermi National Accelerator Laboratory, Batavia, Illinois 60510, USA*

(Received 3 March 2010; published 25 August 2010)

# *Elliptic Potential Properties*

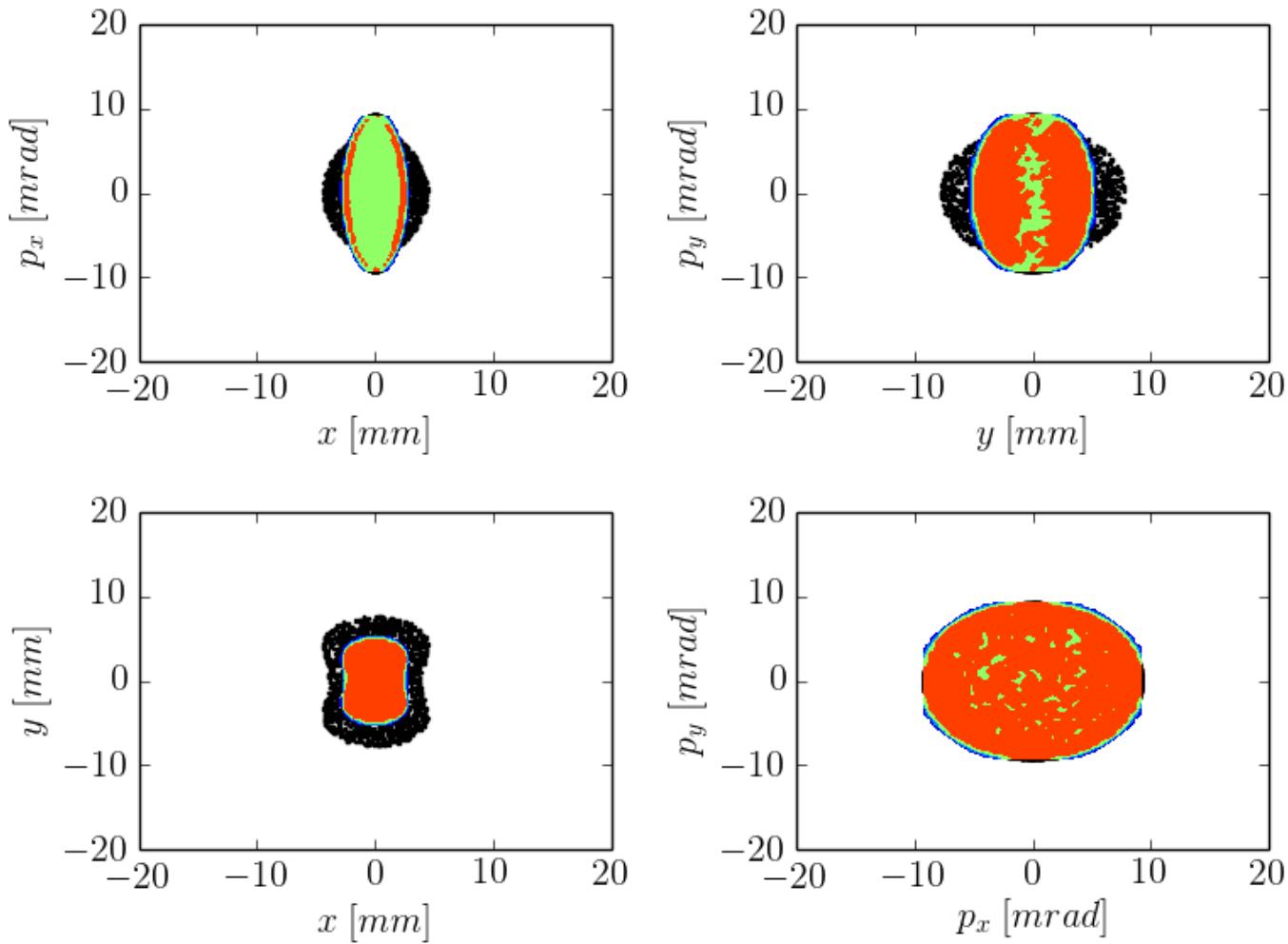


# Lattice Design Requirements

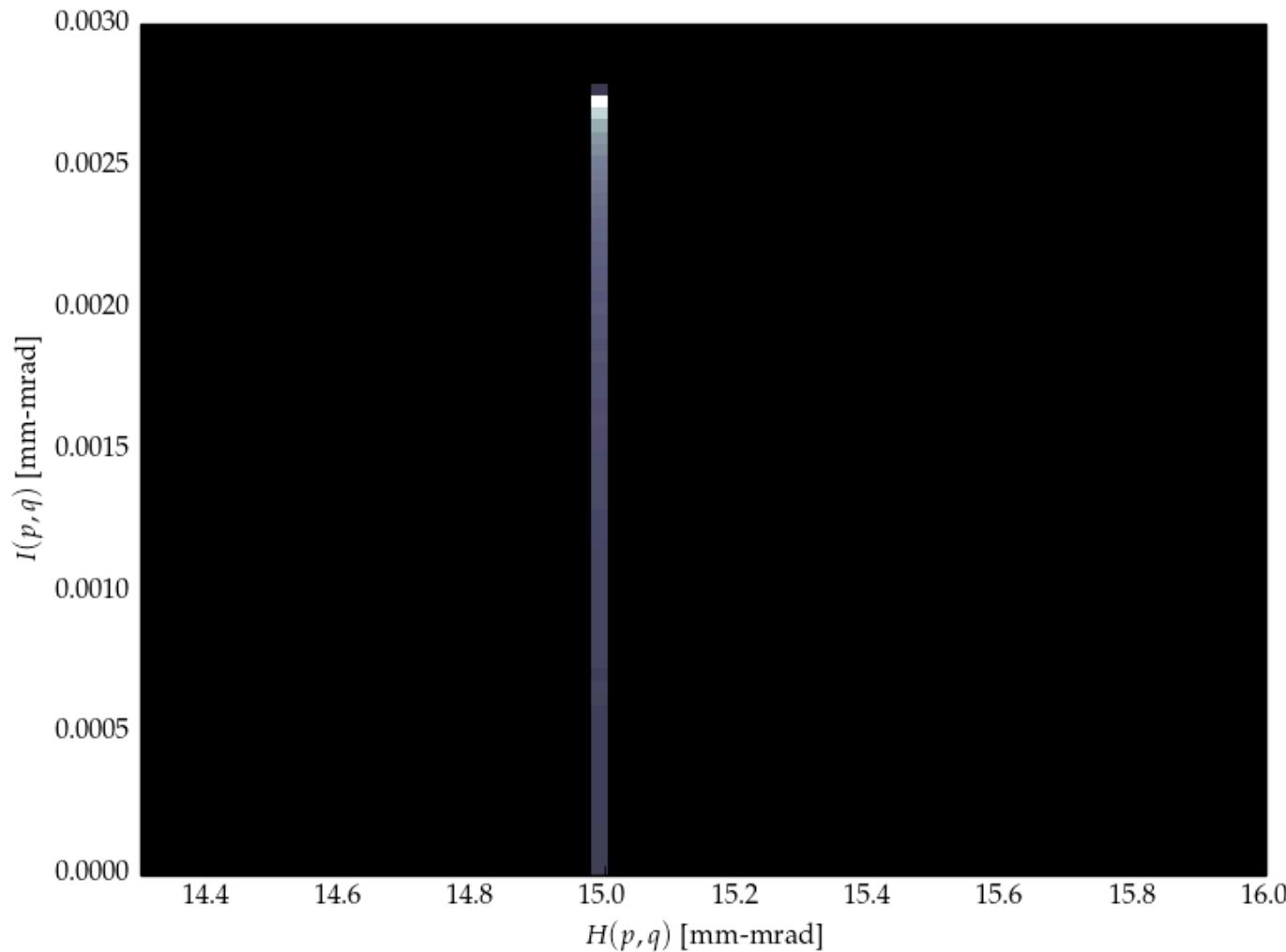


$$\begin{aligned}
H = & \mu_x (1 - C_x(\delta)) \frac{1}{2} (\hat{p}_x^2 + \hat{x}^2) + \\
& \mu_y (1 - C_y(\delta)) \frac{1}{2} (\hat{p}_y^2 + \hat{y}^2) + \\
& t \int_0^{\ell_{\text{drift}}} ds' V(\hat{x} - \delta(\eta/\sqrt{\beta_x}), \hat{y})
\end{aligned}$$

