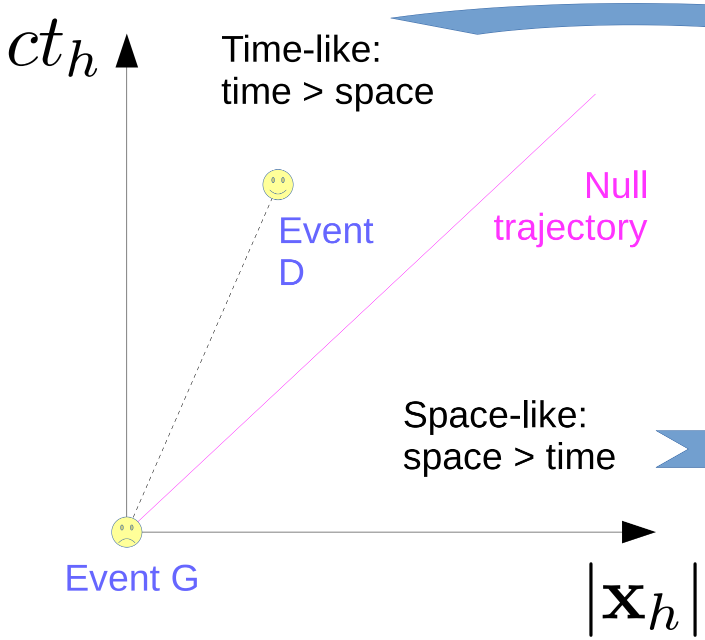
B2
Symmetry and Relativity
Revision 1
TT 2024

“Open book” B2 2018 Q1

1. (a) Two events in the laboratory frame S are characterized by 4-vectors $\mathbf{D} = (ct_d, \mathbf{x}_d)$ and $\mathbf{G} = (ct_g, \mathbf{x}_g)$ where $\mathbf{x}_d$ and $\mathbf{x}_g$ are the corresponding 3-vectors. Write down the conditions for these events to be connected by space-like and time-like intervals. Can two events be connected by a null vector? Explain the answer and draw a schematic (ct, x) diagram indicating all significant features.

Form an interval: $$H^\mu = D^\mu - G^\mu = (c(t_d - t_g), \mathbf{x}_d - \mathbf{x}_g) = (ct_h, \mathbf{x}_h)$$



$$c^2(t_d - t_g)^2 > (\mathbf{x}_d - \mathbf{x}_g)^2$$

$$c^2(t_d - t_g)^2 = (\mathbf{x}_d - \mathbf{x}_g)^2$$

$$(\mathbf{x}_d - \mathbf{x}_g)^2 > c^2(t_d - t_g)^2$$

Q1 (a)

Consider two arbitrary 4-vectors whose dot product is zero, i.e. $Y^\mu X_\mu = 0$. ~~List the conditions under which the dot product of the 4-vectors is equal to zero.~~ Under what conditions is the inner product zero when the vectors are different combinations of timelike, spacelike, and null?

~~Consider the following 4-vectors: the 4-acceleration $\mathbf{A}$, the 4-wavevector $\mathbf{K}$, and the 4-current $\mathbf{J}$. For which pairs of these vectors are the dot products equal to zero?~~ Justify your answers. [7]

$$X^\mu = (x^0, \mathbf{x})$$

Note: no requirement that time components are positive!

$$Y^\mu = (y^0, \mathbf{y})$$

$$0 = Y^\mu X_\mu = -x^0 y^0 + \mathbf{x}\cdot\mathbf{y}$$

$$x^0 y^0 = \mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|\cdot|\mathbf{y}|\cos\theta$$

Ratios make it easy to categorize vectors into null, time-like, space-like

$$\frac{(x^0)^2}{|\mathbf{x}|^2} = \frac{|\mathbf{y}|^2}{(y^0)^2}\cos^2\theta$$

Q1 (a)

$$R = \frac{(x^0)^2}{|\mathbf{x}|^2} = \frac{|\mathbf{y}|^2}{(y^0)^2}\cos^2\theta$$

$$0 \leq \cos^2\theta \leq 1$$

Angle between **x** and **y**

X is time-like: $1 < R \;\Rightarrow\; \dfrac{(y^0)^2}{|\mathbf{y}|^2} < \cos^2\theta \leq 1$

→ Y must be space-like
(any vector orthogonal to a time-like vector must be space-like)

Problem 2.1(v)

X is space-like: $1 > R \;\Rightarrow\; \dfrac{(y^0)^2}{|\mathbf{y}|^2} > \cos^2\theta$

→ Y can be time-like, space-like, or null

Problem 2.1(vi)

X is null: $1 = R \;\Rightarrow\; \dfrac{(y^0)^2}{|\mathbf{y}|^2} = \cos^2\theta \leq 1$

→ Y can be space-like or null

Only null if $\cos\theta = 1$
(same spatial direction)

Q1 (b)

(b) For a particle of mass m moving along a world-line X^μ in the inertial reference frame S, define the 4-momentum $\mathbf{P}$ and 4-acceleration $\mathbf{A}$. Using components of 4-momentum $\mathbf{P}$ calculate the dot product $P^\mu P_\mu$. Show that if 3-velocity and 3-acceleration are parallel to each other then the 4 acceleration invariant is $A^\mu A_\mu = \gamma^6 a^2$. [6]

$$X^\mu = (\ ct(\zeta), x^i(\zeta)\ )$$

Event time and position a function of some parameter

$$P^\mu \equiv m\frac{dX^\mu}{d\tau}$$

$$A^\mu \equiv \frac{dU^\mu}{d\tau} = \frac{d^2X^\mu}{d\tau^2}$$

$$dt = \gamma d\tau \Rightarrow \frac{d}{d\tau} = \gamma\frac{d}{dt}$$

$$P^0 = m\frac{dX^0}{d\tau} = \gamma mc$$

$$P^i = m\frac{dX^i}{d\tau} = \gamma m\mathbf{u}$$

$$\begin{aligned} P^\mu P_\mu &= -\gamma^2 m^2 c^2 + \gamma^2 m^2 u^2 \\ &= -\gamma^2 m^2 c^2 (1-\beta^2) \\ &= -m^2 c^2 \end{aligned}$$

Q1 (b)

