This recipe gives a matrix element, \(-iM_{fi}\), which is trivially integrated because of the delta function,

\[
-iM_{fi} = \int \left[ \bar{u}(p_3) ig_\mu \gamma^\mu u(p_1) \right] \frac{-ig_\nu}{q^2} \left[ \bar{u}(p_4) ig_\nu \gamma^\nu u(p_2) \right] \delta^4(p_1 - p_3 - q) \, dq.
\]

The fermion spinors are each have four elements and the \(\gamma^\mu\) are 4 \times 4 matrices. In the majority of cases, we are dealing with the scattering of unpolarised particles, this is, the particles have not specific spin direction. This means that we are interested in a spin-average matrix element squared, \(\langle|M_{fi}|^2\rangle\), which can be shown (e.g. Griffiths) to be

\[
\langle|M_{fi}|^2\rangle = \frac{g_e^4}{4(p_1 - p_3)^4} \text{Tr} \left[ \gamma^\nu(p_1 + m)\gamma^\nu(p_3 + m) \right] \text{Tr} \left[ \gamma_\nu(p_2 + M)\gamma_\nu(p_4 + M) \right],
\]

\[
= \frac{g_e^4}{4(p_1 - p_3)^4} \left[ 4(p_1^\nu p_3^\nu + p_2^\nu p_4^\nu + (m^2 - p_1 \cdot p_3)g_{\mu\nu}) \right] \left[ 4(p_2^\mu p_4^\nu + p_4^\mu p_2^\nu + (M^2 - p_2 \cdot p_4)g_{\mu\nu}) \right].
\]

Which, by multiplying out and summing over repeated indices comes to the general result,

\[
\langle|M_{fi}|^2\rangle = \frac{8g_e^4}{(p_1 - p_3)^4} \left[ (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - m^2(p_2 \cdot p_4) - M^2(p_1 \cdot p_3) + 2m^2M^2 \right].
\]
1.1 Mott scattering

In my Basics notes, there is a derivation of phase space and cross section of point-particle scattering in the lab frame,

\[ \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \int |M|_{ij}|^2 \frac{1}{(M + E(1 - \cos \theta))^2} d\Omega, \]

where \( M \) is mass of the target, \( E \) is the energy of the probe and \( \theta \) is the angle of recoil. Now, consider the specific case of an electron \((m)\), scattering off a much-heavier particle of mass \( M \) (e.g. muon or proton) such that the recoil can be neglected \((\theta \sim 0)\). Continuing with the spin-averaged matrix element,

\[ \frac{d\sigma}{d\Omega} = \int \frac{\langle |M|_{ij}|^2 \rangle}{(8\pi M)^2} d\Omega. \]  \hfill (2)

As the recoil is neglected, the kinematics are simple,

\[ p_2 = p_4 = (M, 0) \quad p_1 = (E, p_1) \quad p_3 = (E, p_3) \quad |p_1| = |p_3|. \]

Point particle, negligible recoil. \( p_2 = p_4 = (M, 0) \)

\[
(p_1 - p_3)^2 = (0, p_{1z} - p_{3z})^2 = |p_{1z}|^2 + |p_{3z}|^2 + 2p_{1z} \cdot p_{3z} = -2|p_{1z}|^2(1 - \cos \theta) = -4|p_{1z}|^2 \sin^2 \frac{\theta}{2}
\]

\[
(p_1 \cdot p_3) = E^2 - p_{1z} \cdot p_{3z} = |p_{1z}|^2 + m^2 - |p_{1z}|^2 \cos \theta = m^2 + 2|p_{1z}|^2 \sin^2 \frac{\theta}{2}
\]

\[
(p_1 \cdot p_2)(p_3 \cdot p_4) = (p_1 \cdot p_4)(p_2 \cdot p_3) = M^2 E^2 \quad \text{and} \quad (p_2 \cdot p_4) = M^2.
\]

Putting this into Eq. 2 above,

\[
\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi M)^2} \int \frac{8g_e^4}{(-4|p_{1z}|^2 \sin^2 \frac{\theta}{2})^2} \left[ 2M^2 E^2 - m^2 M^2 - M^2 (m^2 + 2|p_{1z}|^2 \sin^2 \frac{\theta}{2}) + 2m^2 M^2 \right] d\Omega
\]

With \( g_e = \sqrt{4\pi\alpha} \), the Mott scattering formula is,

\[
\frac{d\sigma}{d\Omega} = \int \left( \frac{\alpha}{2|p_{1z}|^2 \sin^2 \frac{\theta}{2}} \right)^2 \left[ m^2 + |p_{1z}|^2 \cos^2 \frac{\theta}{2} \right] d\Omega. \]  \hfill (3)

It is appropriate for relativistic electrons with energies \( O(1 - 100 \text{ MeV}) \), well below the proton mass, \( M = 938 \text{ MeV}/c^2 \). The non-relativistic limit, \( O(1 - 10) \text{ keV} \) electrons, the probe has momentum \( |p_{1z}|^2 \ll m^2 \) and so can use kinetic energy \( E_K = p^2 / 2m \) and the Rutherford scattering cross section is recovered,

\[
\frac{d\sigma}{d\Omega} = \int \frac{\alpha^2}{16 E_K^2 \sin^4 \frac{\theta}{2}} d\Omega.
\]
1.2 $e^- \rightarrow p^+$ scattering with recoil

Let’s extend Mott scattering to include the recoil but neglect the probe mass, $E_1 \gg m$. With energy being passed to the proton, the electron changes energy as well as direction. As we see below, this gives sensitivity to magnetic spin-spin interaction. With $m = 0$ and the proton (of mass $M$) initially at rest the 4-vectors are,

$$p_3 = (E_3, 0, E_3 \sin \theta, E_3 \cos \theta)$$

$$p_1 = (E_1, 0, 0, E_1)$$

From Eq. 1 for $m = 0$,

$$\langle |M_{ji}|^2 \rangle = \frac{8g_e^4}{(p_1 - p_3)^4} \left[ (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - M^2(p_1 \cdot p_3) \right].$$

with $p_1 \cdot p_2 = E_1 M \quad p_1 \cdot p_3 = E_1 E_3(1 - \cos \theta) \quad p_2 \cdot p_3 = E_3 M$.