# Preliminary Results – Halo Suppression



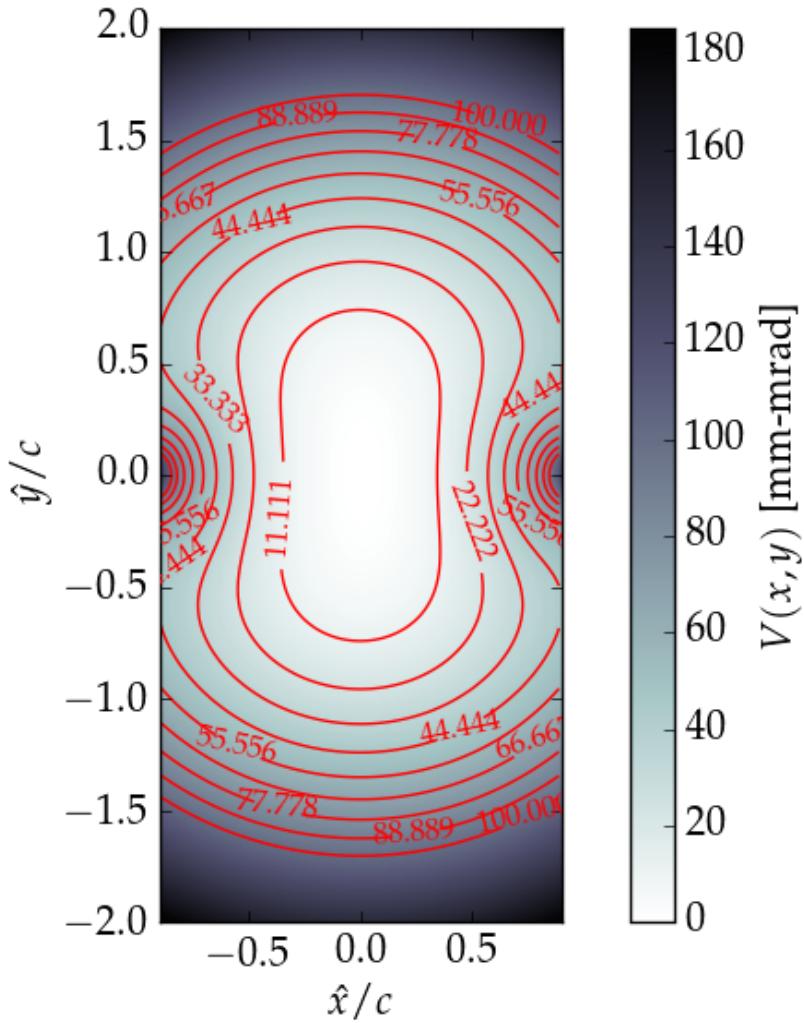
# *Preliminary Results – Phase Space Diffusion*



# *Future Work*

- *Injection matching*
- *Quadrupole wake fields and integrability*
- *Synchrotron oscillations & transverse-longitudinal chromatic coupling*
- ...