(b) For a particle of mass m moving along a world-line X^μ in the inertial reference frame S, define the 4-momentum $\mathbf{P}$ and 4-acceleration $\mathbf{A}$. Using components of 4-momentum $\mathbf{P}$ calculate the dot product $P^\mu P_\mu$. Show that if 3-velocity and 3-acceleration are parallel to each other then the 4 acceleration invariant is $A^\mu A_\mu = \gamma^6 a^2$. [6]

Steane 2.5.2

$$\mathbf{u}\cdot\mathbf{a} = ua$$

$$U^\mu = \gamma(c, \mathbf{u})$$

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{\mathbf{u}\cdot\dot{\mathbf{u}}/c^2}{(1-\mathbf{u}\cdot\mathbf{u}/c^2)^{3/2}} = \gamma^3\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}$$

$$A^\mu = \gamma\frac{dU^\mu}{dt} = \gamma(\dot{\gamma}c, \dot{\gamma}\mathbf{u} + \gamma\mathbf{a})$$

$$\begin{aligned} A^\mu A_\mu &= \gamma^2(-\dot{\gamma}^2c^2 + \dot{\gamma}^2u^2 + 2\gamma\dot{\gamma}\mathbf{u}\cdot\mathbf{a} + \gamma^2a^2) \\ &= \gamma^2(-\gamma^6\beta^2a^2(1-\beta^2) + 2\gamma^4\beta^2a^2 + \gamma^2a^2) \\ &= \gamma^4a^2(-\gamma^2\beta^2 + 2\gamma^2\beta^2 + 1) = \gamma^6a^2 \end{aligned}$$

Q1 (c)

(c) Consider the motion of a particle under a pure (rest mass preserving), inverse square law force $\mathbf{f} = \alpha\mathbf{r}/r^3$ where α is a constant. Taking into account that $\mathbf{f}$ is a central force, derive the energy conservation equation showing that $\gamma mc^2 - \alpha/r$ is constant. ~~Give an example of such a force.~~ Give examples of such a force. [5]

Problem 3.3

Pure force (no rest mass change)

$$\frac{dE}{dt} = \mathbf{f}\cdot\mathbf{v} \;\Rightarrow\; dE = \mathbf{f}\cdot d\mathbf{r} = (-\nabla V)\cdot d\mathbf{r} = -dV$$

$$dE + dV = 0$$

(true for any central potential)

$$E = \gamma mc^2$$

What is the potential?

$$V = -\frac{\alpha}{r} \;\Rightarrow\; \mathbf{f} = -\nabla V = \frac{\alpha\mathbf{r}}{r^3}$$

Therefore E+V is a constant, and hence so is

$$\gamma mc^2 - \frac{\alpha}{r}$$

Examples:
gravity,
electrostatic field

Q1 (d)

(d) An electron is accelerated from rest across a gap of $L = 5\,\mathrm{m}$ by a constant electric field of strength $10\,\mathrm{MV\,m^{-1}}$. Find γ at the other end of the gap. Calculate how long it takes for the electron to reach the other end of the gap. Sketch $\gamma(t)$ and $\beta(t)$, indicating the significant values. How much will β change if the accelerating field is reduced by 20%? If, instead of an electron, a proton is accelerated, how will γ and β change? [7]

Potential across 5m gap is 50 MV

For a single electron:

$$E = E_0 + \Delta E = mc^2 + 50\ \mathrm{MeV}$$

$$\gamma = \frac{mc^2 + 50\ \mathrm{MeV}}{mc^2} = 1 + \frac{50}{0.511} \approx 98.8$$

Q1 (d)

Steane 4.2.4 and 4.2.2

(d) An electron is accelerated from rest across a gap of $L = 5\,\mathrm{m}$ by a constant electric field of strength $10\,\mathrm{MV\,m^{-1}}$. Find γ at the other end of the gap. Calculate how long it takes for the electron to reach the other end of the gap. Sketch $\gamma(t)$ and $\beta(t)$, indicating the significant values. How much will β change if the accelerating field is reduced by 20%? If, instead of an electron, a proton is accelerated, how will γ and β change? [7]

Need to get x(t) from a known force, so start with definition of 3-force

$$\mathbf{f} = \frac{d\mathbf{p}}{dt}$$

Electric field, not energy

$$qE = mc\frac{d}{dt}(\beta\gamma) = mc\frac{d}{dt}\sinh\eta$$

rapidity

$$d\sinh\eta = \frac{qE}{mc}dt$$

sinhη = 0 when t = 0

$$\sinh\eta = \frac{qEt}{mc}$$

But we want β(t) = tanhη

Q1 (d)

$$\tanh \eta = \frac{\alpha t}{\sqrt{1+\alpha^2 t^2}} \qquad \alpha = \frac{qE}{mc}$$

$$L = c\int_0^T \beta dt = c\int_0^T \tanh \eta dt$$

$$= \frac{c}{\alpha}\int_0^{\alpha T} \frac{y}{\sqrt{1+y^2}} dy$$

$$= \frac{c}{\alpha}\left[\sqrt{1+\alpha^2 T^2} - 1\right]$$

$$\left(L + \frac{c}{\alpha}\right)^2 - c^2 T^2 = \left(\frac{c}{\alpha}\right)^2 \longrightarrow \text{hyperbola}$$

Note (T,L) always same spacetime interval to (0,-c/α)

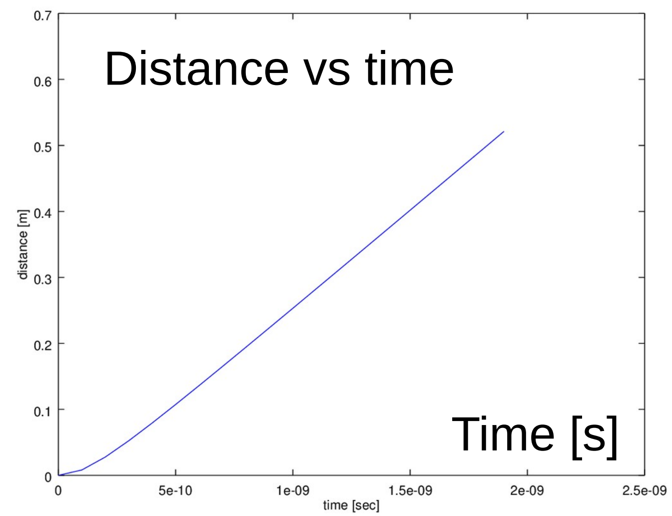
Q1 (d)

