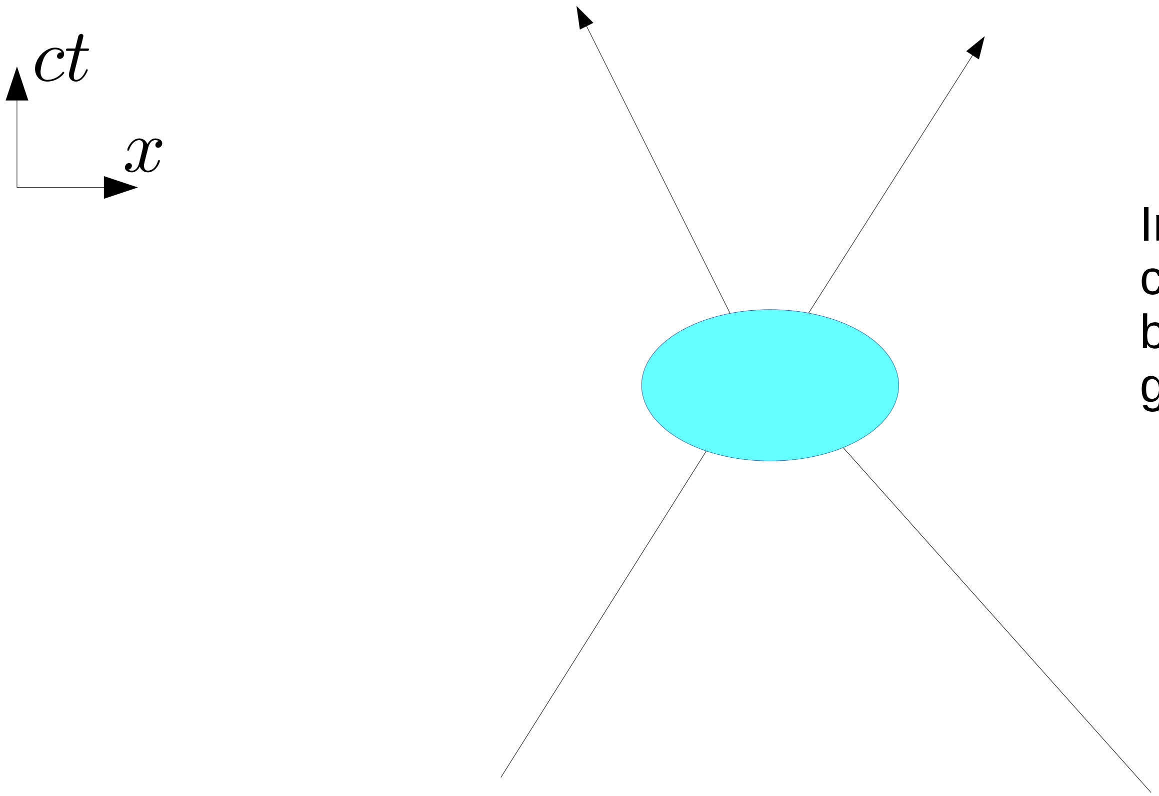
B2
Symmetry and Relativity
Lecture 18

Closer look at collisions

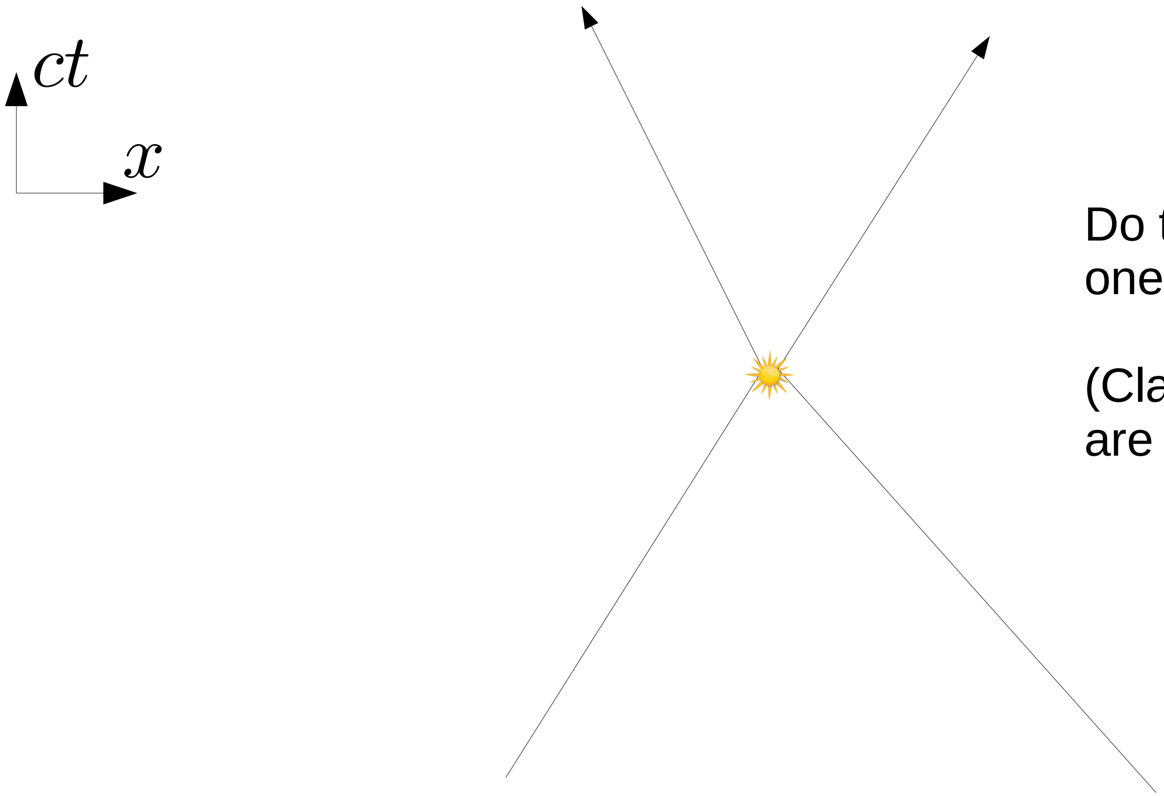
- What's going on during a collision?



In collision problems, we've mostly concerned ourselves with the before and after, ignoring what goes on in between

Closer look at collisions

- What's going on during a collision?