# *Injection Matching*



Matched beam is very not Gaussian

Possible schemes:

- Phase space painting
- Adiabatic up-ramp
- Adiabatic down-ramp

# *Quadrupole wake fields & integrability*

$$H = \mu_x (1 - C_x(\delta)) \frac{1}{2} (\hat{p}_x^2 + \hat{x}^2) +$$

$$\mu_y (1 - C_y(\delta)) \frac{1}{2} (\hat{p}_y^2 + \hat{y}^2) +$$

$$t \int_0^{\ell_{\text{drift}}} ds' V(\hat{x} - \delta(\eta/\sqrt{\beta_x}), \hat{y})$$

$$+ \langle Q_x \rangle \hat{x}^2 + \langle Q_y \rangle \hat{y}^2$$

# *Synchrotron Oscillations*

$$H = \mu_x (1 - C_x(\delta)) \frac{1}{2} (\hat{p}_x^2 + \hat{x}^2) +$$

$$\mu_y (1 - C_y(\delta)) \frac{1}{2} (\hat{p}_y^2 + \hat{y}^2) +$$

$$t \int_0^{\ell_{\text{drift}}} ds' V(\hat{x} - \delta(n/\sqrt{\beta_x}), \hat{y})$$

$$+ U_{\text{rf}}(\phi)$$

# *Nonlinear Integrable Optics as a Route to High Intensity Accelerators*

***Thank you for your attention***

*Stephen D. Webb*

*RadiaSoft, LLC., Boulder, CO*

[swebb@radiasoft.net](mailto:swebb@radiasoft.net)

*7 September, 2015*

*Oxford University*

*This work is supported in part by US DOE  
Office of Science, Office of High Energy  
Physics under SBIR award DE-SC0011340*

# When are sextupoles optically transparent?

- Lie operator approach

$$\mathcal{M} = e^{-S_n :z^n:} e^{-:h_2:} e^{-S_n :z^n:}$$

$$e^{-:h_2:} = \mathcal{A}^{-1} \underbrace{\mathcal{R}(\theta)}_{\text{pure rotation}} \mathcal{A}$$

$$\mathcal{M} = e^{-:h_2:} \exp(-S_n :e^{:h_2:} z^n:) \exp(-S_n :z^n:)$$

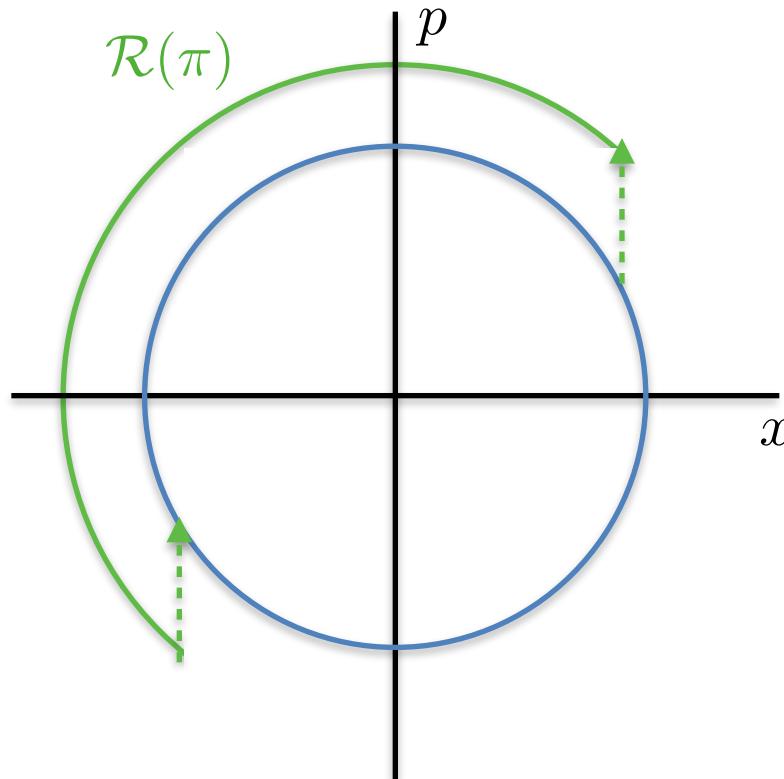
$$\mathcal{M} = \mathcal{A}^{-1} \mathcal{R} \exp(-S_n : \mathcal{R}(\bar{z})^n:) \exp(-S_n : \bar{z}^n:)$$

$$\mathcal{R} \propto -1, \theta = (2n+1)\pi \implies \exp(-S_n : \mathcal{R}(\bar{z})^n:) \exp(-S_n : \bar{z}^n:) \mathcal{A} = 1$$

- Off-momentum particles do not cancel exactly because  $\theta$  is energy-dependent. This is the basis of chromaticity correction.