And $(p_1 - p_3)^4$ comes from,

$$q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2(E_1 E_3 - E_1 E_3 \cos \theta)$$

$$= -4E_1 E_3 \sin^2 \frac{\theta}{2}.$$

Use momentum conservation to side-step using the unmeasured proton recoil, $p_4 = p_1 + p_2 - p_3$ in the following,

$$p_3 \cdot p_4 = p_3 \cdot p_1 + p_3 \cdot p_2 - p_3 \cdot p_3$$

$$= E_1 E_3(1 - \cos \theta) + E_3 M - M^2$$

$$p_1 \cdot p_4 = p_1 \cdot p_1 + p_1 \cdot p_2 - p_1 \cdot p_3$$

$$= p_1^2 + E_1 M - E_1 E_3(1 - \cos \theta)$$

So, \( \langle |M_{ji}|^2 \rangle = \frac{8g_e^4}{16E_1^2 E_3^4 \sin^4 \frac{\theta}{2}} E_1 E_3 M \left[ (E_1 - E_3)(1 - \cos \theta) + M(1 + \cos \theta) \right], \)

$$= \frac{8g_e^4}{16E_1^2 E_3^4 \sin^4 \frac{\theta}{2}} 2E_1 E_3 M \left[ (E_1 - E_3) \sin^2 \frac{\theta}{2} + M \cos^2 \frac{\theta}{2} \right].$$

In contrast to Mott Scattering, the $E_1 - E_3 \neq 0$ and dependent on $\sin^2(\theta/2)$. This is associated with he magnetic interaction of the two spin-\(\frac{1}{2}\) particles. An expression for $E_1 - E_3$ is,

$$q + p_2 = p_4 \Rightarrow q^2 + p_2^2 + 2q \cdot p_2 = p_4^2$$

$$q^2 + M^2 + 2q \cdot p_2 = M^2$$

$$q^2 = -2q \cdot p_2$$

$$q^2 = -2(p_1 - p_3) \cdot p_2$$

$$q^2 = -2(E_1 - E_3) \cdot M \quad (m \approx 0)$$

$$E_1 - E_3 = -\frac{q^2}{2M}$$

Note that two independent expressions for $q^2$ have been derived,
1. $q^2 = -2E_1E_3(1 - \cos \theta)$ conserves the 4-momentum of the electron.

2. $q^2 = -2M(E_1 - E_3)$ conserves 4-momentum of the recoiling proton, and that it does not break up.

Equating these expressions arrives using these equations together arrives at useful results for later,

$$E_3 = \frac{ME_1}{M + E_1(1 - \cos \theta)} \quad q^2 = \frac{ME_1^2(1 - \cos \theta)}{M + E_1(1 - \cos \theta)}$$  \(4\)

Altogether the spin-averaged matrix element squared is,

$$\langle |M_{fi}|^2 \rangle = \frac{g^4M^2}{E_1 E_3 \sin^2 \frac{\theta}{2}} \left[ \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin \frac{\theta}{2} \right].$$

Then using another standard phase-space result,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left( \frac{E_3}{E_1 M} \right)^2 |M|^2$$

we arrive at a formula for $e^-$ scattering off a recoiling $p^+$ point particle,

$$\frac{d\sigma}{d\Omega} = \left( \frac{E_3}{E_1} \right) \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \left[ \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin \frac{\theta}{2} \right]$$  \(5\)


This paper shows early work of electron-nucleon scattering (by Nobel Prize winner Robert Hofstadter). The experimental setup is simple, a spectrometer to measure momentum ($\approx E$ for relativistic electrons)

The energy/angle relationship of Eq. 4 is seen in the data (below left). The deviation from Mott scattering due to magnetic moment term starts to be visible at these probe energies (below right).

![Fig. 2. Arrangement of parts in experiments on electron scattering from a gas target.](image)
1.3 The Rosenbluth equation

The generalisation of Eq. 5 to describe scattering off a target with a finite size, is the Rosenbluth scattering formula,

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \left[ \frac{G_E^2(q^2) + \tau G_{\mu}^2(q^2)}{1 + \tau} \cos^2 \frac{\theta}{2} - 2\tau G_{\mu}^2 \sin^2 \frac{\theta}{2} \right] \quad \text{where} \quad \tau = -\frac{q^2}{4M^2}.
\]  

(6)

\(G_E(q^2)\) and \(G_{\mu}(q^2)\) are structure functions to describe the spatial distribution of the proton’s charge and magnetic moment. These quantities are not derived from first principles but rather measured from data.

Recasting the Rosenbluth equation as a ratio with respect to the Mott scattering expectation of a charge \(e\),

\[
\frac{\frac{d\sigma}{d\Omega}}{\left( \frac{d\sigma}{d\Omega} \right)_\text{Mott}} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \cos^2 \frac{\theta}{2}.
\]


\[
G_p = \frac{d\sigma}{d\Omega} \left/ \left( \frac{d\sigma}{d\Omega} \right)_\text{Mott} \right. = \left( \frac{E_3}{E_1} \right) \left[ \frac{G_E^2(q^2) + \tau G_{\mu}^2(q^2)}{1 + \tau} + \frac{-q^2}{2M^2} \left( 2G_{\mu}^2(q^2) \tan^2 \frac{\theta}{2} \right) \right] \left( f_2(q^2) / f_1(q^2) \right).
\]

(7)

Importantly, \(G_p\) is a linear function of \(\tan^2 \frac{\theta}{2}\) for a fixed value of \(\tau\). The form factors \(G_E(q^2)\) and \(G_{\mu}(q^2)\), are thus deduced.

\footnote{Not the same form factors as in the non-relativistic case. These are dependent on the 4-momentum, \(q^2\) and not the three momentum, \((q^2)\).}
In this paper nicely brings together the topics discussed above to measure $G_E(q^2)$ and $G_\mu(q^2)$ from elastic $e^-\rightarrow p^+$ Rosenbluth scattering. A beam of [relativistic] electrons, with momentum up to 1000 MeV/c is fired onto a thin liquid hydrogen target. The intensity of the scattered electrons, and their energy, is measured as a function of the scattering angle ($\theta$) by a small spectrometer able to rotate by $\theta$ around the hydrogen target.

Elastic scattering is ensured selecting only a small range of scattered electron energies, as calculated by Eq. 4,

$$E_3 = \frac{E_1 M}{M + E_1(1 - \cos \theta)}$$

which assumed elastic scattering. In the plot (right) $E_1 = 401$ MeV and the experiment is rotated to count at 75°.

$$E_3 = \frac{401 \times 938}{938 + 401(1 - \cos 75°)} = 305 \text{ MeV}.$$  

The data shows the collected data to peak, with little background around this value. Conclusion: the data is elastic scatter and Rosenbluth equation is relevant.
The cross section is measured for a variety of angles and incident energies. For each point $E_3, q^2$ is calculated by Eq. 4,

$$E_3 = \frac{E_1 M}{M + E_1 (1 - \cos \theta)}$$

$$q^2 = \frac{M E_1^2 (1 - \cos \theta)}{M + E_1 (1 - \cos \theta)}$$

And with this input, the cross section measurements are used to plot $G_p$ as a function of $\tan^2 \frac{\theta}{2}$. The slot and incidents (for each fixed value of $q^2$) can then be extracted to calculate the structure functions, $G_E(q^2)$ and $G_\mu(q^2)$ as a function of $q^2$.

The plot shows $F_{\text{ch}} = G_E(q^2)$ and $F_{\text{mag}} = G_\mu(q^2)/2.79$. The proton magnetic moment, $\mu_p = 2.79 \mu_{\text{Dirac}}$.

This work shows the structure of the proton is such that the electrostatics charge and its magnetic moment are distributed spatially in the same way.

### 2 Deep inelastic scattering

With sufficiently high-energy electrons ($E_1 > 1$ GeV), the proton breaks up forming a shower of hadrons. By baryon-number conservation there at least one baryon in this shower and as the proton is the lowest mass baryon, the invariant mass of this shower must be greater than that of the proton, $p_4^2 > M^2$.