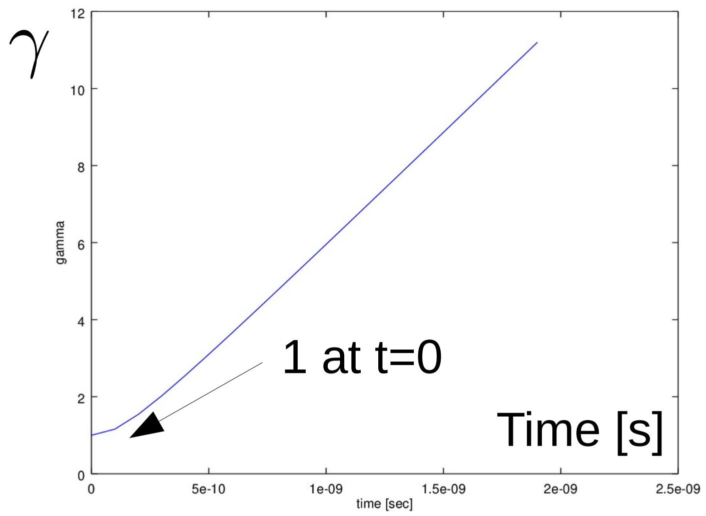
$$L = 5\ \mathrm{m}$$

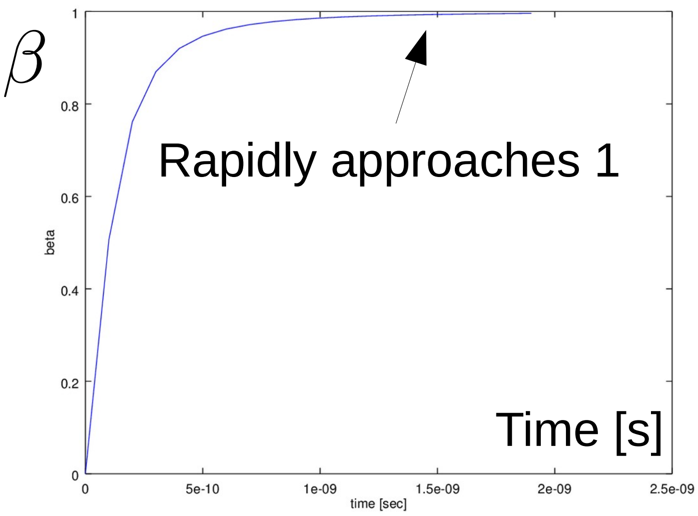
Plugging in...

$$\frac{c}{\alpha} = \frac{mc^2}{qE} = \frac{0.511\ \mathrm{MeV}}{10\ \mathrm{MeV/m}} = 0.0511\ \mathrm{m}$$

$$\boxed{T = \frac{1}{\alpha}\left[\left(1+\frac{\alpha L}{c}\right)^2 - 1\right]^{1/2} \approx 16.8\ \mathrm{ns}}$$







Q1 (d)

(d) An electron is accelerated from rest across a gap of $L = 5\,\mathrm{m}$ by a constant electric field of strength $10\,\mathrm{MV\,m^{-1}}$. Find γ at the other end of the gap. Calculate how long it takes for the electron to reach the other end of the gap. Sketch $\gamma(t)$ and $\beta(t)$, indicating the significant values. How much will β change if the accelerating field is reduced by 20%? If, instead of an electron, a proton is accelerated, how will γ and β change? [7]

Potential across 5m gap is now 40 MV

$$\gamma = 1 + \frac{40}{0.511} \approx 79.3$$

Final velocity

$$\boxed{\beta \approx 0.99992}$$

Proton across same gap, original field magnitude:

Opposite charge, mass 938 MeV/c^2

$$\gamma = 1 + \frac{50}{938} \approx 1.05$$

$$\beta \approx 0.314$$

Done with Q1!

Q2 (a)

2. (a) A photon can be defined by a 4-wavevector $\mathbf{K}$. Write down the components of this 4-vector and the relationship between them. Considering a photon propagating in a vacuum, define its phase and group velocities in terms of 4-wavevector $\mathbf{K}$ components. Show that the phase ϕ of the wave is Lorentz invariant. Is the phase velocity $v_{\rm ph}$ Lorentz invariant? Explain your answer. [6]

frequency wavenumber

$$K^\mu = (\omega/c, \mathbf{k})$$

Components related by phase velocity

$$v_p = \omega/|\mathbf{k}| \qquad \mathbf{v}_p = \frac{\mathbf{k}}{|\mathbf{k}|^2}\omega$$

In vacuum, $v_p = c$

$$\omega = |\mathbf{k}|c$$

General expressions for group velocity

$$v_g = \frac{d\omega}{dk} \qquad \mathbf{v}_g = \nabla_{\mathbf{k}}\omega$$

$$v_g = c$$

$$\mathbf{v}_g = \frac{\mathbf{k}}{|\mathbf{k}|}c$$

Really looking for this form, since asking for group velocity in terms of components

Q2 (a)

2. (a) A photon can be defined by a 4-wavevector **K**. Write down the components of this 4-vector and the relationship between them. Considering a photon propagating in a vacuum, define its phase and group velocities in terms of 4-wavevector **K** components. Show that the phase ϕ of the wave is Lorentz invariant. Is the phase velocity v_{ph} Lorentz invariant? Explain your answer. [6]

wave

$$\psi(ct, \mathbf{x}) \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

phase

$$\phi \equiv \mathbf{k}\cdot\mathbf{x} - \omega t$$

Space-time event (already know it's a 4-vector)

$$X^\mu \equiv (ct, \mathbf{x})$$

4-wave vector

$$K^\mu \equiv (\omega/c, \mathbf{k})$$

$$K^\mu X_\mu = -\omega t + \mathbf{k}\cdot\mathbf{x} = \phi$$

Phase is therefore a contraction of two 4-vectors → a scalar Lorentz invariant

Q2 (a)

Steane 6.4

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Cheap answer: phase velocity is the magnitude of a 3-vector → not a Lorentz invariant

Could it be nonetheless a scalar invariant?