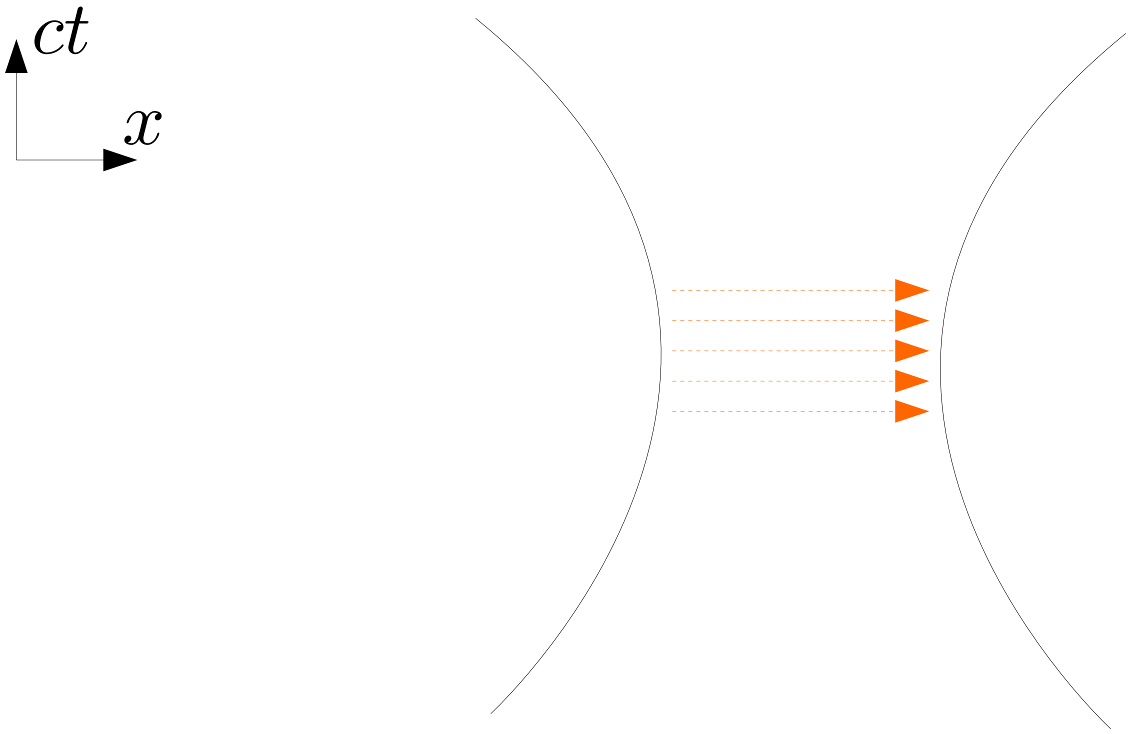


Do the particles come into contact with one another?

(Classical physics likes particles which are point-like, or hard objects)

Closer look at collisions

- Particles exert force on one another (no need for actual contact)