#### 2.1 DIS kinematics

We introduce four useful Lorentz invariant variables,

$$Q^2 = -q^2$$

$$x = \frac{Q^2}{2p_2 \cdot q}$$

$$y = \frac{p_2 \cdot q}{M}$$

$$y = \frac{p_2 \cdot q}{p_2 \cdot p_1}$$

Let’s consider the frame where the target proton is at rest, $p_2 = (M, 0, 0, 0)$,

$$p_4^2 = (p_2 + q)^2 = M^2 - Q^2 + 2p_2 \cdot q$$

$$p_4^2 - M^2 \geq 0$$

so, $0 < x < 1$. 

![Fig. 5. The experimental values found for the charge and magnetic form factors of the proton as a function of $q^2$.](image)
In such a frame, \( p_1 = (E_1, 0, 0, E_1) \) and \( q = (E_1 - E_3, p_1 - p_3) \), hence \( y \) is the fractional energy loss of the probe,

\[
\frac{p_1 \cdot q = M(E_1 - E_3)}{p_1 \cdot p_1 = M^2 E_1^2} \quad \text{so} \quad y = \frac{E_1 - E_3}{E_1} \quad 0 < y < 1.
\]

Similarly, \( \nu = E_1 - E_3 \) is the energy loss, in the frame where the proton target is at rest.

The CoM energy-squared, \( s = (p_1 + p_2)^2 = M^2 + 2p_1 \cdot p_2 \), neglecting the electron mass. Thus,

\[
p_1 \cdot p_2 = \frac{1}{2}(s - M^2), \quad \text{so} \quad y = \frac{2M\nu}{s - M^2}.
\]

And as \( x = \frac{Q^2}{2M\nu} \), \( Q^2 = (s - M^2)xy \).

The \( s \) is predefined by the probe and target 4-momenta, \( M \) is fixed, so the momentum transfer of the collision is dependent on two kinematic observables, \( Q^2 = f(x, y) \). Or in terms of the experimental quantities, \( Q^2 = f(E_3, \theta) \). This compares with the elastic case (where \( x \equiv 1 \)) which is dependent on just one kinematic observable; Eq. 4 is of the form \( q^2 = f(\theta) \).

Phys. Lett. B 28(2) (1968) “Electroproduction of pions near the \( \Delta(1236) \) isobar and the form factor \( G_M^*(q^2) \)”

In this example, the differential cross section for electrons, \( p_1 = (E_1, 0, 0, E_1) \) scattering off protons, \( p_2 = (M, 0, 0, 0) \) is measured as a function of scattered electron energy and angle. In this setup, \( s - M^2 = 2ME_1 \), so,

\[
Q^2 = 2ME_1 xy = 4E_1E_3 \sin^2(\theta/2).
\]

Similarly, the invariant mass squared of the proton remnants, \( p_3^2 \) (labelled \( W \) in the plot below), is fully determined,

\[
W = p_3^2 = (p_2 + q)^2 = M^2 + 2p_2 \cdot q - Q^2 = M^2 + 2ME_1y(1 - x) \quad \text{in terms of } x, y
\]

\[
= M^2 + 2ME_1 - 2[E_1(1 - \cos \theta) + M]E_3 \quad \text{in terms of } \theta, E_3
\]

For known \( E_1 = 4.879 \) GeV and specified angle, \( \theta \), the scattered electron energy, \( E_3 \) gives the remnant mass \( W \) is directly.

### 2.2 Structure functions

Due to the dependence on two kinematic variables, the differential cross section of deep inelastic scattering is written as,

\[
\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ \frac{1}{x} \left( 1 - y - \frac{M^2x^2y^2}{Q^2} \right) F_2(x, Q^2) + y^2 F_1(x, Q^2) \right].
\]

This is the Lorentz invariant equivalent of the Rosenbluth formula for high \( Q^2 \) and \( p_3^2 \neq M^2 \). In experiments where the proton are initially at rest, we can recast Eq. 8 in terms of the directly measurable quantities, \( E_3, \Omega \):

\[
\frac{d^2\sigma}{dE_3 d\Omega} = \frac{dx}{dE_3} \cdot \frac{dQ^2}{d\Omega} \cdot \frac{(d\Omega)^{-1}}{d\Omega} \cdot \frac{d^2\sigma}{dx dQ^2}
\]

as

\[
\frac{2Mx^2}{Q^2} \quad \frac{2E_1E_3 \sin \theta}{Q^2} \quad (2\pi \sin \theta)^{-1}
\]

from \( x = \frac{Q^2}{2M(E_1 - E_3)} \), \( Q^2 = 4E_1E_3 \sin^2 \frac{\theta}{2} \), \( \Omega = 2\pi(1 - \cos \theta) \).
Figure 1: Cross section for electron-proton interaction as a function of scattered probe energy for initial electron energy $E_1 = 4.879$ GeV and scattering angle $\theta = 10^\circ$. The elastic scattering peak is at high $E_3 = 4.522$ GeV at the right of the plot (value calculable from Eq. 4). Just to the left is the resonance enhancement where $J = I = \frac{3}{2}$ $(1232)$ baryons are produced. Left of that, as more energy is transferred into the target, the proton breaks up into several particles; this is DIS though some resonance enhancement is still seen.

\[
\frac{d^2\sigma}{dE_3 d\Omega} = \frac{2Mx^2 E_1 E_3}{\pi Q^2} \frac{d^2\sigma}{dx dQ^2}.
\]

With careful but trivial manipulation, the following simplifications are achieved,

\[
\frac{2Mx^2 y^2}{(1-y)Q^2} = \frac{2}{M} \sin^2 \theta \frac{\theta}{2}
\]

so that,

\[
\frac{d^2\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \left[ \frac{1}{x} \left( 1 - y - \frac{M^2 x^2 y^2}{Q^2} \right) F_2(x, Q^2) + y^2 F_1(x, Q^2) \right].
\]

An experiment to measure the structure functions can proceed by forming a ratio of cross section with an expectation of such for Mott Scattering for probe energy $E_1$. Like with the elastic case, this forming a linear relationship with dependent on $\tan^2(\theta/2)$. The structure functions are extracted from the coefficients and the measurement of $E_3$.

\[
\frac{d^2\sigma}{dE_3 d\Omega} \Bigg| \frac{d\sigma}{dQ}_{\text{Mott}} = \frac{1}{E_1 - E_3} F_2(x, Q^2) + \frac{2}{M} F_1(x, Q^2) \tan^2 \frac{\theta}{2}.
\]

Many measurements of cross section and are needed at different scattering angles and $E_3$ energy bins for the linear relationship to be examined and the structure functions extracted.
2.3 Bjorken scaling and the Callan-Gross relation

Such measurements of $F_1(x, Q^2)$ and $F_2(x, Q^2)$ are usually shown versus $Q^2$ in subsets of fixed $x$ as in Fig. 2. The structure functions show a remarkable property: that, for a given $x$ the structure function, and hence scattering cross section, is independent of $Q^2$. From the definition of $x$, for constant $x$, $(E_1 - E_3) \propto Q^2$, meaning that this result says that no matter how much energy is transferred into proton from the electron, the probability to scatter is the same. This is known as Bjorken scaling and was the first evidence that the high-energy electron was scattering off point-like objects in the proton.