Sometimes not obvious – see end of Steane 8.2.2

- Does it change with a Lorentz transformation?
- Calculate an invariant and see whether phase velocity changes with frame

$$K^\mu K_\mu = -\frac{\omega^2}{c^2} + k^2$$

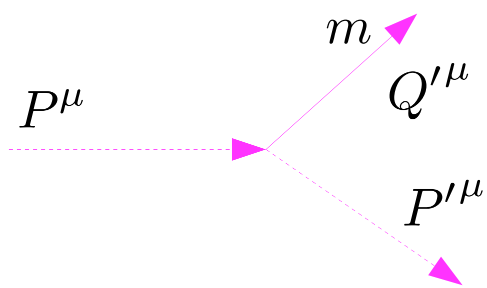
$$= \boxed{\omega^2}\left(\frac{1}{\boxed{v_p^2}} - \frac{1}{c^2}\right)$$

We know ω changes with frame, but neither c nor $K^\mu K_\mu$ do

→ v_p must change with frame. It is not Lorentz invariant.

Q2 (b)

(b) Can a single photon in a vacuum decay into a single massive particle with mass $m \neq 0$ and another photon? Prove your answer. Show that an electron-positron pair can be produced during collisions of photons. Find the minimum number of the photons required and find the minimal energy required for the electron-positron pair to appear, assuming that the photons participating in the collision have the same frequency. [7]



Start with 4-momentum conservation...

$$P^\mu = P'^\mu + Q'^\mu$$

$$P^\mu - P'^\mu = Q'^\mu$$

Isolate one of the 4-momenta

$$P^\mu P_\mu + P'^\mu P'_\mu - 2P^\mu P'_\mu = Q'^\mu Q'_\mu$$

Both zero

$$P^\mu P'_\mu = \frac{m^2c^2}{2}$$

Still true in all frames

Q2 (b)

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Choose some frame (in any case, both are null vectors)

$$P^\mu = (k, \mathbf{k})$$

$$P'^\mu = (k', \mathbf{k}')$$

$$k, k' > 0$$

$$\frac{m^2c^2}{2} = P^\mu P'_\mu = -kk' + \mathbf{k} \cdot \mathbf{k}'$$

$$= kk'(\cos\theta - 1) \leq 0$$

But m > 0, so...no

Q2 (b)

(b) Can a single photon in a vacuum decay into a single massive particle with mass $m \neq 0$ and another photon? Prove your answer. Show that an electron-positron pair can be produced during collisions of photons. Find the minimum number of the photons required and find the minimal energy required for the electron-positron pair to appear, assuming that the photons participating in the collision have the same frequency. [7]

Sufficient to prove that it doesn't work for 1 photon, but **can** work for 2.

Electron and positron

N photons

$$P^\mu = \sum_i (P_i)^\mu$$

$$P^\mu = (Q_1)^\mu + (Q_2)^\mu$$

$$P^\mu - (Q_1)^\mu = (Q_2)^\mu$$

$$P^\mu P_\mu + (Q_1)^\mu (Q_1)_\mu - 2P^\mu (Q_1)_\mu = (Q_2)^\mu (Q_2)_\mu$$

If only 1 photon, LHS is 0.

$$P^\mu P_\mu = 2P^\mu (Q_1)_\mu$$

P and Q_1 are thus orthogonal.

But we know Q_1 is time-like → P can only be space-like.

Question 1(a)!

Yet P must be a null vector → contradiction.

1 photon cannot produce an electron-positron pair.

Q2 (b)

(b) Can a single photon in a vacuum decay into a single massive particle with mass $m \neq 0$ and another photon? Prove your answer. Show that an electron-positron pair can be produced during collisions of photons. Find the minimum number of the photons required and find the minimal energy required for the electron-positron pair to appear, assuming that the photons participating in the collision have the same frequency. [7]

minimum number of photons producing e^+e^- pair in some frame at minimum energy is sufficient to show e^+e^- pair can be produced

$$
\begin{aligned}
(P_1)^\mu + (P_2)^\mu &= (Q_1)^\mu + (Q_2)^\mu \\
(Q_1)^\mu &= (mc, 0) \\
(Q_2)^\mu &= (mc, 0) \\
(P_1)^\mu &= \hbar(\omega/c, \mathbf{k}) \\
(P_2)^\mu &= \hbar(\omega/c, -\mathbf{k})
\end{aligned}
$$

$$\Rightarrow \boxed{\hbar\omega = mc^2}$$

Q2 (c)

(c) A free electron, with initial velocity in the x direction $\mathbf{v}_0 = (v_0, 0, 0)$ is injected into a vacuum vessel. Show and prove which components of the electron 4-momentum are conserved if inside the vessel there is: (i) a constant electric field (no magnetic field) $\mathbf{E} = (E_x, 0, 0)$ directed along the electron initial velocity; (ii) the magnetic field $\mathbf{B} = (0, B_y, 0)$ with magnetic field lines perpendicular to the electron initial velocity and directed along the y coordinate (no electric field). [5]

(i) Lorentz force, E only:

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = q\mathbf{E} = q(E_x, 0, 0)$$

$$\frac{dp_x}{dt} = qE_x$$

$$\frac{dp_y}{dt} = 0$$

$$\frac{dp_z}{dt} = 0$$

Clearly p_y and p_z are the conserved components

$$\frac{E^2}{c^2} = |\mathbf{p}|^2$$

Energy is not conserved

Q2 (c)

(c) A free electron, with initial velocity in the x direction $\mathbf{v}_0 = (v_0, 0, 0)$ is injected into a vacuum vessel. Show and prove which components of the electron 4-momentum are conserved if inside the vessel there is: (i) a constant electric field (no magnetic field) $\mathbf{E} = (E_x, 0, 0)$ directed along the electron initial velocity; (ii) the magnetic field $\mathbf{B} = (0, B_y, 0)$ with magnetic field lines perpendicular to the electron initial velocity and directed along the y coordinate (no electric field). [5]

(ii) Lorentz force, B only:

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = q\mathbf{v} \wedge \mathbf{B} = q(-v_z B_y, 0, v_x B_y)$$

p_y clearly stays the same, while p_x and p_z don't

Work along path:

$$dW = \mathbf{f} \cdot d\mathbf{x}$$

$$\frac{dW}{dt} = \mathbf{f} \cdot \frac{d\mathbf{x}}{dt} = q(\mathbf{v} \wedge \mathbf{B}) \cdot \mathbf{v} = 0$$

Energy is conserved

Q2 (d)

(d) A plane mirror moves uniformly in the direction of its normal $\mathbf{x}_0$ in a laboratory frame S with velocity $\beta_x = v_x/c = 0.99$. A photon has wavelength $\lambda_1 = 1\,\mu\text{m}$ in the laboratory stationary frame and 3-wavevector $\mathbf{k} = (-k_x, k_y, 0)$ with $|k_x| = |k_y|$ in the frame co-moving with the mirror. Along the x-coordinate it moves in the opposite direction to the mirror. The photon is reflected by the mirror. Find (in the laboratory frame) the angle of reflection and the measured wavelength of the reflected photon, λ_2. [7]

(Need to assume system is in a vacuum!)

