B2
Symmetry and Relativity
Lecture 22

Caltech exams

- Always “take home”, usually “open book”
  - Professor hands out the exam paper with due date
  - Also gives instructions on what resources allowed, e.g., textbook, lecture notes, written work
  - No one allowed to watch
  - Students decided when/where to sit exam
  - Responsible for timing themselves, marking “overtime” sections honestly

Caltech exams

- Typical instruction (Phys 1 quizzes):

Time: 1 hour

You can use your lecture notes, your homework, Ohanian and Feynman

Liénard-Wiechert potentials

- Field of a moving charge:
  - Expect it to depend on charge, speed, distance from source
  - Form-invariant combination which matches known behaviors (charge at rest, uniform motion)

$$A^\mu = \frac{q}{4\pi\epsilon_0} \frac{U^\mu/c}{(-R^\nu_{\rm sf} U_\nu)} \qquad R^\nu_{\rm sf} \equiv R^\nu - R^\nu_s$$

Liénard-Wiechert potentials

- Generalization to a 4-current field

$$A^\mu(t,\mathbf{x}) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{d\mathbf{x}_s}{r_{\rm sf}} J^\mu(t - r_{\rm sf}/c, \mathbf{x}_s) \qquad r_{\rm sf} \equiv |\mathbf{x} - \mathbf{x}_s|$$

(scalar invariant)

- Anyone want to guess the form for gravitational radiation?
  - Einstein field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Curvature (Ricci) tensor and scalar

Source term: the stress-energy tensor!

Liénard-Wiechert potentials

- If you guessed (within factors)

Linearized metric perturbation

$$h^{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int \frac{d\mathbf{x}_s}{r_{\mathrm{sf}}} T^{\mu\nu}(t - r_{\mathrm{sf}}/c, \mathbf{x}_s)$$

you'd be right

- One might ask, how could you be wrong?
- Congratulations: you've mastered the art of the symmetry-informed guess

NON-SYLLABUS

In the wild

The above condition is also known as *Einstein gauge, de Donder gauge, Fock gauge, or Lorentz gauge[17]*; in particular, the latter name refers to the analogy with the correspondent condition used in electromagnetism (see below). Then, from (2) we get

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{4}$$

Notice that the condition (3) can be always achieved by a gauge transformation; in fact, Einstein equations are invariant with respect to the infinitesimal transformations

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} \tag{5}$$

which, in terms of $\bar{h}_{\mu\nu}$ becomes

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} \tag{6}$$

So, if $\bar{h}^{\mu\nu}{}_{,\nu} \neq 0$, it is sufficient to choose ξ^μ to be a solution of $\Box \xi^\mu = -\bar{h}^{\mu\nu}{}_{,\nu}$.

Eqs. (4) are in clear analogy with Maxwell equations for the electromagnetic four-potential: so, they can be solved in the same way (see e.g. Ruggiero and Tartaglia [2], Mashhoon [3], Mashhoon et al. [18], Mashhoon [19], Padmanabhan [20]). In fact, neglecting the solution of the homogeneous wave equations associated to (4), the general solution is given in terms of retarded potentials

$$\bar{h}_{\mu\nu} = \frac{4G}{c^4} \int_V \frac{T_{\mu\nu}(ct - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3x' \ , \tag{7}$$

where integration is extended to the volume V, containing the source. We may set $T^{00} = \rho c^2$ and $T^{0i} = cj^i$, in

In analogy with the corresponding solutions of electromagnetism, it is possible to introduce the *gravitoelectromagnetic potentials*: namely, the gravitoelectric Φ and gravitomagnetic A^i potentials are defined by

$$\bar{h}_{00} \doteq 4\frac{\Phi}{c^2}, \quad \bar{h}_{0i} = -2\frac{A_i}{c^2}, \tag{11}$$

which, taking into account Eqs. (9) and (10), take the form

$$\Phi = G\int_V \frac{\rho(ct - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3x' \ , \tag{12}$$

$$A_i = \frac{2G}{c}\int_V \frac{j^i(ct - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3x' \ . \tag{13}$$

Eventually, the spacetime metric describing the solutions of Einstein's equation in weak-field approximation is written in the form [2, 3]

$$\mathrm{d}s^2 = -c^2\left(1 - 2\frac{\Phi}{c^2}\right)\mathrm{d}t^2 - \frac{4}{c}A_i \mathrm{d}x^i \mathrm{d}t + \left(1 + 2\frac{\Phi}{c^2}\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{14}$$

Now that we have defined the gravitoelectromagnetic potentials, it is possible to reconsider the Hilbert gauge condition (3) and express it in terms of Φ and A^i. From (3) we obtain indeed two conditions: setting $\mu = 0$ we get

$$\bar{h}^{00}{}_{,0} + \bar{h}^{0i}{}_{,i} = 0 \to \frac{1}{c}\frac{\partial \Phi}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}\cdot\mathbf{A} = 0, \tag{15}$$

ML Ruggiero, “A note on the gravitoelectromagnetic analogy”, arxiv:2111.09008 [gr-qc], 17 Nov 2021

Poincaré group

- Add translations to Lorentz group

$$P_\mu = -i\partial_\mu$$

Translation generator: momentum operator

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$$

Total angular momentum: orbital + internal

$$J_i = \frac{1}{2}\epsilon_{ijk}J_{jk}$$

$$K_i = J_{0i}$$

Boosts

$$[P_\mu, P_\nu] = 0$$

$$[J_{\mu\nu}, P_\rho] = i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu)$$

$$[J_{\mu\nu}, J_{\kappa\lambda}] = i(J_{\mu\lambda}g_{\nu\kappa} + J_{\nu\kappa}g_{\mu\lambda} - J_{\mu\kappa}g_{\nu\lambda} - J_{\nu\lambda}g_{\mu\kappa})$$

Poincaré group representations

Bargman, Wigner, 1946

- $P_\mu P^\mu = -m^2$
  - Finite mass particles, spin 0, ½, 1, …
  - Eigenvalues of $W_\mu W^\mu$: $m^2 s(s+1)$
  - State labels: s_3, p (continuous 3-momentum)
- $P_\mu P^\mu = 0$, $W_\mu W^\mu = 0$
  - Massless particles, two helicities $\pm s$
- $P_\mu P^\mu = 0$, $W_\mu W^\mu > 0$
  - Massless particles, continous spin
- $P_\mu P^\mu > 0$
  - tachyons

$$P_\mu = -i\partial_\mu$$

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda} P_\nu J_{\kappa\lambda}$$

(Paul-Lubanski vector)

Reminder

- Last two B2 lectures (Thursday, Friday) will be at noon as usual