It is also insightful to alter Eq. 10 to make a comparison with the equivalent ratio from the elastic case,

$$\frac{d^2\sigma}{dE_3d\Omega} \bigg/ \left( \frac{d^2\sigma}{dE_3d\Omega} \right)_{\text{Mott}} = \frac{F_2(x, Q^2)}{\nu} \left[ 1 + \frac{2\nu}{M} \frac{F_1(x, Q^2)}{F_2(x, Q^2)} \tan^2 \frac{\theta}{2} \right],$$

$$\left[ 1 + \frac{2Q^2}{2xM^2} \frac{F_1(x, Q^2)}{F_2(x, Q^2)} \tan^2 \frac{\theta}{2} \right] \text{ using the definition } x = \frac{Q^2}{2M\nu},$$

$$\left[ 1 + \frac{Q^2x}{\mu^2} \frac{F_1(x, Q^2)}{F_2(x, Q^2)} \tan^2 \frac{\theta}{2} \right] \text{ introducing a fractional mass, } \mu = xM.$$

$$\left[ 1 + \frac{Q^2}{2\mu^2} \tan^2 \frac{\theta}{2} \right] \text{ postulating a relationship, } \frac{F_1}{F_2} = \frac{1}{2x},$$

which is the Callan-Gross relation. This last formula should be compared with Eq. 5,

$$\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \left[ 1 - \frac{q^2}{2M^2} \tan^2 \frac{\theta}{2} \right],$$

which described the elastic scattering off the spin-$\frac{1}{2}$, point-particle proton of mass $M$. In that equation, the second term arises from the “magnetic” spin-spin interaction of electron (spin-$\frac{1}{2}$) and proton (spin-$\frac{1}{2}$). This relation between the two
structure functions,

\[
\frac{2x F_1(x, Q^2)}{F_2(x, Q^2)} = 1,
\]
bears out in the data and thus strongly indicates the elastic scattering off point-like, spin-\(\frac{1}{2}\) particles of mass \(\mu = xM\). This was the first evidence that quarks are real constituents, rather than just a mathematical convenience in describing the symmetries in hadron multiplets.

Figure 3: Data supporting the Callan-Gross relation for a large range of \(x\) and \(Q^2\).

3 The quark-parton model

This section develops from the concept of scattering off spin-\(\frac{1}{2}\) point particles to an understanding of the momentum distributions of the various parts of the proton. We consider the “infinite momentum limit” (IML) where \(Q^2 \gg M^2\) so mass terms can be neglected.

So, in the “quark parton model”, we imagine the electron is scattered off a quark that is carrying a fraction \(a\) of the proton’s four-momentum, \(p_q = ap_2\), \(0 < a < 1\). The scattered quark has four-momentum (squared is),

\[
(p_q')^2 = a^2p_2^2 + q^2
\]

so

\[
(p_q')^2 = a^2p_2^2 + q^2 + 2a \cdot 2 \cdot q \quad \text{neglecting masses.}
\]

Thus

\[
a = \frac{Q^2}{2p_2 \cdot q}.
\]

This justifies the earlier assertion that Bjorken \(x\) is the four-momentum fraction (in the IML).
A quick comparison of the Lorentz Invariant variables for the quark and proton is:

\[
\begin{align*}
\text{proton:} & \quad & \text{quark:} \\
x &= \frac{Q^2}{2p_2 \cdot q} & x_q = 1 \quad \text{(assume a point particle)} \\
y &= \frac{p_2 \cdot q}{p_2 \cdot p_1} & y_q = \frac{xp_2 \cdot q}{xp_2 \cdot p_1} = y \\
s = (p_1 + p_2)^2 \approx 2p_1 \cdot p_2 & \quad s_q = xp_1 \cdot p_2 = xs
\end{align*}
\]

3.1 The electron-quark scattering

Consider a high-energy electron-quark collision in the CoM frame such that each fermion has \(|p_1| = E = \sqrt{s}/2\).

\[
p_1 = (E, 0, 0, E) \quad p_2 = (E, 0, 0, -E) \quad p_3 = (E, E \sin \theta', 0, E \cos \theta') \quad p_4 = (E, -E \sin \theta', 0, -E \cos \theta')
\]

where \(\theta'\) is the CoM scattering angle. From Eq. 1, we take the matrix element for two fermions where the masses are neglected but remembering a quark has fractional charge, \(e_q\),

\[
|\langle M_{fi} \rangle|^2 = 8e_q^2 \alpha e^4 \frac{(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4)}{(p_1 - p_3)^4} = 2e_q^2 \alpha e^4 \frac{4 + (1 + \cos \theta')^2}{(1 - \cos \theta')^2}.
\]

The cross section in the CoM frame is taken from the Basics lectures, so the \(eq \rightarrow eq\) differential cross section is,

\[
\sigma = \int \frac{|M_{fi}|^2}{64 \pi^2 s_q} dQ = \frac{d\sigma}{dQ} = \frac{e_q^2 \alpha e^4}{32 \pi^2 s_q} \frac{4 + (1 + \cos \theta')^2}{(1 - \cos \theta')^2}.
\]

Plugging in \(g_e = \sqrt{4\pi\alpha}\), this expression is recast in a Lorentz invariant form using,

\[
q^2 \approx -2p_1 \cdot p_3 = -2E^2(1 - \cos \theta') = -s_q(1 - \cos \theta')/2 \quad \text{or} \quad \frac{-q^2}{s_q} = \sin^2 \frac{\theta'}{2}.
\]

\[
\frac{d\sigma}{dq^2} = \left(\frac{dq^2}{d\theta}\right)^{-1} \cdot \frac{d\alpha}{d\theta} \cdot \frac{d\sigma}{dQ} = \frac{-2}{s_q \sin \theta'} \frac{2\pi \sin \theta'}{16 \pi^2 \alpha^2 e_q^2} \frac{4(1 + \frac{1}{2}(1 + \cos \theta')^2)}{32 \pi^2 s_q} \frac{4q^2}{s_q^2} = -\frac{2\pi\alpha^2 e_q^2}{q^4} \left(1 + \frac{1}{4}(1 + \cos \theta')^2\right)
\]

Then, \(\frac{1}{4}(1 + \cos \theta')^2 = \frac{1}{4} \left(1 - \cos \theta'\right)^2 + 4 \cos \theta' = \left(1 + \frac{\theta'}{2}\right)^2\) because, \(\cos \theta' = 1 + \frac{2\theta'}{\pi}\),

\[
\frac{d\sigma}{dq^2} = -\frac{2\pi\alpha^2 e_q^2}{q^4} \left(1 + \left(1 + \frac{q^2}{s_q}\right)^2\right) \quad (11)
\]

The last step is to use the invariant relation, \(Q^2 = (s - M^2)_{xy}\), applied to the quark: \(M = 0, x \equiv 1\). Thus, \(y = \frac{-q^2}{s_q}\) so,

\[
\frac{d\sigma}{dq^2} = -\frac{4\pi\alpha^2 e_q^2}{q^4} \left|1 - y + \frac{y^2}{2}\right| \quad (12)
\]
3.2 Relating the QPM to the structure functions

