Breit-Wigner cross section and angular momentum

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What we will cover

• The Breit-Wigner resonance curve
• Addition of angular momentum
• Clebsch-Gordon coefficients
  – Their derivation with raising and lowering operators
• Examples
  – Combining spin $\frac{1}{2}$
  – Nucleon wave function
  – Decay of the $\Lambda$
  – The $\Delta(1232)$ resonance
Non-relativistic Breit-Wigner

• Broad states which can be formed by collisions between the particles into which the decay are referred to as resonances.
• They are described by the Breit-Wigner formula:

\[
\sigma(E) = \frac{4\pi\lambda^2 g \Gamma_i \Gamma_j / 4}{(E - E_R)^2 + \Gamma^2/4}
\]

where:
\[
\lambda = \frac{\hbar}{p}
\]

\(\Gamma_i\) is partial width of formation channel \(i\)
\(\Gamma_f\) is partial width of decay channel \(f\)
\(\Gamma = \sum_j \Gamma_j\) is the total width = FWHM
\(E_R\) is mass of the resonance

\[
g = \frac{(2J + 1)}{(2s_1 + 1)(2s_2 + 1)}
\]

\(J\) is the spin of the resonance
\(s_1, s_2\) are the spins of the two particles forming the resonance
The form of the Breit Wigner

- The form derives from the exponential time dependence of the decay
  - The energy dependence can then be found by taking the Fourier transform of the time domain
- The wave function of a non-stationary decaying state of central angular frequency $\omega_R = E_R / \hbar$ and lifetime $\tau = \hbar / \Gamma$ is:
  \[ \psi(t) = \psi(0)e^{-i\omega t} e^{-t/2\tau} = \psi(0)e^{-t(i\omega_R + \Gamma/2)}, \text{ where } \hbar = c = 1 \]
- The intensity $\psi \psi^*$ follows the exponential decay law $I \propto e^{-\Gamma t}$
- The Fourier transform of the decay is:
  \[ g(\omega) = \int_{0}^{\infty} \psi(t)e^{i\omega t} dt \text{ where } \omega = E / \hbar \]
- The amplitude as a function of $E$ is:
  \[ \chi(E) = \int \psi(t)e^{iEt} dt = \psi(0) \int e^{-t[(\Gamma/2)+i(E_R-E)]} dt \propto \frac{1}{(E-E_R)-i\Gamma/2} \]
Form of the Breit Wigner

- The cross section $\sigma(E)$ is proportional to $\chi\chi^*$:

\[
\sigma(E) = \sigma_{\text{max}} \frac{\Gamma^2 / 4}{(E - E_R^2) + \Gamma^2 / 4}
\]

- The cross section falls to half the maximum at $E - E_R = \pm \Gamma / 2$

- Optical theorem arguments lead to $\sigma_{\text{max}} = 4\pi\lambda^2$ see Perkins 4th edition, pp57 for more details.

- Spin multiplicity factors for the initial and final state are $(2s_1+1)(2s_2+1)$ and $(2J+1)$, respectively. The cross section is scaled by final/initial.

- The width scale if you are looking at formation in $i$ and decay in $j$ is $\Gamma_i \Gamma_j / \Gamma^2$

- The Lorentz invariant relativistic form is:

\[
\sigma(E) = \sigma_{\text{max}} \frac{M_0^2 \Gamma^2}{(s - M_0^2)^2 + \Gamma(s)^2 M_0^2} \quad \text{where} \quad M_0 = E_R
\]
Example: $\Delta^{++}(1232)$

Asymmetric width due to phase space variation over resonance

Formed in elastic $\pi p$ scattering ($\Gamma = \Gamma_i = \Gamma_f$)
Addition of angular momentum

• Let $J_1$ and $J_2$ be 2 angular momentum operators i.e.
  – The orbital angular momentum ($J_1$) and the spin ($J_2$) of the same particle
  – The angular momentum of spinless particle 1 ($J_1$) and spinless particle 2 ($J_2$)

• These operators squared and their projection operators follow these relations

\[ J_1^2 Y(j_1, m_1) = j_1(j_1 + 1)Y(j_1, m_1) \text{ and } J_{1z} Y(j_1, m_1) = m_1 Y(j_1, m_1) \]
\[ J_2^2 Y(j_2, m_2) = j_2(j_2 + 1)Y(j_2, m_2) \text{ and } J_{2z} Y(j_2, m_2) = m_2 Y(j_2, m_2) \]
Addition of angular momentum