Go to some frame S' we understand: where the mirror is at rest.

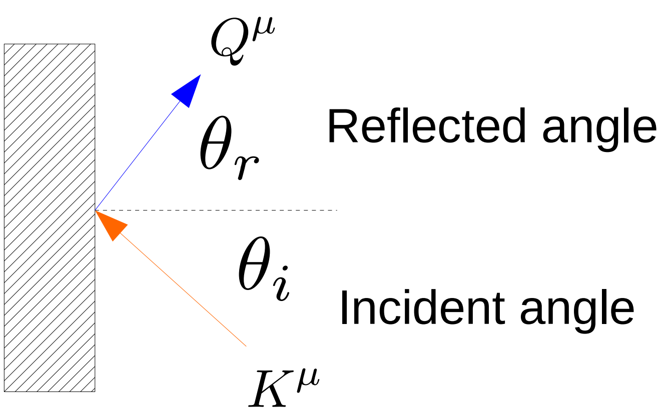
Incoming photon: $$K^\mu = (\omega/c, -k_x, k_x, 0)$$

Outgoing photon (simple reflection): $$Q^\mu = (\omega/c, k_x, k_x, 0)$$

$$\omega/c = \sqrt{2}k_x$$

(Unfortunately question defines unprimed components in S'!)

Frame S'



Q2 (d)

Boost from S' (mirror frame) to S (lab frame)

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

(Check: origin of S' travels in +x direction in S)

$$x^* = \gamma(x + \beta ct)$$

$$\omega/c = \sqrt{2}k_x$$

Incoming:

$$K^{*\mu} = \Lambda^{\mu}{}_{\nu} K^{\nu} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \omega/c \\ -k_x \\ k_x \\ 0 \end{pmatrix} = k_x \begin{pmatrix} \gamma(\sqrt{2} - \beta) \\ \gamma(\sqrt{2}\beta - 1) \\ 1 \\ 0 \end{pmatrix}$$

Outgoing:

$$Q^{*\mu} = k_x \begin{pmatrix} \gamma(\sqrt{2} + \beta) \\ \gamma(\sqrt{2}\beta + 1) \\ 1 \\ 0 \end{pmatrix}$$

Can now read off components to get final energy and reflected angle in lab frame

Q2 (d)

Can now read off components to get final energy and reflected angle in lab frame

$$\lambda_1 \propto 1/E_i^*$$

$$\lambda_2 \propto 1/E_r^*$$

$$E_i^* = k_x \gamma(\sqrt{2} - \beta)$$

$$E_r^* = k_x \gamma(\sqrt{2} + \beta)$$

$$= \frac{\sqrt{2} + \beta}{\sqrt{2} - \beta} E_i^*$$

$$\beta = 0.99$$

$$\lambda_1 = 1\ \mu\text{m}$$

$$\lambda_2 = \frac{\sqrt{2} - \beta}{\sqrt{2} + \beta} \lambda_1$$

$$\boxed{\approx 0.18\ \mu\text{m}}$$

Q2 (d)

Outgoing:

$$Q^{*\mu} = k_x \begin{pmatrix} \gamma(\sqrt{2}+\beta) \\ \gamma(\sqrt{2}\beta+1) \\ 1 \\ 0 \end{pmatrix}$$

Read off components to get reflected angle in lab frame

$$\tan\theta_r^* = \frac{Q^{*2}}{Q^{*1}} = \frac{1}{\gamma(\sqrt{2}\beta+1)} \approx 0.059$$

$$\boxed{\theta_r^* \approx 0.059}$$

Q2 (d)

(d) A plane mirror moves uniformly in the direction of its normal $\mathbf{x}_0$ in a laboratory frame S with velocity $\beta_x = v_x/c = 0.99$. A photon has wavelength $\lambda_1 = 1\,\mu$m in the laboratory stationary frame and 3-wavevector $\mathbf{k} = (-k_x, k_y, 0)$ with $|k_x| = |k_y|$ in the frame co-moving with the mirror. Along the x-coordinate it moves in the opposite direction to the mirror. The photon is reflected by the mirror. Find (in the laboratory frame) the angle of reflection and the measured wavelength of the reflected photon, λ_2. [7]

Can also check against "standard" results...if you wish...

$$\omega_i \sin\theta_i = \omega_r \sin\theta_r$$

$$\frac{\tan\frac{\theta_i}{2}}{\tan\frac{\theta_r}{2}} = \frac{1+\beta}{1-\beta}$$

Using this standard result instead of deriving result:
doesn't really show much you know about symmetry and relativity

Q3 (a)

3. (a) A particle moves along world-line $X^\mu = X^\mu(\tau)$ in the inertial reference frame S. Does special relativity place any bounds on the possible sizes of forces and accelerations for particles of mass m? Justify your answer. Does special relativity place any bounds on the phase velocity of the electromagnetic wave? Derive an expression for the photon's frequency shift during Compton scattering. [6]

For a particle with non-zero mass:

$$A^\mu = \frac{dU^\mu}{d\tau} = \gamma(c\dot{\gamma}, \dot{\gamma}\mathbf{u} + \gamma\mathbf{a})$$

$$\dot{\gamma} = \frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\gamma^3$$

$$= \gamma^2\left(\gamma^2\frac{\mathbf{u}\cdot\mathbf{a}}{c}, \gamma^2\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\mathbf{u} + \mathbf{a}\right)$$

$$\gamma \in [1, \infty)$$

→ no upper bound on 4-acceleration

Note also no upper limit on proper acceleration: $A^\mu A_\mu = a_0^2$

For m=0, speed is always *c*, so acceleration is constrained.