Thinking of the proton again, Eq. 12 is the valid for the differential cross section (vs. $Q^2 = -q^2$) for a quark with given $x$, 
\[
\left( \frac{d\sigma}{dQ^2} \right)_x = \frac{4\pi\alpha^2 e^2_q}{Q^4} \left[ (1 - y) + y^2 \right].
\]
Then we sum over all quarks in the proton with momentum fraction $x$ and integrate over all $x$,
\[
\frac{d\sigma}{dQ^2} = \int_x \left( \frac{d\sigma}{dQ^2} \right)_x q^p(x) \ dx,
\]
where we introduce a **parton distribution function** $q^p(x)$, which is the number density of partons of type $q$ in the proton ($p$) with momentum fraction $x$. Note that this is a function of $x$, not $Q^2$ because the parton density is a property of the proton, not the scattering process. Thus the double-differential cross section for the whole proton, in the quark-parton model becomes,
\[
\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ (1 - y) + y^2 \right] \times \sum_q e^2_q q^p(x),
\]
which we compare to Eq. 8 (the expression with structure functions from deep inelastic scattering) with $M^2 y^2 \ll Q^2$:
\[
\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ \frac{1}{x} (1 - y) F_2(x, Q^2) + y^2 F_1(x, Q^2) \right],
\]
to arrive at a fundamental definition of the structure functions in the quark-parton model for high energy collisions,
\[
2x F_1(x, Q^2) \xrightarrow{\text{Callan-Gross}} F_2(x, Q^2) \xrightarrow{\text{Bjorken scaling}} F_2(x) = x \sum_q e^2_q q^p(x)
\]
(14)
The quark-parton model of nucleons can thus be tested against experimental measurements of structure functions.

- **$e \rightarrow p$ scattering:**
  \[
  F_2^p(x) = x \left[ \frac{4}{9} u^p(x) + \frac{1}{9} d^p(x) \right],
  \]
  where $u^p(x)$ and $d^p(x)$ are the parton distribution functions for the up and down valence quarks.

- **$e \rightarrow n$ scattering:**
  \[
  F_2^n(x) = x \left[ \frac{4}{9} u^n(x) + \frac{1}{9} d^n(x) \right],
  \]
  but if we apply isospin symmetry and say the proton is identical to the neutron except for the $u \leftrightarrow d$ switch, we can say the up quarks will have the same momentum distribution in the proton as down quarks have in the neutron. So,
  \[
  u^n(x) = d^p(x) \quad \text{and} \quad d^n(x) = u^p(x).
  \]

Integrating over all $x$,

- **proton:**
  \[
  \int_0^1 F_2^p(x) \ dx = \int_0^1 x \left[ \frac{4}{9} u^p(x) + \frac{1}{9} d^p(x) \right] \ dx = \frac{4}{9} f_u + \frac{1}{9} f_d,
  \]
  (15)

- **neutron:**
  \[
  \int_0^1 F_2^n(x) \ dx = \int_0^1 x \left[ \frac{4}{9} d^n(x) + \frac{1}{9} u^n(x) \right] \ dx = \frac{1}{9} f_u + \frac{4}{9} f_d.
  \]
  (16)
Here, \( f_{u,d} \) are the fractions of the proton momentum carried by the up and down valence quarks.

To recap: from scattering cross section measurements (vs. scattering angle, \( \theta \) and scattered electron energy, \( E_3 \)), the structure function, \( F_2(x, Q^2) \) is measured across a wide range of \( x \) (\( x, y, v, Q^2 \) calculated from \( M, E_1, E_3, \theta \)). A plot of \( F_2(x, Q^2) \) vs. \( x \) is shown on the right, for low \( Q^2 \) (\( Q^2 \ll m_W^2 \)). From the quark parton model, the integral of this distribution is the fraction of proton four-momentum carried by the charged partons (the quarks).

\[
\text{proton: } \int_0^1 F_2^p(x) \, dx \approx 0.18
\]

From similar measurements with a deuterium target the same estimate is made for the neutron, once the protons contribution to the scattering is subtracted,

\[
\text{neutron: } \int_0^1 F_2^n(x) \, dx \approx 0.12.
\]

Plugging these results into Eq. 15 and 16 and important result is achieved,

\[
f_u \approx 0.36 \quad f_d \approx 0.18 \quad \Rightarrow \quad f_u + f_d \approx 0.52 \quad (17)
\]

That is, only half of the proton’s four momentum is carried by the quarks.

### 3.3 Sea quarks and total PDFs

The proton is a complicated object where gluons exchange with between the valence quarks as well as giving rise to quantum fluctuations of quark currents. These spontaneous \( q\bar{q} \) pairs can interact with the electron probe so count towards the momentum fraction calculated above. If we separate out the sea quarks explicitly, and making an assumption of isospin symmetry for the parton distributions,

\[
u(x) = \bar{u}(x) = d(x) = \bar{d}(x) = S(x),
\]

the parton model becomes,

\[
\text{proton: } \int_0^1 F_2^p(x) \, dx = \int_0^1 x \left[ \frac{4}{9} u^p(x) + \frac{1}{9} d^p(x) + \frac{10}{9} S(x) \right] \, dx
\]

\[
\text{neutron: } \int_0^1 F_2^n(x) \, dx = \int_0^1 x \left[ \frac{4}{9} d^n(x) + \frac{1}{9} u^n(x) + \frac{10}{9} S(x) \right] \, dx
\]
Where the subscript $v$ explicitly identifies the valance quarks. The sea quark pairs are produced by a massless gluon propagator (with Feynman propagator term $\propto 1/q^2$) means that the contribution to the nucleon form factors from sea quarks should be dominant at low-$x$. This means we expect the ratio of proton to neutron form factors to approach unity at small Bjorken-$x$ as seen in this plot where,

$$\frac{F_2^u(x)}{F_2^p(x)} \rightarrow 1 \text{ as } x \rightarrow 0,$$

which is evidence for the dominance of sea quark interactions in the electron scattering at low-$x$.

PDF fits

The modern representation of the proton structure shows the structure function versus Bjorken-$x$ for each component of the proton. One such summary is shown here; it is derived from the HERAPDF model fitted to all deep inelastic scattering data. The red band shows the fit uncertainty.

As the influence of the sea quarks comes in a small $x$, the PDFs are shown along on a logarithmic scale. The integral of each component PDF is, nevertheless, the total fraction of the four-momentum carried by that parton.

From the plot we see that the integral of the up quark PDF about twice that of the down quark and has a marginally harder momentum spectrum.

The sea quark PDF rises rapidly at low $x$. Their PDFs are summed together $\Sigma = 2(\bar{u}(x) + \bar{d}(x) + \bar{s}(x) + \bar{c}(x))$. The result is graphically scaled by $1/20$ to fit on the plot.