• Consider the operators:
  – $J_z = J_{1z} + J_{2z}$ and
  – $J^2 = (J_{1x} + J_{2x})^2 + (J_{1y} + J_{2y})^2 + (J_{1z} + J_{2z})^2$

acting on the wave function
  – $Y(j_1, m_1; j_2, m_2) = Y(j_1, m_1)Y(j_2, m_2)$

• $Y(j_1, m_1; j_2, m_2)$ is an eigenfunction of $J_z$ with eigenvalue $M = m_1 + m_2$

• $Y(j_1, m_1; j_2, m_2)$ is not in general an eigenfunction of $J^2$

• However, linear combinations of $Y(j_1, m_1; j_2, m_2)$ can produce eigenfunctions $Y(J, M, j_1, j_2)$ of $J^2$ with eigenvalues $J(J+1)$

\[ Y(J, M, j_1, j_2) = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C_{j_1 j_2} (J, M, m_1, m_2) Y(j_1, m_1; j_2, m_2) \]
Raising and lowering operators

- To derive Clebsh Gordon (CG) coefficients we need to make use of raising and lowering operators.
- Angular momentum operators obey the commutation relation:
  \[ [J_i, J_j] = i \varepsilon_{ijk} J_k \]
  where \( \varepsilon_{ijk} = +1 \) or \( -1 \) if \( i, j, k \) are even (i.e., yzx) or odd permutations (i.e., zyx), respectively, and 0 otherwise (i.e., zzx).
- The only operator that commutes with \( J_x, J_y \) and \( J_z \) is:

\[
J^2 = J_x^2 + J_y^2 + J_z^2 \quad \text{and} \quad [J^2, J_i] = 0
\]

- We define states as eigenfunctions of \( J^2 \) and \( J_z \) with eigenvalues:

\[
J^2 | j, m \rangle = j(j + 1) | j, m \rangle \quad \text{and} \quad J_z | j, m \rangle = m | j, m \rangle
\]

where \( m = -j, -j+1, \ldots, j-1, j \).
Raising and lowering operators

• Now we define raising and lowering operators:

\[ J_\pm = J_x \pm iJ_y \]

which satisfy: \[ [J_z, J_\pm] = \pm J_\pm \]

• Using this commutation relation:

\[ J_z J_+ | j, m \rangle = (J_- J_z - J_-) | j, m \rangle \]
\[ = J_- (J_z - 1) | j, m \rangle \]
\[ = (m-1) J_- | j, m \rangle \]

\[ J_z (J_- | j, m \rangle) = (m-1)(J_- | j, m \rangle) \]

In an analogous fashion it is possible to show that \( J_+ | j, m \rangle \) is an eigenfunction of \( J_z \) with eigenvalue \( m+1 \)

\[ J_+ | j, m \rangle = C_+ | j, m + 1 \rangle \]

Generic constants that are dependent on \( j \) and \( m \)
Raising and lowering operators

• Noting that $J_+$ is the complex conjugate of $J_-$

$$\langle j, m | J_- | j, m + 1 \rangle = \mathcal{C}_-(m + 1) \langle j, m | j, m \rangle = C_- (m + 1)$$

$$\langle j, m | J_+^* | j, m + 1 \rangle = C_+^*(m) \langle j, m + 1 | j, m + 1 \rangle = C_+^*(m)$$

therefore $C_- (m + 1) = C_+ (m) = C$ ignoring an arbitrary phase so,

$$J_- J_+ | j, m \rangle = C^2 | j, m \rangle$$

but

$$J_- J_+ = J_x^2 + J_y^2 + i(J_x J_y - J_y J_x)$$

$$J_- J_+ = J_x^2 + J_y^2 + J_z^2 - J_z^2 + i[J_x, J_y]$$

$$J_- J_+ = J_x^2 - J_z^2 - J_z$$, therefore

$$C^2 = j(j + 1) - m(m - 1) \text{ so } C^2_\pm (j, m) = \sqrt{j(j + 1) - m(m \pm 1)}$$
Clebsch Gordon coefficients

• We can now use the expression for $C^2$ to derive the C-G coefficients.