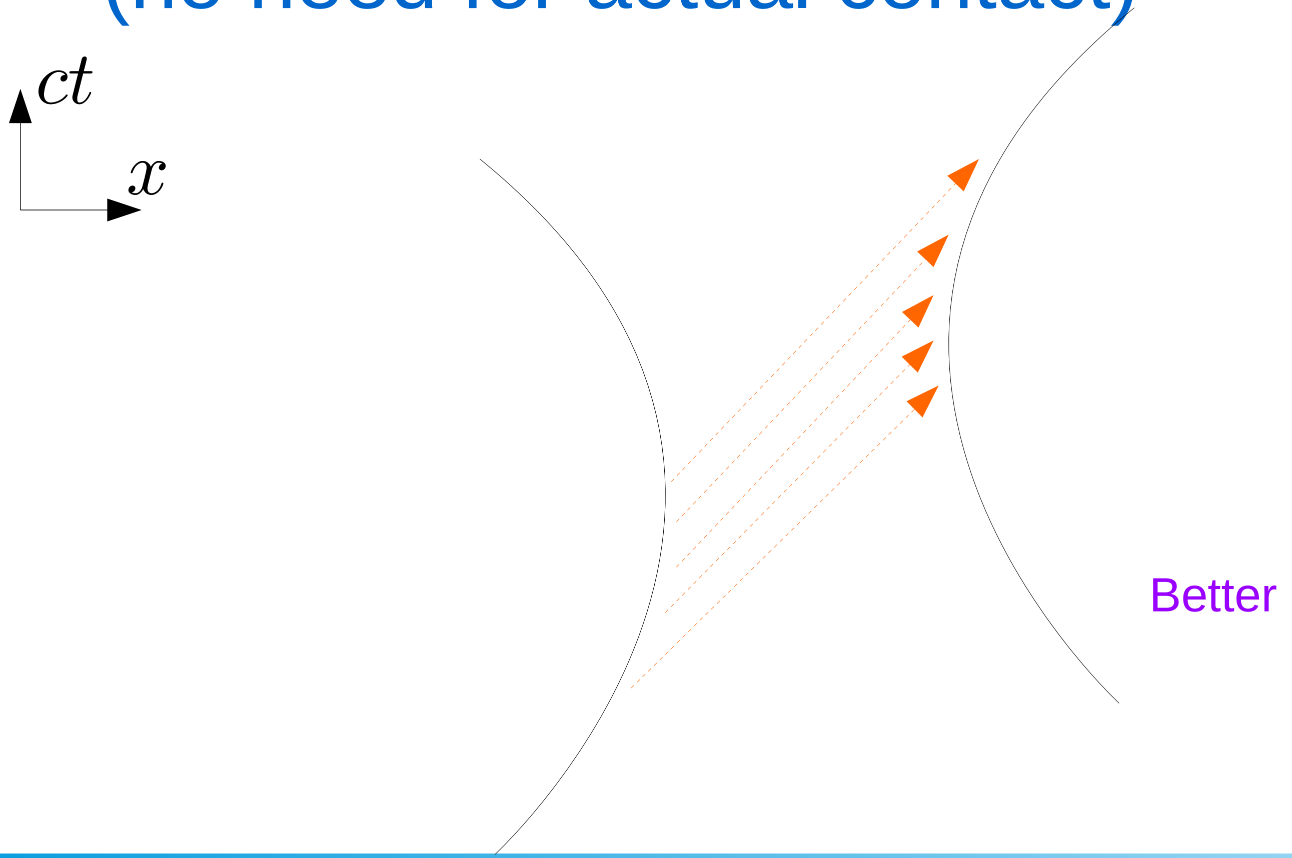


Whoops – forces need to signal from one particle to the other

There's a maximum speed!

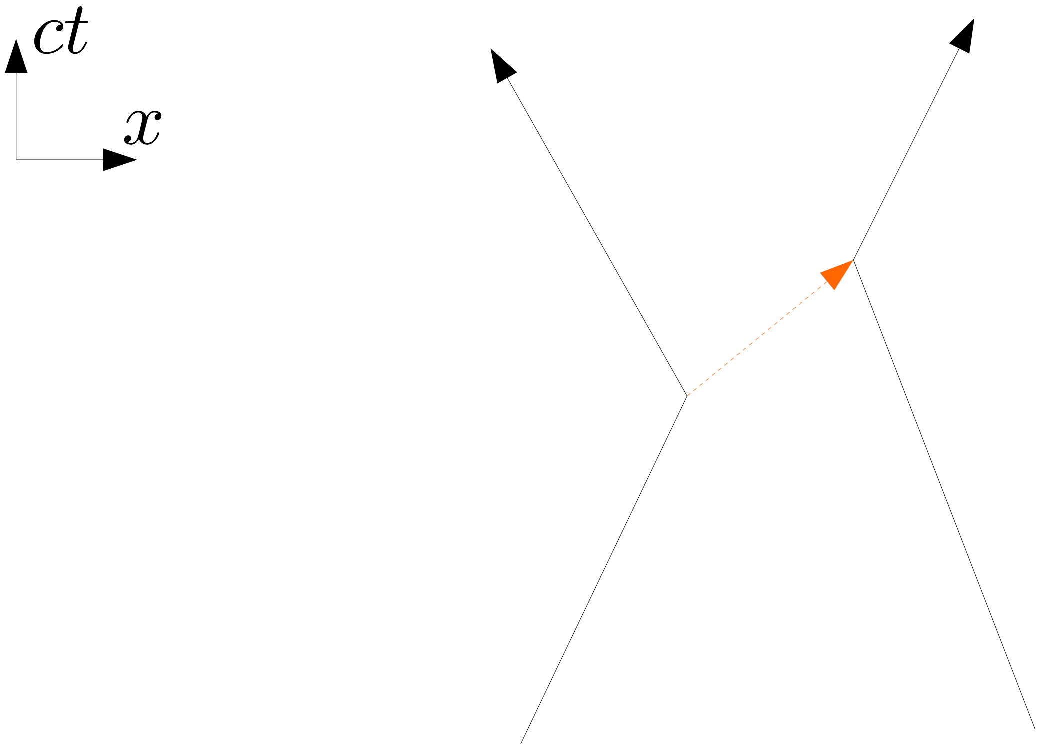
Closer look at collisions

- Particles exert force on one another (no need for actual contact)



Closer look at collisions

- Individual “signal”
  (like particle decays and formations)