4 Application of PDFs to proton-antiproton cross sections

The understanding of the proton structure is used in the calculation of cross sections at hadron colliders. In the case of the Tevatron, which collided protons and antiprotons at $\sqrt{s} = 2$ TeV until 2009, two important processes are studied in detail. The colliding quarks will have momentum share, $x_+$ and $x_-$ in the proton and antiproton respectively. Where the quarks are of the same flavour, a neutral boson exchange with the final state occurs. This is most readily studied with dimuon production as there are no leptons in the initial state. Where the quarks are of different flavour, $W^\pm$ boson production occurs.
The CoM-energy-squared, \( (P_p + P_{\bar{p}})^2 \approx 2P_p \cdot P_{\bar{p}} = s \), and neglecting any transverse component, the two colliding quarks have 4-momentum,
\[
P_+ = \frac{\sqrt{s}}{2}(x_+, 0, 0, x_+) \quad P_- = \frac{\sqrt{s}}{2}(x_-, 0, 0, -x_-)
\]
The energy and [longitudinal] momentum of the quark-antiquark collision is,
\[
E_{q\bar{q}} = \frac{\sqrt{s}}{2}(x_+ + x_-), \quad P_{q\bar{q}} = \frac{\sqrt{s}}{2}(x_+ - x_-).
\]
Thus the invariant mass, \( M_{q\bar{q}}^2 = x_+ x_- s \), and the rapidity, \( y \),
\[
y = \frac{1}{2} \ln \left( \frac{E + P_z}{E - P_z} \right) = \frac{1}{2} \ln \left( \frac{x_+}{x_-} \right) \quad \text{or} \quad \frac{x_+}{x_-} = e^{2y}.
\]
a simple rearrangement relates the rapidity to Bjorken-\( x \),
\[
\frac{(x_+)^2 s}{M_{q\bar{q}}^2} = e^{2y} \Rightarrow x_+ = \sqrt{e^{2y}}. \quad \frac{M_{q\bar{q}}^2}{(x_-)^2 s} = e^{2y} \Rightarrow x_- = \sqrt{e^{-2y}}.
\]
where a variable \( \tau = \frac{M_{q\bar{q}}^2}{s} \) is introduced; \( \sqrt{\tau} \) is the fraction of the total CoM energy in the \( q\bar{q} \) collision.

The CDF detector at the Tevatron instruments the central region of the \( p\bar{p} \) collisions with rapidity range \(-2 < y < 2\). From the above relations, the range of assessable Bjorken-\( x \) that the CDF detector can study is calculated.

<table>
<thead>
<tr>
<th>( y )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_+ )</td>
<td>0.005</td>
<td>0.015</td>
<td>0.040</td>
<td>0.109</td>
<td>0.297</td>
</tr>
<tr>
<td>( x_- )</td>
<td>0.297</td>
<td>0.109</td>
<td>0.040</td>
<td>0.015</td>
<td>0.005</td>
</tr>
</tbody>
</table>

if one considers \( W^\pm \) production: \( M_{q\bar{q}} = m_W = 80.35 \text{ GeV}/c^2 \).

Example: \( W^\pm \) production asymmetry

As there is a direct relationship between Bjorken-\( x \) and rapidity, an expression for differential cross section can be,
\[
\frac{d\sigma}{dy}(W^+) \propto \left[ u_p(x_+) \bar{d}_{\bar{p}}(x_-) + \bar{u}_p(x_+) u_{\bar{p}}(x_-) \right],
\]
\[
\frac{d\sigma}{dy}(W^-) \propto \left[ d_p(x_+) \bar{u}_{\bar{p}}(x_-) + \bar{d}_p(x_+) u_{\bar{p}}(x_-) \right].
\]
The first term in each expression is dominated by the valance quarks at high \( x \) but the sea quarks must be included at low \( x \). The second term describes both partons coming from sea. As one of the partons always has high \( x \), this term can be neglected and we continue with first term only.
In this demonstration, we can simple read off \( u_+(x) \), \( d_-(x) \) and \( \Sigma(x) \) for each \( x_+ \) and \( x_- \), remembering to divide the \( \Sigma(x) \) by eight to get the individual quark or antiquark contribution. The final line (marked with an *) is proportional to the differential cross section. The equivalent PDF numbers for \( W^- \) are the same, but reverse across the row, from the symmetry of \( p\bar{p} \rightarrow W^\pm X \) production. These numbers are sketched below and an asymmetry and from which an asymmetry can be seen. This asymmetry, studied at CDF, is shown in the plot.

<table>
<thead>
<tr>
<th>( y )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_+ )</td>
<td>0.005</td>
<td>0.015</td>
<td>0.040</td>
<td>0.109</td>
<td>0.297</td>
</tr>
<tr>
<td>( xu_+ ) (valance)</td>
<td>0.16</td>
<td>0.27</td>
<td>0.41</td>
<td>0.54</td>
<td>0.35</td>
</tr>
<tr>
<td>( x\Sigma/8 ) (sea)</td>
<td>0.37</td>
<td>0.23</td>
<td>0.10</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>( u_+(x_+) = x(u_+ + \Sigma/8) )</td>
<td>0.54</td>
<td>0.50</td>
<td>0.51</td>
<td>0.57</td>
<td>0.35</td>
</tr>
<tr>
<td>( x_- )</td>
<td>0.297</td>
<td>0.109</td>
<td>0.040</td>
<td>0.015</td>
<td>0.005</td>
</tr>
<tr>
<td>( xd_- ) (valance)</td>
<td>0.12</td>
<td>0.27</td>
<td>0.25</td>
<td>0.15</td>
<td>0.08</td>
</tr>
<tr>
<td>( x\Sigma/8 ) (sea)</td>
<td>0.00</td>
<td>0.01</td>
<td>0.10</td>
<td>0.23</td>
<td>0.37</td>
</tr>
<tr>
<td>( \bar{d} \bar{p} (x_-) = x(d_- + \Sigma/8) )</td>
<td>0.12</td>
<td>0.30</td>
<td>0.35</td>
<td>0.38</td>
<td>0.45</td>
</tr>
<tr>
<td>( W^+: u_+(x_+) \bar{d} \bar{p} (x_-) )</td>
<td>0.64</td>
<td>0.15</td>
<td>0.18</td>
<td>0.21</td>
<td>0.16 *</td>
</tr>
</tbody>
</table>

The backward-forward asymmetry, the humped peaking structure and the fact that the \( W^+ \) distribution peaks in positive rapidity, vice-versa for \( W^- \) production. The blue data point at from \( p\bar{p} \) collisions and shows excellent agreement with the prediction from the PDF fits to deep inelastic scattering data (red line).
Quantum ChromoDynamics proposes that quarks carry one of three colour charges (labelled: red, green, blue) in addition to, and independent of, their fractional electric charge. In reciprocation, antiquarks carry anticolour charge (labelled: red-bar, green-bar, blue-bar). Strong force interactions occur by the exchange of massless electrically-neutral bosons called *gluons*. In contrast to the electromagnetic force carrier (the photon), gluons carry a mix of colour-charges and thus can couple to themselves as well as to quarks. Some examples of colour-conserving strong interactions are,

\[
q(g)q(b)g(g\bar{b})
\]

Unlike the photon, which has infinite range, the strong force effect is confined to within hadrons and cannot be sensed directly. The strong force potential for a \(q\bar{q}\) bound state is modelled with two parts,

\[
V(r) \approx -C \frac{\alpha_s}{r} + \lambda r
\]

where \(C\) is the *colour factor* (see later), \(\alpha_s\) is the strong coupling constant and \(\lambda \sim 1\) GeV/fm. The first term is reminiscent of a Coulomb potential; this is justified by, e.g. the observed quarkonia spectra. The second term is a simple model of the confining potential. As the \(\pi\pi\) invariant mass around 270 MeV/c^2, the gluon field will find it energetically favourable to pair-produce hadrons rather than allow a free quark to escape a bound state (proton radius \(\approx 0.8\) fm).