Q3 (a)

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4-force:

$$F^\mu \equiv \frac{dP^\mu}{d\tau}$$

$$P^\mu = (E/c, \mathbf{p})$$

$$= \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\mathbf{p}}{dt}\right)$$

$\mathbf{f}$ 3-force

Similarly, no limit on sizes of components of 4-force or 3-force

Q3 (a)

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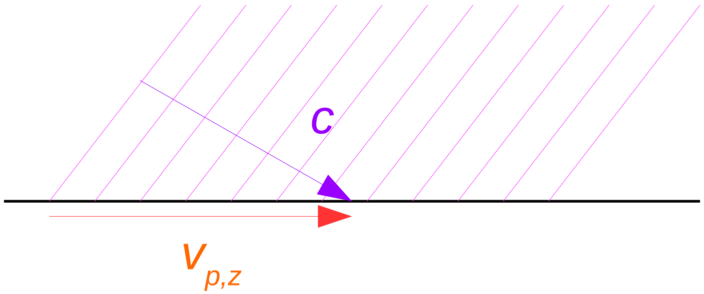
In general, $\mathbf{v}_p = \dfrac{\omega}{|\mathbf{k}|}\hat{\mathbf{k}}$ In free space, magnitude is always *c*.

Phase velocity may be greater than *c*, *e.g.*, in a wave guide:

$$v_{p,z} = \frac{\omega}{k_z} \geq c \quad \text{with} \quad k_z < |\mathbf{k}|$$

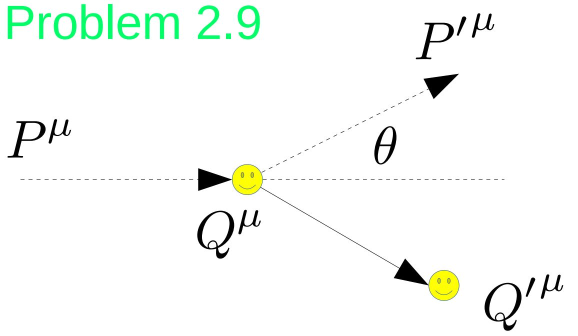


It turns out special relativity doesn't limit phase velocity

Q3 (a)

3. (a) A particle moves along world-line $X^\mu = X^\mu(\tau)$ in the inertial reference frame S. Does special relativity place any bounds on the possible sizes of forces and accelerations for particles of mass m? Justify your answer. Does special relativity place any bounds on the phase velocity of the electromagnetic wave? Derive an expression for the photon's frequency shift during Compton scattering. [6]

Problem 2.9



$$P^\mu + Q^\mu = P'^\mu + Q'^\mu$$
$$Q'^\mu = P^\mu + Q^\mu - P'^\mu$$
$$-m^2c^2 = -m^2c^2 + 2P^\mu Q_\mu - 2P^\mu P'_\mu - 2Q^\mu P'_\mu$$
$$0 = P^\mu Q_\mu - P^\mu P'_\mu - Q^\mu P'_\mu$$

Lab frame:

$$P^\mu = \hbar(\omega/c, \mathbf{k})$$
$$Q^\mu = (mc, 0)$$
$$P'^\mu = \hbar(\omega'/c, \mathbf{k}')$$

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{\hbar}{mc^2}(1 - \cos\theta)$$

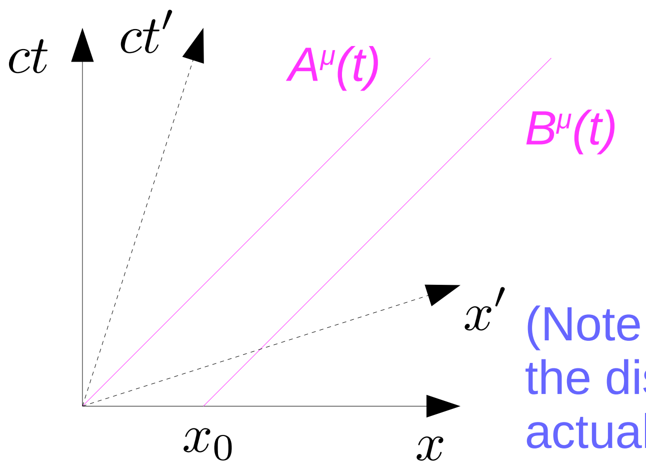
Q3 (b)

(b) In the laboratory frame S two photons propagate along the x-coordinate, separated by a distance x_0. Calculate the distance between the photons in frame S' moving along the x-coordinate with velocity $\mathbf{v} = (v_x, 0, 0)$. [3]

You may be tempted to use the usual length contraction – but that's if you know the proper length of the travelling span. In this case, the span is travelling at *c*.
No rest frame → no proper length!

$$\cancel{L = \frac{L_0}{\gamma}}$$

Go back to the photon worldlines in S:



(Note this indicates the distance should actually grow)

$$A^\mu = (ct_A, x_A) = (ct, ct)$$
$$B^\mu = (ct_B, x_B) = (ct, x_0 + ct)$$

Q3 (b)

(b) In the laboratory frame S two photons propagate along the x-coordinate, separated by a distance x_0. Calculate the distance between the photons in frame S' moving along the x-coordinate with velocity $\mathbf{v} = (v_x, 0, 0)$. [3]

Transform to S':

$$\beta \equiv \frac{v_x}{c}$$

$$ct'_A = \gamma(ct_A - \beta x_A) = \gamma(1-\beta)ct$$

$$x'_A = \gamma(x_A - \beta ct_A) = \gamma(1-\beta)ct = ct'_A$$

$$ct'_B = \gamma(ct - \beta(ct + x_0))$$

$$x'_B = \gamma(x_0 - (1-\beta)ct)$$

Find t_B in terms of t_A when $t'_B(t_B) = t'_A(t_A)$:

$$\gamma(ct_B - \beta(ct_B + x_0)) = \gamma(1-\beta)ct_A$$

$$\boxed{t_B = t_A + \frac{\beta}{1+\beta}x_0}$$

Q3 (b)

(b) In the laboratory frame S two photons propagate along the x-coordinate, separated by a distance x_0. Calculate the distance between the photons in frame S' moving along the x-coordinate with velocity $\mathbf{v} = (v_x, 0, 0)$. [3]

$$x'_B(t_B) - x'_A(t_A) = \gamma(1-\beta)t_B + \gamma x_0 - \gamma(1-\beta)t_A$$

$$= \gamma(1+\beta)x_0$$

$$\boxed{= \sqrt{\frac{1+\beta}{1-\beta}}x_0}$$

This is the distance in S' we're looking for

If I was an Examiner, I might be tempted to ask the candidate to relate this result to the usual length contraction (though for more than 3 marks!).