• Consider 2 particles $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ forming a combined state $|j, m\rangle$ where $j_1$ and $j_2$ are 1 and 1/2, respectively. Therefore, $j$ is 1/2 or 3/2.

• Clearly the maximum (3/2) and minimum (-3/2) m states are:

$$|\frac{3}{2}, \frac{3}{2}\rangle = |1, 1\rangle |\frac{1}{2}, \frac{1}{2}\rangle$$

$$|\frac{3}{2}, -\frac{3}{2}\rangle = |1, -1\rangle |\frac{1}{2}, -\frac{1}{2}\rangle$$

using the raising and lower operators and the definition of $C_-$:

$J_-|\frac{1}{2}, \frac{1}{2}\rangle = 1|\frac{1}{2}, -\frac{1}{2}\rangle$ and $J_-|\frac{1}{2}, -\frac{1}{2}\rangle = 0$

$J_-|1,1\rangle = \sqrt{2}|1,0\rangle$, $J_-|1,0\rangle = \sqrt{2}|1,-1\rangle$ and $J_-|1,-1\rangle = 0$
CG coefficients

- Now operating on the combined state

\[ J_- \left| \frac{3}{2}, \frac{3}{2} \right> = \sqrt{3} \left| \frac{3}{2}, \frac{1}{2} \right> = J_- \left| 1,1 \right> \left| \frac{1}{2}, \frac{1}{2} \right> \]

\[ = \sqrt{2} \left| 1,0 \right> \left| \frac{1}{2}, \frac{1}{2} \right> + \left| 1,1 \right> \left| \frac{1}{2}, -\frac{1}{2} \right> \]

\[ \left| \frac{3}{2}, \frac{1}{2} \right> = \sqrt{\frac{2}{3}} \left| 1,0 \right> \left| \frac{1}{2}, \frac{1}{2} \right> + \sqrt{\frac{1}{3}} \left| 1,1 \right> \left| \frac{1}{2}, -\frac{1}{2} \right> \]

- In analogous fashion one finds:

\[ \left| \frac{3}{2}, -\frac{1}{2} \right> = \sqrt{\frac{2}{3}} \left| 1,0 \right> \left| \frac{1}{2}, -\frac{1}{2} \right> + \sqrt{\frac{1}{3}} \left| 1,-1 \right> \left| \frac{1}{2}, \frac{1}{2} \right> \]

- So we have all the \( j=3/2 \) states and the CG coefficients
Clebsch Gordon coefficients

• To find the $j=1/2$ states we note that:

\[ |\frac{1}{2}, \frac{1}{2}\rangle = a|1,1\rangle |\frac{1}{2}, -\frac{1}{2}\rangle + b|1,0\rangle |\frac{1}{2}, \frac{1}{2}\rangle \]

with $a^2 + b^2 = 1$. Now apply $J_+$

\[ J_+ |\frac{1}{2}, \frac{1}{2}\rangle = 0 = a|1,1\rangle |\frac{1}{2}, \frac{1}{2}\rangle + b\sqrt{2}|1,1\rangle |\frac{1}{2}, \frac{1}{2}\rangle \]

therefore

\[ a + b\sqrt{2} = 0 \]

leading to

\[ a = \sqrt{\frac{2}{3}} \quad \text{and} \quad b = -\sqrt{\frac{1}{3}} \]

• Similarly with $J_-$:

\[ |\frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}} |1,0\rangle |\frac{1}{2}, \frac{1}{2}\rangle - \sqrt{\frac{2}{3}} |1,-1\rangle |\frac{1}{2}, \frac{1}{2}\rangle \]

General case:

\[ |j,m\rangle = \sum_{m_1m_2} C_{m_1m_2}^{jm} |j_1,m_1\rangle |j_2,m_2\rangle \]

Can be computed by applying $J_\pm$ operators
35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND $d$ FUNCTIONS

Note: A square-root sign is to be understood over every coefficient, e.g., for $-i/\sqrt{3}$ read $-i/\sqrt{3}$.