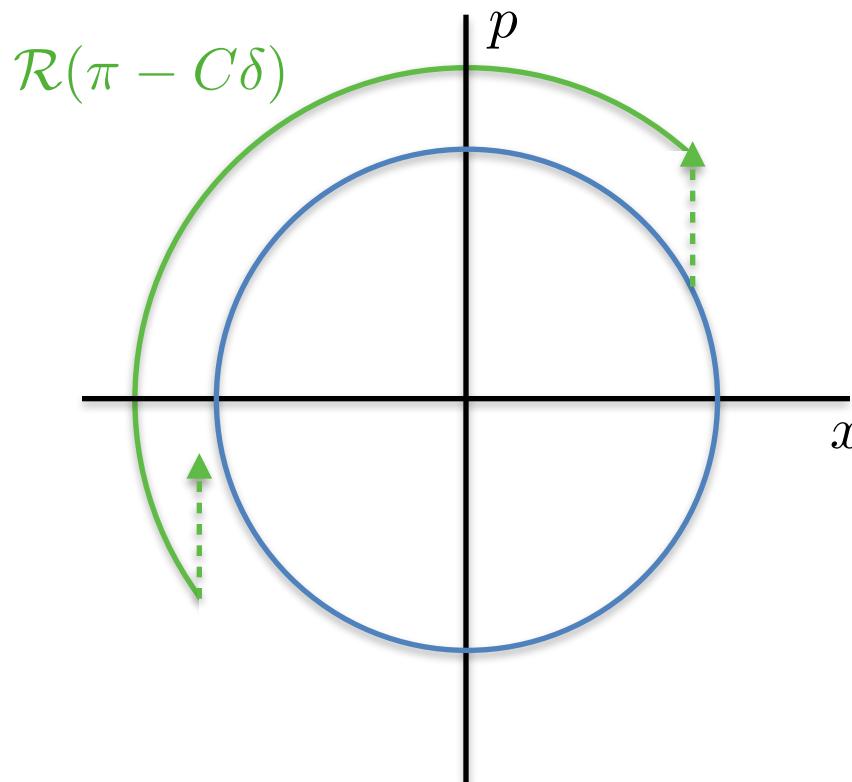
# When are sextupoles optically transparent?

- Pictorial approach (design momentum)



# When are sextupoles optically transparent?

- Pictorial approach (off-momentum)



# The horror...

$$\begin{aligned}\mathcal{M} &= \left( \prod_{i=0}^{N/2} \exp \left\{ - : \frac{p^2}{2} + t_i \mathcal{V}_i(x, y) : \Delta s \right\} \right) e^{- : h_0 :} \left( \prod_{i=N/2}^N \exp \left\{ - : \frac{p^2}{2} + t_i \mathcal{V}_i(x, y) : \Delta s \right\} \right) \\ &= \left( \prod_{i=0}^{N/2} \exp \left\{ - t_i : e^{-(i+1/2) : p^2/2 : \Delta s} \mathcal{V}_i(x, y) : \Delta s \right\} \right) \circ \\ &\quad \underbrace{e^{- : p^2/2 : \ell/2} e^{- : h_0 :} e^{- : p^2/2 : \ell/2}}_{e^{- : h_2 :}} \circ \\ &\quad \left( \prod_{i=N/2}^N \exp \left\{ - t_i : e^{(i+1/2) : p^2/2 : \Delta s} \mathcal{V}_i(x, y) : \Delta s \right\} \right)\end{aligned}$$