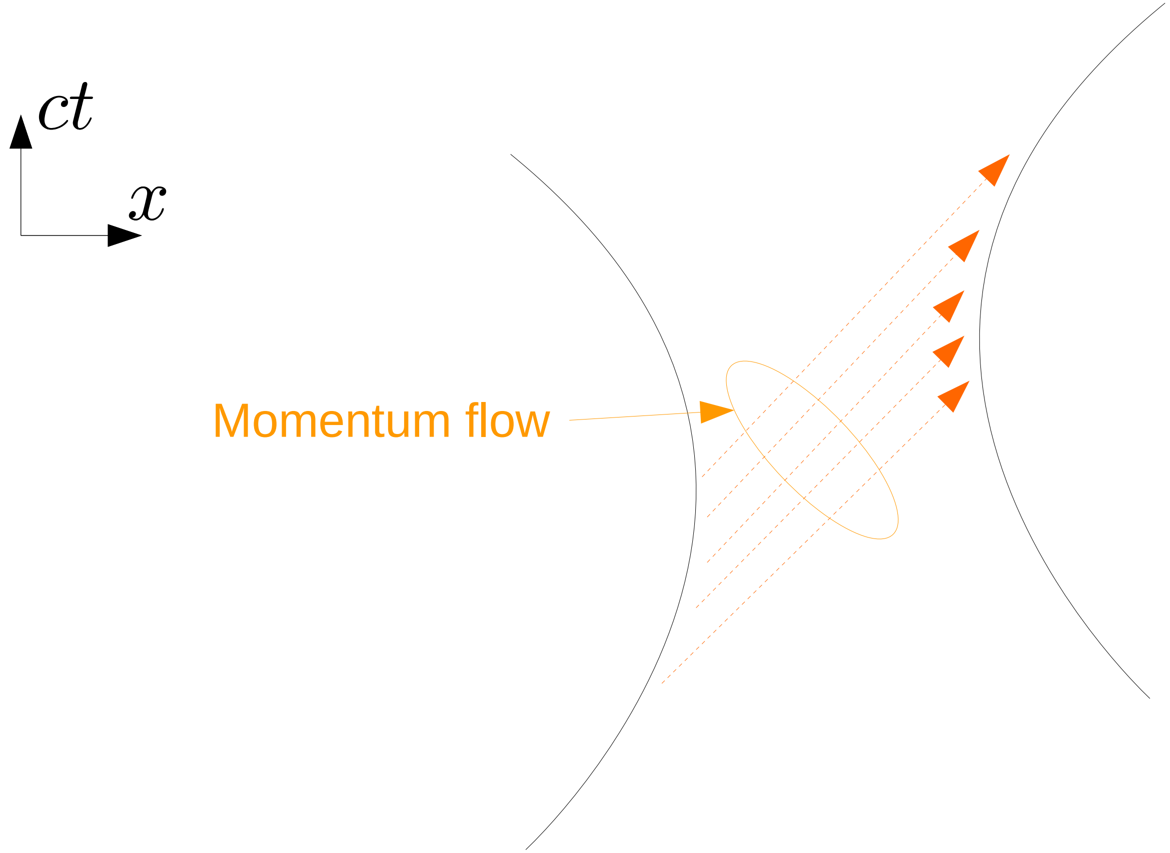
We can conserve 4-momentum and satisfy relativistic symmetries with each signal

But each signal must carry momentum!

(Note: haven’t identified signal as electromagnetic)

Closer look at collisions

- Force as flow of signals



Flow is defined at all space-time points
→ **fields**

It must also propagate in space-time
→ **field currents**

Fields

- Individual current conservation:

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = \partial_\mu J^\mu$$

- 4-momentum conservation (no external forces):

$$\sum_i (P_i)^\mu = \sum_i (Q_i)^\mu$$

4 equations!