$$Y_0^m = \delta_{00} \delta_{m0}$$

$$Y_1^m = \sqrt{\frac{2}{3}} \delta_{1m} \delta_{m0}$$

$$Y_2^m = \sqrt{\frac{1}{2}} \left[ \delta_{2m} \delta_{m0} + \sqrt{3} \delta_{2(-m)} \delta_{m0} \right]$$

$$d_{m,m'}^{1/2 \times 1/2} = \sqrt{\frac{2}{3}} \delta_{m,m'}$$

$$d_{m,m'}^{2 \times 1/2} = \sqrt{\frac{1}{2}} \left[ \delta_{2m} \delta_{m0} + \sqrt{3} \delta_{2(-m)} \delta_{m0} \right]$$

$$d_{m,m'}^{2 \times 1/2} = \sqrt{\frac{1}{2}} \left[ \delta_{m,m'} + \sqrt{3} \delta_{(-m),m'} \right]$$

$$d_{m,m'}^{2 \times 2} = \sqrt{\frac{1}{2}} \left[ \delta_{m,m'} - i \delta_{(-m),m'} \right]$$

$$d_{m,m'}^{2 \times 2} = \sqrt{\frac{1}{2}} \left[ \delta_{m,m'} + i \delta_{(-m),m'} \right]$$

$$d_{m,m'}^{2 \times 2} = \sqrt{\frac{1}{2}} \left[ \delta_{m,m'} - i \delta_{(-m),m'} \right]$$

$$d_{m,m'}^{2 \times 2} = \sqrt{\frac{1}{2}} \left[ \delta_{m,m'} + i \delta_{(-m),m'} \right]$$

Figure 35.1: The sign convention is that of Wigner (Group Theory, Academic Press, New York, 1959), also used by Condon and Shortley (The Theory of Atomic Spectra, Cambridge, 1935). For tables of Clebsch-Gordan Coefficients, see Wigner (1959) and Cohen (Tables of the Clebsch-Gordan Coefficients, North American Rockwell, Vitens Center, Thousand Oaks, Calif., 1954). The coefficients here have been calculated using computer programs written independently by Cohen and at LBL.
### Table of CG coefficients: $1\times1/2$ and $1/2\times1/2$

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<thead>
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<th>$m_1$</th>
<th>$m_2$</th>
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<th>$\frac{3}{2}$</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{3}{2}$</th>
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<td>$M$</td>
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<tr>
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<td>0</td>
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<td>-1</td>
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<th>$J$</th>
<th>1  1  0  1</th>
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<tbody>
<tr>
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<td>+1/2</td>
<td>$M$</td>
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<td>$-\frac{1}{2}$</td>
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Example: 2 spin $\frac{1}{2}$ particles

- Two spin $\frac{1}{2}$ particles can combine to form 4 possible spin states:
  - $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$ (particle 1 and particle 2)
- The total spin can be $s=1$ a triplet state or $s=0$ a singlet with respect to $s_z$
- Using CG coefficients we can write the irreducible representation:

$$|1,1\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle = |\uparrow\uparrow\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}, \frac{1}{2} \rangle |\frac{1}{2}, -\frac{1}{2} \rangle + |\frac{1}{2}, -\frac{1}{2} \rangle |\frac{1}{2}, \frac{1}{2} \rangle \right) = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right)$$

$$\begin{cases} \text{Symmetric triplet} \\
|1,-1\rangle = |\frac{1}{2}, -\frac{1}{2} \rangle |\frac{1}{2}, -\frac{1}{2} \rangle = |\downarrow\downarrow\rangle \\
|1,0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}, \frac{1}{2} \rangle |\frac{1}{2}, -\frac{1}{2} \rangle - |\frac{1}{2}, -\frac{1}{2} \rangle |\frac{1}{2}, \frac{1}{2} \rangle \right) = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \end{cases}$$

Antisymmetric singlet
Example: the nucleon wavefunction

- Three quarks of spin $\frac{1}{2}$ produce a baryon with spin $\frac{1}{2}$
  - Therefore, one quark’s spin antiparallel to the other two: $\uparrow\uparrow\downarrow$
- The spin state is a mixture of symmetric and antisymmetric states
  - The flavour wavefunction also has mixed symmetric, but total spin-flavour wavefunction is symmetric
- Flavour asymmetry impossible with uuu, ddd and sss states
  - No spin a $\frac{1}{2}$ states of this type observed
- The spin wavefunction of a proton with spin $+1/2$ written as the product of one quark’s spin wavefunction and that of the remaining pair
  \[
  \chi_p \left( \frac{1}{2}, \frac{1}{2} \right) = \sqrt{\frac{2}{3}} \chi_{uu}(1,1) \chi_d \left( \frac{1}{2}, -\frac{1}{2} \right) + \sqrt{\frac{2}{3}} \chi_{uu}(1,0) \chi_d \left( \frac{1}{2}, \frac{1}{2} \right)
  \]
  - Arbritrary choice of u-quark pair
  - Using our previously derived spin $1/2 \times 1$ CG coefficients
Example: nucleon wave functions