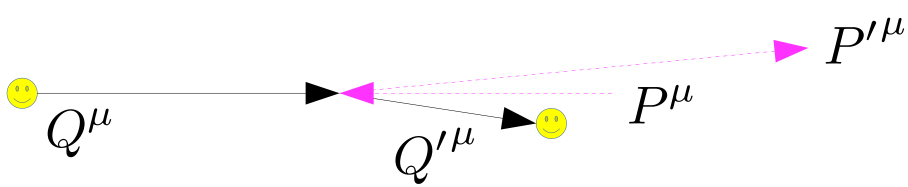
Q3 (c)

(c) An electron and a photon of wavelength $\lambda_1 = 8\,\mu$m are moving toward each other along the x-coordinate and at some point the photon is scattered in a head on collision. Show that the wavelength of the scattered photon can be estimated as $\lambda_2 \approx \lambda_1/(4\gamma^2)$. Calculate the wavelength of the scattered photon when the initial velocity of the electron in the laboratory frame is $\beta_x = 0.999$. [6]

[*Hint: Use the fact that the energies of the photons are much smaller than the rest energy of the electron.*]

"Inverse Compton scattering": an electron hits a "soft" photon, giving energy to the photon

"Head on":
mostly linear,
small angles



$$Q^\mu = (E/c, p)$$

$$P^\mu = (k, -k)$$

$$P'^\mu = (k', k')$$

$$0 = P^\mu Q_\mu - P^\mu P'_\mu - Q^\mu P'_\mu$$

Q3 (c)

(c) An electron and a photon of wavelength $\lambda_1 = 8\,\mu$m are moving toward each other along the x-coordinate and at some point the photon is scattered in a head on collision. Show that the wavelength of the scattered photon can be estimated as $\lambda_2 \approx \lambda_1/(4\gamma^2)$. Calculate the wavelength of the scattered photon when the initial velocity of the electron in the laboratory frame is $\beta_x = 0.999$. [6]

[*Hint: Use the fact that the energies of the photons are much smaller than the rest energy of the electron.*]

$$\begin{aligned} 0 &= P^\mu Q_\mu - P^\mu P'_\mu - Q^\mu P'_\mu \\ &= -k\left(\tfrac{E}{c} + p\right) + k'\left(\tfrac{E}{c} - p + 2k\right) \end{aligned}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$$

$$k' = \frac{\frac{E}{c} + p}{\frac{E}{c} - p + 2k}k = \frac{k(1 + \frac{pc}{E})}{1 - \frac{pc}{E} + \boxed{\frac{2kc}{E}}} \approx k\frac{E + pc}{E - pc} = k\frac{1+\beta}{1-\beta}$$

$$\frac{kc}{E} \ll 1$$

$$\lambda_2 \approx \lambda_1 \frac{1-\beta}{1+\beta} \approx \frac{\lambda_1}{4\gamma^2} \approx 0.004\ \mu\text{m}$$

Q3 (d)

(d) In the laboratory frame S, a plane monochromatic electromagnetic wave with angular frequency ω and 3-wavevector $\mathbf{k} = (k_x, 0, 0)$ propagates in vacuum. Write down a possible form of the 4-vector potential $\mathbf{A}$. Use this 4-vector potential $\mathbf{A}$ to find the components of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. [4]

$$K^\mu = (\omega/c, k_x, 0, 0)$$

$$A^\mu = (\phi/c, A_x, A_y, A_z)$$

- For an EM wave, **A** is perpendicular to **k**
- Use gauge invariance to choose simple form

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi$$

- No charges or currents → Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0$$

- Can also fix φ to be constant over time

$$\phi = 0$$

$$\boxed{\mathrm{A}^\mu = (0, 0, A_0 \cos(\omega t - k_x x), 0)}$$

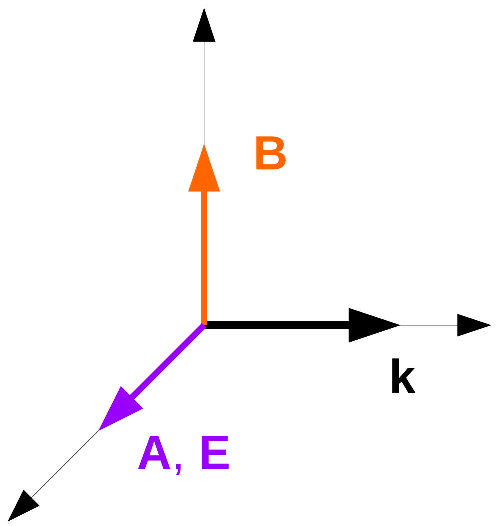
Q3 (d)

(d) In the laboratory frame S, a plane monochromatic electromagnetic wave with angular frequency ω and 3-wavevector $\mathbf{k} = (k_x, 0, 0)$ propagates in vacuum. Write down a possible form of the 4-vector potential $\mathbf{A}$. Use this 4-vector potential $\mathbf{A}$ to find the components of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. [4]

$$\mathrm{A}^{\mu} = (0, 0, A_0 \cos(\omega t - k_x x), 0)$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -A_0 \omega \sin(\omega t - k_x x)\hat{\mathbf{y}}$$

$$\mathbf{B} = \nabla \wedge \mathbf{A} = A_0 k_x \sin(\omega t - k_x x)\hat{\mathbf{z}}$$



Note that **A** and **E** are parallel to one another,
B perpendicular to them,
and all perpendicular to **k**

Q3 (e)

(e) An ultra-relativistic electron propagates with constant velocity $\mathbf{v} = (0, 0, v_z)$ along the z-axis through a periodic field. The field is defined in the laboratory frame S by a 4-vector potential with only one non-vanishing temporal component $A^\mu = (\frac{\phi}{c}, 0, 0, 0)$ where $\phi = \phi_0 \cos(k_u z)$, $k_u = 2\pi/d$ and d is the period of the field. Find the fields $\mathbf{E}$ and $\mathbf{B}$ in the laboratory frame and in the rest frame of the electron. ~~Can the field observed in the rest frame of the electron be considered as an electromagnetic wave?~~ Compare the field observed in the rest frame of the electron to that of an electromagnetic wave. [6]