# ... the horror

$$e^{-:h_2:} = \mathcal{A} e^{-:\overline{h_2}:} \mathcal{A}^{-1}$$

Normalized coördinates

$$\mathcal{A}^{-1} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 & 0 & 0 & 0 & -\eta/\sqrt{\beta_x} \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} & 0 & 0 & 0 & -\alpha_x\eta + \beta_x\eta'/\sqrt{\beta_x} \\ 0 & 0 & 1/\sqrt{\beta_y} & 0 & 0 & 0 \\ 0 & 0 & \alpha_x/\sqrt{\beta_y} & \sqrt{\beta_y} & 0 & 0 \\ \eta' & \eta & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{Courant-Snyder Parameterization}$$

$$\mathcal{M} = \left( \prod_{i=0}^{N/2} \exp \left\{ -t_i : e^{-(i+1/2) : p^2/2 : \Delta s} \mathcal{V}_i(x, y) : \Delta s \right\} \right) \circ \\ \left( \mathcal{A} e^{-:\overline{h_2}:} \mathcal{A}^{-1} \right) \circ \\ \left( \prod_{i=N/2}^N \exp \left\{ -t_i : e^{(i+1/2) : p^2/2 : \Delta s} \mathcal{V}_i(x, y) : \Delta s \right\} \right)$$

$$\begin{aligned}
& \mathcal{A}^{-1} \exp \left\{ -t_i : e^{(i+1/2) : p^2 : 2} \mathcal{V}_i(x, y) : \Delta s \right\} = \\
& \mathcal{A}^{-1} \exp \left\{ -t_i : e^{(i+1/2) : p^2 : 2} \mathcal{V}_i(x, y) : \Delta s \right\} \mathcal{A} \mathcal{A}^{-1} = \\
& \underbrace{\mathcal{A}^{-1} e^{-(i+1/2) : p^2 / 2 : \Delta s}}_{\mathcal{A}_i^{-1}} \exp \left\{ -t_i : \mathcal{V}_i(x, y) : \Delta s \right\} \underbrace{e^{(i+1/2) : p^2 / 2 : \Delta s}}_{\mathcal{A}_i} \mathcal{A}
\end{aligned}$$

The Danilov-Nagaitsev potential normalizing trick as follows:

$$A_0^{(i)} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 & 0 & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} & 0 & 0 \\ 0 & 0 & 1/\sqrt{\beta_y} & 0 \\ 0 & 0 & \alpha_x/\sqrt{\beta_y} & \sqrt{\beta_y} \end{pmatrix}$$

$$\mathcal{V}_i(x, y) = \mathcal{V}_i \left( A_0^{(i)}(x, y) \right)$$

$$\mathcal{A}^{-1} e^{-(i+1/2) : p^2/2 : \Delta s} \exp \{-t_i : \mathcal{V}_i(x, y) : \Delta s\} e^{(i+1/2) : p^2/2 : \Delta s} \mathcal{A} =$$

$$\exp \left\{ -t : \mathcal{V} \left( \bar{x} - \delta \frac{\eta}{\sqrt{\beta_x}}, \bar{y} \right) : \Delta s \right\}$$

*Final transfer map in normalized coordinates*

$$\left( \prod_{i=0}^{N/2} \exp \left\{ - : \frac{p^2}{2} + t_i \mathcal{V}_i(x, y) : \Delta s \right\} \right) e^{- : h_0 :} \left( \prod_{i=N/2}^N \exp \left\{ - : \frac{p^2}{2} + t_i \mathcal{V}_i(x, y) : \Delta s \right\} \right) =$$

$$\mathcal{A} \exp \left\{ \sum_i -(1-\delta)t : \mathcal{V} \left( \bar{x} - \delta \frac{\eta_i}{\sqrt{\beta_i}}, \bar{y} \right) : \right\} e^{- : \bar{h}_2 :} \exp \left\{ \sum_i -(1-\delta)t : \mathcal{V} \left( \bar{x} - \delta \frac{\eta_i}{\sqrt{\beta_i}}, \bar{y} \right) : \right\} \mathcal{A}^{-1}$$

$$\bar{h}_2 = \frac{\mu_0}{2} [(1 - C_x \delta) (\bar{p}_x^2 + \bar{x}^2) + (1 - C_y \delta) (\bar{p}_y^2 + \bar{y}^2)]$$