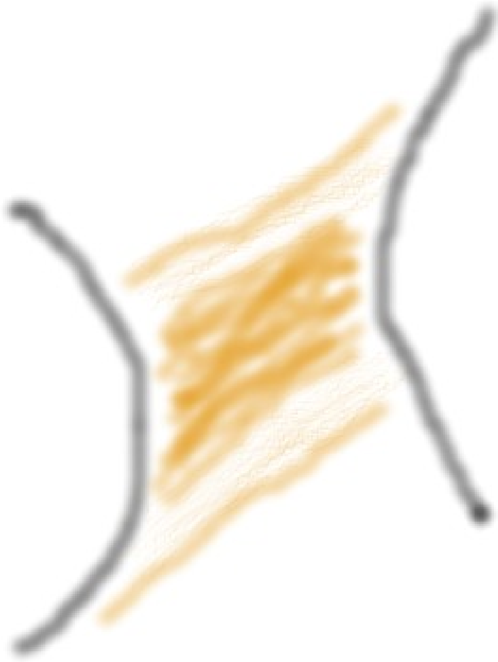
➔ We need 4 currents with continuity equations: stress-energy tensor

$$0 = \partial_\nu T^{\mu\nu}$$

This has to be a rank-2 field tensor so the 4 currents form a 4-vector

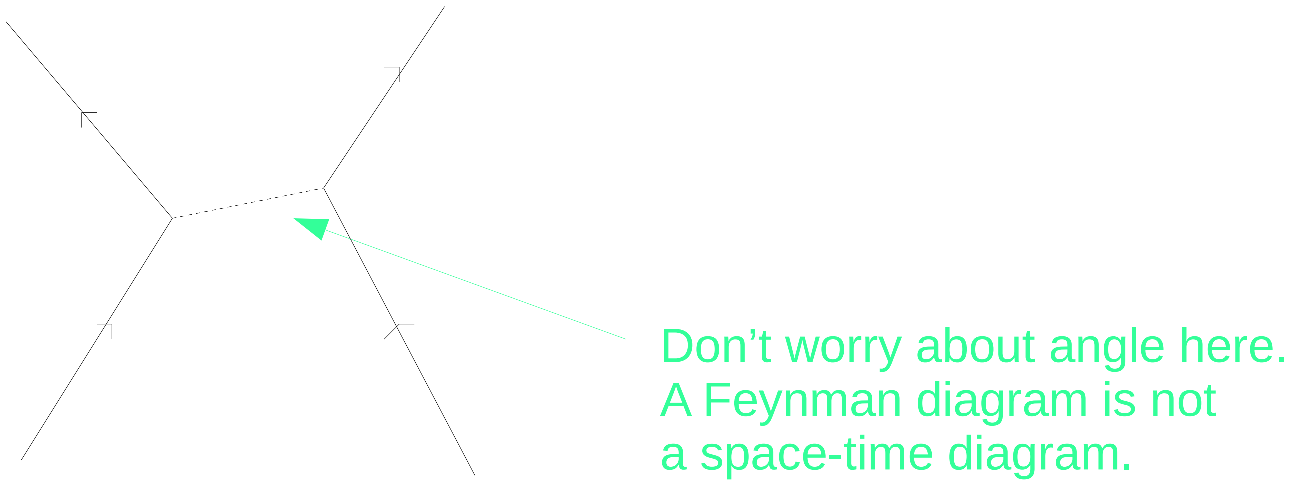
Fields

- Special Relativity requires forces to be treated as fields
  - Field needs to propagate: **radiation**
  - Also must carry momentum
- Add quantum mechanics:
  - Particles have a width
    - overlap not a problem
  - Fields propagate like particles
    - Interesting details here...
  - Side note: how many exchanged?
    - good thing they're bosons



Fields

- Calculating amplitudes in B4:



- Looks very classical – don’t be fooled!
  - Lines represent quantum fields
  - Quantum theory shares symmetries, e.g., form invariance,
    because what else can it do?