The three charges of QCD respect the SU(3) symmetry exactly. This is contrast to the broken SU(3) symmetry in the flavour structure of \(u, d, s\) mesons and baryons in the *Basics* lectures. Mathematically, this means that the strong interaction is invariant under rotation in colour space. Or practically speaking, that there is no way of telling which charge any given quark is carrying.

In the *Basics* lectures, Young’s tableaux are used to combine three SU(3) particles and a SU(3) particle-antiparticle pair,

**Baryons** :

\[
\begin{array}{c}
\begin{array}{c}
\Box \otimes \Box \otimes \Box = \left( \Box \oplus \Box \right) \otimes \Box \\
= \Box \oplus \Box \oplus \Box \oplus \Box \\
10 \oplus 8 \oplus 8 \oplus 1
\end{array}
\end{array}
\]

**Mesons** :

\[
\begin{array}{c}
\begin{array}{c}
\Box \otimes \Box = \Box \oplus \Box \\
\bar{3} \otimes 3 = 1 \oplus \bar{8}
\end{array}
\end{array}
\]

In each case a colour singlet occurs which defines the colour wavefunctions,

- **Baryons**: \(\frac{1}{\sqrt{6}} (rgb - rgb + gbr + grb + bgr - bgr)\) \\
  Antisymmetric wavefunction
- **Mesons**: \(\frac{1}{\sqrt{8}} (r\bar{r} + g\bar{g} + b\bar{b})\) \\
  Symmetric wavefunction

which are the same for all mesons and baryons. Note that other colour-neutral combinations, e.g. \(\frac{1}{\sqrt{8}} (r\bar{r} - g\bar{g})\), which one might identify as the SU(3) colour analogy of the \(\pi^0\), is not colourless as it is part of the octet (like the \(\pi^0\)). The singlet states colourless in the sense that when applying either raising or lowering operators, the wavefunction vanishes.
5.1 Gluons

An exchange of gluons will rotate the colour state. The basis generators of SU(3) rotations, taken from the Basics lectures are the Gell-Mann matrices. As expected for a SU(n) group there are \((n^2 - 1) = 8\) traceless, orthogonal matrices defined as \(T_i = \frac{1}{2} \lambda_i\) where \(\lambda_i\) are,

\[
\lambda_1 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_2 = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_4 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
\lambda_5 = \begin{pmatrix}
0 & 0 & -i \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_6 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\lambda_7 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix}
\]

Each QCD exchange is a colour-conserving linear superposition of these eight, orthogonal gluons.

5.2 Comparison of the electron-quark (QED) and quark-quark (QCD) interaction

The lowest-order QCD matrix element is built by analogy to the lowest-order QED interaction in the derivation of Eq. 1. The modification for QCD takes into account the three possible colour charges of each quark \((i = 1, 2, 3)\) and the eight possible gluons that connect them \((a, b = 1 : 8)\).

\[
\begin{align*}
\text{QED} & \\
\text{propagator} & = -ig_{\mu\nu}/q^2 \\
\text{vertex factor} & = ig_\gamma^\mu \\
\text{coupling constant} & = g_e = \sqrt{4\pi\alpha}
\end{align*}
\]

\[
\begin{align*}
\text{QCD} & \\
\text{propagator} & = -ig_{ab}/q^2 \\
\text{vertex factor} & = ig_\gamma^\mu \lambda_{abji}/2 \\
\text{coupling constant} & = g_s = \sqrt{4\pi\alpha_s}
\end{align*}
\]

The modifications with respect to the QED matrix element are highlighted.

- The delta function in the propagator, \(\delta^{ab}\) requires the gluon charge at each vertex to be the same.
- The Gell-Mann matrices, \(\lambda_{abji}\) in the vertex term defines the probability of gluon \(a\) connecting colour charges \(i\) and \(j\).
- The strength of the QCD coupling is governed by \(\alpha_s\) which is 1-2 orders of magnitude larger than the QED coupling.

The lowest-order QCD quark-quark scattering matrix element is built in the same way as for the QED scattering,

\[
-iM_{ij} = \int \left[ \bar{u}(p_3) i g_\gamma^\mu \lambda_{abji}^\mu u(p_1) \right] -ig_{ab} g_{\mu\nu} \frac{1}{q^2} \left[ \bar{u}(p_4) i g_\gamma^\nu \lambda_{abjk}^\nu u(p_2) \right] \delta^4(p_1 - p_3 - q) d^4q.
\]

\[
M_{ij} = -\frac{\lambda_{abji}^\mu \lambda_{abjk}^\nu}{4} g_s^2 \frac{1}{(p_1 - p_3)^2} \left[ \bar{u}(p_3) \gamma^\mu u(p_1) \right] \left[ \bar{u}(p_4) \gamma_\nu u(p_2) \right],
\]

which is identical to the QED case except for the coupling strength and a colour factor from the Gell-Mann matrices.
5.3 Colour factors

The colour factor is thus defined for quark-quark scattering for $a = \{1 - 8\}$ and $i, j = \{1, 2, 3\},$

$$C = \frac{1}{4} \sum_a \lambda^a_{ij} \lambda^a_{jk} \quad (19)$$

Next we calculate the colour factors for three types of colour exchange from inspection of the Gell-Mann basis matrices,

\[ i = j = k = l \]

\[ \begin{align*}
    r & \rightarrow r & i = k & = j = l & \frac{1}{4} \left( \lambda^1_{11} \lambda^3_{11} + \lambda^8_{11} \lambda^8_{11} \right) &= \frac{1}{4} \left( 1 - \frac{1}{\sqrt{3}} \right) \\
    g & \rightarrow g & r & \rightarrow r & i = k & = j = l & \frac{1}{4} \left( \lambda^3_{12} \lambda^3_{12} + \lambda^8_{12} \lambda^8_{12} \right) &= \frac{1}{4} \left( 1 - \frac{1}{\sqrt{3}} \right) \\
    b & \rightarrow b & g & \rightarrow g & r & \rightarrow r & i = k & = j = l & \frac{1}{4} \left( \lambda^3_{33} \lambda^3_{33} \right) &= \frac{1}{4} \left( \frac{-2}{\sqrt{3}} \right) \\
\end{align*} \]

\[ i = j, k = l \]

\[ \begin{align*}
    r & \rightarrow r & g \rightarrow g & i = j, k = l & \frac{1}{4} \left( \lambda^3_{11} \lambda^3_{11} + \lambda^8_{11} \lambda^8_{11} \right) &= \frac{1}{4} \left( 1 - \frac{1}{\sqrt{3}} \right) \\
    r & \rightarrow r & b \rightarrow b & i = j, k = l & \frac{1}{4} \left( \lambda^8_{11} \lambda^8_{11} \right) &= \frac{1}{4} \left( \frac{1}{\sqrt{3}} \right) \\
    g & \rightarrow g & b \rightarrow b & i = j, k = l & \frac{1}{4} \left( \lambda^3_{33} \lambda^3_{33} \right) &= \frac{1}{4} \left( \frac{-2}{\sqrt{3}} \right) \\
\end{align*} \]

\[ i = l, j = k \]

\[ \begin{align*}
    r & \rightarrow g & r & \rightarrow g & i = l, j = k & \frac{1}{4} \left( \lambda^1_{12} \lambda^1_{21} + \lambda^2_{12} \lambda^2_{21} \right) &= \frac{1}{4} \left( 1 - 1 - i \right) \\
    g & \rightarrow r & b & \rightarrow b & i = l, j = k & \frac{1}{4} \left( \lambda^8_{13} \lambda^8_{31} + \lambda^8_{13} \lambda^8_{31} \right) &= \frac{1}{4} \left( 1 - 1 - i \right) \\
    b & \rightarrow g & b & \rightarrow g & i = l, j = k & \frac{1}{4} \left( \lambda^8_{23} \lambda^8_{32} + \lambda^8_{23} \lambda^8_{32} \right) &= \frac{1}{4} \left( 1 - 1 - i \right) \\
\end{align*} \]