- Now substituting the wave functions for 2 spin ½ particles:
  \[ \chi_{uu}(1,1) = \uparrow\uparrow \text{ and } \chi_{uu}(1,0) = \sqrt{\frac{1}{2}} (\uparrow\downarrow + \uparrow\downarrow) \]
  leads to:
  \[ |p^\uparrow\rangle = \sqrt{\frac{2}{3}} |u^\uparrow u^\uparrow d^\downarrow\rangle - \sqrt{\frac{1}{6}} |u^\uparrow u^\downarrow d^\uparrow\rangle - \sqrt{\frac{1}{6}} |u^\downarrow u^\uparrow d^\uparrow\rangle \]
  - Symmetric under interchange of the 2 u quarks
- To make the other symmetric pairings: swap 1\textsuperscript{st} with 3\textsuperscript{rd} and 2\textsuperscript{nd} with 3\textsuperscript{rd} quark and add the new terms. Normalised to 1.
  \[ |p^\uparrow\rangle = \sqrt{\frac{1}{18}} \left( 2 |u^\uparrow u^\uparrow d^\downarrow\rangle - |u^\uparrow u^\downarrow d^\uparrow\rangle - |u^\downarrow u^\uparrow d^\uparrow\rangle \right) \]
  \[ \quad + 2 |u^\uparrow d^\downarrow u^\uparrow\rangle - |u^\downarrow d^\uparrow u^\uparrow\rangle - |u^\downarrow d^\uparrow u^\downarrow\rangle \]
  \[ \quad + 2 |d^\downarrow u^\uparrow u^\uparrow\rangle - |d^\uparrow u^\downarrow u^\uparrow\rangle - |d^\uparrow u^\uparrow u^\downarrow\rangle \]  
  Swap 1\textsuperscript{st} and 3\textsuperscript{rd} quarks
  Swap 2\textsuperscript{nd} and 3\textsuperscript{rd} quarks
- Other examples:
  - Make neutron with a u→d
  - Use two s quarks and a u or d to make Ξ\textsuperscript{0} and Ξ\textsuperscript{-}, respectively
  - Wave functions for Σ triplet and Λ
Example: decay of the $\Lambda$

- The $\Lambda$ is an isospin singlet ($I=0$) and it decays almost solely to $p\pi^-$ and $n\pi^0$
- To a first approximation the decay is a $s\to u$ quark transition which changes the isospin by $\frac{1}{2}$
- The pion-nucleon state is in a state $I=1/2$ with a third component $I_3=-1/2$
- The CG coefficients for spin 1 (pion) and $1/2$ (nucleon) give

$$\left| \frac{1}{2}, -\frac{1}{2} \right> = \frac{1}{\sqrt{3}} |1,0\rangle \left| \frac{1}{2}, -\frac{1}{2} \right> - \sqrt{\frac{2}{3}} |1,-1\rangle \left| \frac{1}{2}, \frac{1}{2} \right>$$

$$= \frac{1}{\sqrt{3}} \left| \pi^0 \right> |n\rangle - \sqrt{\frac{2}{3}} \left| \pi^- \right> |p\rangle$$

Therefore

$$\frac{\sigma(\Lambda \to p\pi^-)}{\sigma(\Lambda \to n\pi^0)} = \frac{\left< p\pi^- \left| \frac{1}{2}, -\frac{1}{2} \right> \right|^2}{\left< n\pi^0 \left| \frac{1}{2}, -\frac{1}{2} \right> \right|^2} = \left( \frac{-\sqrt{\frac{2}{3}}}{\frac{1}{\sqrt{3}}} \right)^2 = 2$$

Assuming phase space the same due to the mass of $n$ and $p$ being approximately equal and the masses of the $\pi$ being approximately equal

- The measured ratio is 1.8 isospin not an exact symmetry especially in weak decay
Summary

• Breit Wigner resonance form
• Derivation of Clebsch Gordon coefficients
• Some examples of their use