Emmy Noether (1882-1935)

- Wide-ranging mathematician at Göttingen
- “Guest lectured” for David Hilbert and others
- Ended up by Bryn Mawr
- Most famous for *Noether’s Theorem* connecting symmetries and conservation laws

Noether's Theorem

- Noether's Theorem in brief: a continuous symmetry → a conserved current
  - We're going to talk about Noether's Theorem for fields
  - It's actually a bit easier than the usual one for discrete particles
- Some definitions we'll use:
  - Path: a field configuration $\{\varphi_i(x)\}$
  - Classical path: a stationary path with respect to the action (which is a functional of the fields)
  - Action invariance: the new action has the same classical path
  - Symmetry: a change in the path which leaves the action invariant (same classical path)
- Basic idea: two kinds of symmetry in the action
  - General invariance: a change from any path which leaves the action invariant
    - No requirement that the original path is stationary/classical
  - Invariance of action around classical path itself
- Noether relates the two

Noether's Theorem

- Simplest case: cyclic field
  - Lagrangian doesn't depend on the field explicitly

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \boxed{\frac{\partial \mathcal{L}}{\partial \phi}} \quad \longleftarrow \quad \boxed{\frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu J^\mu$$

$$0 = \partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}$$

This is the form of a conserved current (doesn't have to be electric charge)

General invariance

- The action is a functional of the fields

$$I[\phi(x)] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

Consider just one field for simplicity

- First step: keep the action identically the same at all points

$$I[\phi(x)] - I[\phi(x) + \delta\phi_s(x)] = 0$$

A function of x, added to the original field (not necessarily a small change)

  - The action clearly has the same extrema
  - Note also this is true for cyclic fields

General invariance

- But remember that action is an integral

$$I[\phi(x)] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

- Adding a divergence term only affects the surface – not the interior, where the action is varied to find a stationary configuration

$$\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu(x)$$

Analogous to 3D

$$\int_V d^3x \nabla \cdot \mathbf{E} = \int_S \mathbf{E} \cdot d\mathbf{a}$$

$$\delta I[\phi, \delta\phi_s] = I[\phi(x) + \delta\phi_s(x)] - I[\phi(x)] = \int d^4x \partial_\mu K^\mu$$

Variation around classical path

- Now consider a classical field configuration

$$\bar{\phi} \equiv \bar{\phi}(x) \longrightarrow 0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\phi}}$$

- Small variation around that configuration

$$\delta I[\bar{\phi}, \delta\phi] = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right) + \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$$= \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

=0 because it's the classical path

Another divergence!

Relating the two

- If the variation around the classical path is a symmetry (leaves the action invariant), then it can take the form of a general invariance as well

Assign $\delta\phi(x) = \delta\phi_s(x)$

Classical path

$$\delta I[\bar{\phi}, \delta\phi_s] = \int d^4x \partial_\mu K^\mu = \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi_s \right)$$

From general invariance – any other sort of invariance

Change close to classical path

Noether's Theorem

- Gather both divergences → conserved current

$$0 = \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_s \phi - K^\mu \right) = \int d^4x \partial_\mu J^\mu$$

$$0 = \partial_\mu J^\mu \qquad J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi(x) - K^\mu$$

Note: for cyclic coordinate $K^\mu=0$ (reduces to previous result)

- Now let's look at the EM field

Maxwell stress-energy tensor

- Where did this come from?

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[-\frac{1}{4}(F_{\alpha\beta}F^{\alpha\beta})g^{\mu\nu} - F^{\mu}{}_{\gamma}F^{\gamma\nu}\right]$$

- In lectures, we do this:

  - Defence: there aren't that many rank-2 symmetric tensors we can make out of the fields
- We will see it's a conserved Noether current

Maxwell field tensor

- Field Lagrangian, no sources

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

Plug into field Euler-Lagrange equations to get source-free Maxwell equations

- Translation: $x^\mu \to x^\mu + \epsilon^\mu$

(For normal Lagrangians, translational invariance results in momentum conservation)

$$F'_{\mu\nu}(x) = F_{\mu\nu}(x - \epsilon) = \partial_\mu A_\nu(x - \epsilon) - \partial_\nu A_\mu(x - \epsilon)$$

- Noether current:

$$J^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} \delta A_\beta - K^\alpha$$

Field shift

$$A'_\mu(x) = A_\mu(x-\epsilon) = A_\mu(x) - \epsilon^\alpha \partial_\alpha A_\mu(x)$$
$$\delta A_\mu(x) = A'_\mu(x) - A_\mu(x) = -\epsilon^\alpha \partial_\alpha A_\mu(x)$$