All other combinations will not be colour conserving so the colour factor is zero. A non-colour-conserving example:

\[ C(rg \rightarrow gb) = \frac{1}{4} \sum_a \lambda^a_{rg} \lambda^a_{gb} \quad \frac{1}{4} \left( 1 \cdot 0 - i \cdot 0 + 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 \cdot i + 0 \cdot 0 \right) = 0. \]

For quark-antiquark interactions, the same factors come out of the Gell-Mann matrices, though with labels $k$ and $l$ swapped as the antiquark colour current is traveling backwards in time. The colour factors thus apply to the following colour-anticolour currents moving forward in time.

\[ \begin{align*}
    i & \rightarrow j & C = +\frac{1}{2} : & \quad g \rightarrow g \quad g \rightarrow g \quad b \rightarrow b \quad b \rightarrow b \quad r \rightarrow g \quad g \rightarrow r \\
    l & \rightarrow k & \quad C = -\frac{1}{6} : & \quad g \rightarrow g \quad g \rightarrow g \quad b \rightarrow b \quad b \rightarrow b \quad r \rightarrow r \\
\end{align*} \]

In a meson, the bound $q\bar{q}$ state must be colourless so the second group are not present. The colour factor in the meson potential of Eq. 18, averaged over the three possible colour-anticolour combinations is,

$$C(q\bar{q})_{\text{meson}} = \frac{1}{3} \left( 3 \cdot \frac{1}{3} + 6 \cdot \frac{1}{2} \right) = \frac{4}{3} .$$

For the s-channel, quark-antiquark annihilation process, the same three colour factors are used but $i$ and $j$ label the initial state, $k$ and $l$ the final state. To calculate a cross section we must sum over possible colour exchange amplitudes (squared)
and average over the possible initial colour states of the colliding quarks \((3 \times 3 = 9\) colour combinations\). The colour factors and the number of exchanges they refer to are listed above. So for \(q\bar{q}\) annihilation the total colour factor is,

\[
|C(q\bar{q} \to q\bar{q})|^2 = \frac{1}{9} \left( 3 \left| \frac{1}{3} \right|^2 + 6 \left| -\frac{1}{6} \right|^2 + 6 \left| \frac{1}{2} \right|^2 \right) = \frac{2}{9}.
\]

With these factors calculated, we borrow from \(eq \to eq\) differential cross section, Eq. 12, we postulate a lowest-order differential cross section for \(q\bar{q} \to q\bar{q}\) where the QCD modifications are highlighted,

\[
\frac{d\sigma}{dq^2} = -\frac{2}{9} \frac{4\pi\alpha_s^2}{q^4} \left[ 1 - y + \frac{y^2}{2} \right].
\]  

(20)

In experimental tests of QCD scattering, e.g. \(p\bar{p}\) collisions at the Tevatron, such cross section calculations are repeated for all categories of parton collision \((q\bar{q}, qg, \bar{q}g, gg)\) and to NNLO accuracy; excellent agreement with the data is seen.

### 5.4 Experimental validation of QCD

Since the action of the strong force is confined to within hadrons the experimental proof of QCD must come from results in the data that are predicted by QCD. Here we speak qualitatively about three.

**Three jet event**

In the 1970s, QCD was a considered only a candidate theory of the of the strong force but its prediction of the force-carrying spin-1 particle was unique. A series of experiments examined the production of hadronic jets from high-energy \(e^+e^-\) collisions \(\sqrt{s} \approx 20\) GeV in particular to look for rare three jet events corresponding to the diagram,

Moreover the rate of such should depend on the strong coupling constant. The experiments found,

\[
\frac{N(3\text{-jet})}{N(2\text{-jet})} \approx 0.15 \text{ at } \sqrt{s} \approx 20\text{ GeV}.
\]

Four-jet events are also important as they come from the self-interaction of the gluon as it splits.
Running coupling constant

The QED coupling constant, $\alpha$ is known to strengthen at high $Q^2$ where the distances scales reduce and a probe charge can approach more closely the target charge. At short distances the QED screening that dilutes the bare charge strength, diminishes and the true bare charge becomes more apparent. $\alpha = \frac{1}{137}$ at low energies and rises to $\alpha = \frac{1}{128}$ around 170 GeV. In QCD, because the gluons self-interact, there is screening but there is a more-powerful anti-screening effect. that enhances the colour charge strength at large distances. At progressively higher energies (shorter distance scales) the strength of the strong coupling diminishes. This effect can only arise from a self-interacting field, as required by QCD. The magnitude of $\alpha_s$ as shown on the plot is predicted to be,

$$
\alpha_s(Q^2) = \frac{\alpha_s(\Lambda)}{1 + \alpha_s(\Lambda) \frac{12N_c - 2N_f}{12\pi} \ln \left( \frac{Q^2}{\Lambda^2} \right)},
$$

where $N_c$ and $N_f$ are the number of charges and the number of quark flavours in the theory and $\Lambda \approx 300$ MeV.

Scaling violations

For a photon to resolve structure, its wavelength must be as least as small as that structure. From,

$$
\lambda = \frac{\hbar}{|q|} = \frac{197}{\text{MeV}/c \text{ fm}}
$$

we see that the energy to resolve the proton structure (1 fm) needs $|q| \approx 200$ MeV and the energy to resolve quark structure ($10^{-18} \text{m}$) is $|q| \approx 200$ GeV. The smallest distance scales were probed by the HERA facility that operated near Hamburg in the 1990s and 2000s, colliding $e^- p$ at $\sqrt{s} \approx 300$ GeV. The accumulated $F_2$ structure function data is shown in the plot and these data show Blorken scaling down to momentum fractions $x > 0.05$. However, at lower $x$, increasingly the Bjorken-scaling seems violated, but not in a manner that is consistent with quark structure (rapid decrease in structure with $Q^2$). Rather, the electron probe sees increasing structure at larger $Q^2$ for the smallest $x$!

QCD can explain this as follows: at low $x$ the proton is all gluons and sea quarks flashing into existence for a short period. For this short period, the electron probe can scatter off the charged quark or antiquark. At higher $Q^2$ the distance (and thus time) scales that the electron can probe gets smaller, so the more short-lived $q\bar{q}$ pairs are visible, thus more scattering occurs and the measured structure function rises. The data shows excellent agreement with the QCD prediction.

Understanding of the proton structure at low-$x$ has been critical in predicting and understanding the LHC cross sections. As the LHC is a proton-proton collider, there are no valance anti-quarks to collide. The LHC is entirely reliant on the gluons and sea-quarks for every LHC collision. Studies of the proton structure is critical to the success of the LHC.