- Problem: not gauge invariant
  - We can press on, but then we'll have to "fix" later on
  - Simpler approach here: gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \chi$$

$$\delta A_\mu = -\epsilon^\alpha(\partial_\alpha A_\mu) + \partial_\mu(\epsilon^\alpha A_\alpha) = F_{\mu\alpha}\epsilon^\alpha$$

  - Now field shift is manifestly gauge invariant

Field tensor shift

- We shift field values rather than the fields themselves

$$
\begin{aligned}
F'_{\mu\nu} &= \partial_\mu(A_\nu + F_{\nu\alpha}\epsilon^\alpha) - \partial_\nu(A_\mu + F_{\mu\alpha}\epsilon^\alpha) \\
&= F_{\mu\nu} + \epsilon^\alpha(\partial_\mu F_{\nu\alpha} - \partial_\nu F_{\mu\alpha}) \\
&= F_{\mu\nu} - \epsilon^\alpha \partial_\alpha F_{\mu\nu} \\
\delta F_{\mu\nu} &= -\epsilon^\alpha \partial_\alpha F_{\mu\nu}
\end{aligned}
$$

Bianchi identity

$$0 = \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu}$$

$$\delta(F_{\mu\nu}F^{\mu\nu}) = 2F^{\mu\nu}(\delta F_{\mu\nu}) = -2\epsilon^\alpha F^{\mu\nu}\partial_\alpha F_{\mu\nu} = -\epsilon^\alpha \partial_\alpha (F^{\mu\nu}F_{\mu\nu})$$

General invariance

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

- Calculate the change in action

$$\delta I = -\frac{1}{4\mu_0} \int d^4x \delta(F_{\mu\nu} F^{\mu\nu}) = \frac{1}{4\mu_0} \int d^4x \partial_\alpha (\epsilon^\alpha F_{\mu\nu} F^{\mu\nu})$$

- Surface term (divergence):

$$\Rightarrow K^\alpha = \frac{1}{4\mu_0} \epsilon^\alpha F_{\mu\nu} F^{\mu\nu} = -\epsilon^\alpha \mathcal{L}$$

On-shell variation

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} &= -\frac{1}{2\mu_0} F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial(\partial_\alpha A_\beta)} \\
&= -\frac{1}{2\mu_0} F^{\mu\nu} \frac{\partial}{\partial(\partial_\alpha A_\beta)} (\partial_\mu A_\nu - \partial_\nu A_\mu) \\
&= -\frac{1}{2\mu_0} F^{\mu\nu} (\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu) \\
&= \frac{1}{2\mu_0} (F^{\beta\alpha} - F^{\alpha\beta}) \\
&= \frac{1}{\mu_0} F^{\beta\alpha}
\end{aligned}
$$

Conserved current

$$J^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)}\delta A_\beta - K^\alpha$$

$$= \frac{1}{\mu_0}F^{\beta\alpha}F_{\beta\gamma}\epsilon^\gamma - \frac{1}{4\mu_0}\epsilon^\alpha F_{\mu\nu}F^{\mu\nu}$$

Gather free parameters ε

$$= -\frac{1}{\mu_0}F^{\alpha\beta}F_{\beta\gamma}\epsilon^\gamma - \frac{1}{4\mu_0}\epsilon^\gamma\delta^\alpha_\gamma F_{\mu\nu}F^{\mu\nu} = T^\alpha{}_\gamma\epsilon^\gamma$$

$$T^{\alpha\gamma} = \frac{1}{\mu_0}\left[-F^\alpha{}_\beta F^{\beta\gamma} - \frac{1}{4}g^{\alpha\gamma}F_{\mu\nu}F^{\mu\nu}\right]$$

The stress-energy tensor!

Summary

- Recall: particle interactions need a field tensor
- Translational symmetry → momentum conservation
- Applied to fields, this gives us stress-energy tensor
  - The complicated form didn't pop up out of nowhere