Lab frame

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = \phi_0 k_u \sin(k_u z)\hat{\mathbf{z}}$$

$$\mathbf{B} = \nabla \wedge \mathbf{A} = 0$$

Electron frame

$$E'_{\|} = E_{\|} = \phi_0 k_u \sin(k_u z)$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B}) = 0$$

$$B'_{\|} = B_{\|} = 0$$

$$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2) = 0$$

What does the electron "see"?
Need to use local coordinates

Q3 (e)

(e) An ultra-relativistic electron propagates with constant velocity $\mathbf{v} = (0, 0, v_z)$ along the z-axis through a periodic field. The field is defined in the laboratory frame S by a 4-vector potential with only one non-vanishing temporal component $A^\mu = (\frac{\phi}{c}, 0, 0, 0)$ where $\phi = \phi_0 \cos(k_u z)$, $k_u = 2\pi/d$ and d is the period of the field. Find the fields $\mathbf{E}$ and $\mathbf{B}$ in the laboratory frame and in the rest frame of the electron. ~~Can the field observed in the rest frame of the electron be considered as an electromagnetic wave?~~ Compare the field observed in the rest frame of the electron to that of an electromagnetic wave. [6]

Use local coordinates:

$$z = \gamma(z' + v_z t')$$

(zero because it's the electron rest frame)

$$\mathbf{E}' = \phi_0 k_u \sin(k_u z)\hat{\mathbf{z}}$$
$$= \phi_0 k_u \sin(\gamma v_z k_u t')\hat{\mathbf{z}}$$

Frequency "blue shifted"

- Similar to EM wave:
  - electron experiences an oscillating **E** field
- Contrast:
  - **E** parallel to motion, not perpendicular
  - No **B** field
  - no momentum transfer (**E**x**B**)

End of Q3!

Q4 (a)

4. (a) ~~Define polar and axial 3-vectors. Are the electric and magnetic fields polar or axial vectors? Making reference to the Lorentz force law justify your answer.~~ Compare and contrast the Lorentz transformations of vector fields describing momentum, force, current density, and electric and magnetic fields. Explain why the electric field 3-vector is not the space part of a Lorentz 4-vector. [3]

Lorentz transformation can be represented by a 4x4 matrix $\Lambda^{\mu}{}_{\nu}$
(Linear transformation)

$$P'^{\mu} = \Lambda^{\mu}{}_{\nu} P^{\nu}$$

Momentum, force, and current density have 4-vector analogues

$$P^{\mu} = mU^{\mu} = \gamma m(c, \mathbf{v}) = (\gamma mc, \mathbf{p})$$

$$F^{\mu} = m\frac{dP^{\mu}}{d\tau}$$

$$J^{\mu} = (\rho c, \mathbf{j})$$

Longitudinal components are modified along with time.
Transverse components are unchanged.

Q4 (a)

4. (a) ~~Define polar and axial 3-vectors. Are the electric and magnetic fields polar or axial vectors? Making reference to the Lorentz force law justify your answer.~~ Compare and contrast the Lorentz transformations of vector fields describing momentum, force, current density, and electric and magnetic fields. Explain why the electric field 3-vector is not the space part of a Lorentz 4-vector. [3]

Note that 3-momentum is part of 4-momentum,
but 3-force isn't (by itself)

$$\mathbf{p} = \gamma m \mathbf{v}$$

$$\mathbf{f} = \frac{d\mathbf{p}}{dt}$$

$$F^{\mu} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \mathbf{f}\right)$$

Transformation of 4-force leads to Lorentz transformation of 3-force:

$$f'_{\|} = \frac{f_{\|} - \frac{\beta_v}{c}\frac{dE}{dt}}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}$$

$$\mathbf{f}'_{\perp} = \frac{\mathbf{f}_{\perp}}{\gamma_v\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)}$$

Both longitudinal and transverse components modified

Q4 (a)

4. (a) ~~Define polar and axial 3-vectors. Are the electric and magnetic fields polar or axial vectors? Making reference to the Lorentz force law justify your answer.~~ Compare and contrast the Lorentz transformations of vector fields describing momentum, force, current density, and electric and magnetic fields. Explain why the electric field 3-vector is not the space part of a Lorentz 4-vector. [3]

Electric and magnetic fields:

$$\mathbf{E}'_{\|} = \mathbf{E}_{\|}$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B})$$

$$\mathbf{B}'_{\|} = \mathbf{B}_{\|}$$

$$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{\mathbf{v} \wedge \mathbf{E}}{c^2})$$

Apparently opposite 4-vector behavior: longitudinal components unchanged. Transverse components look “boosted”. (Compare with 4-vector transformation)

$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(x - \beta ct)$$

E and **B** transform as part of a rank-2 antisymmetric field tensor

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$$

Q4 (b)

(b) ~~Define the 4-current J^μ and write down the 4-current continuity condition in 4-vector form. Show that the Lorentz force acting on a unit volume of charge density ρ can be written as $f_\mu = J^\nu F_{\nu\mu}$. What is the physical meaning of the f_0 component of this 4-vector?~~ Show that, and explain why, the Lorentz force law is invariant with respect to Lorentz transformations. [4]

$$\mathbf{f} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

Tensor argument: show that it's the space part of a tensor equation $F^\mu = qF^{\mu\nu}U_\nu$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad U_\nu = \gamma \begin{pmatrix} -c \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

Remember use metric to lower index!

Seems pretty obvious except for this

Q4 (b)

(b) ~~Define the 4-current J^μ and write down the 4-current continuity condition in 4-vector form. Show that the Lorentz force acting on a unit volume of charge density ρ can be written as $f_\mu = J^\nu F_{\nu\mu}$. What is the physical meaning of the f_0 component of this 4-vector?~~ Show that, and explain why, the Lorentz force law is invariant with respect to Lorentz transformations. [4]

$$F^\mu = qF^{\mu\nu}U_\nu = (\gamma q(\mathbf{v}\cdot\mathbf{E})/c, \boxed{\gamma q(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})})$$

Recall that 3-force isn't identical
to the space part of the 4-force

$$\mathbf{f} = q(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})$$

$$F^\mu = \left(\frac{\gamma}{c}\frac{dE}{dt}, \boxed{\gamma\mathbf{f}}\right)$$

Lorentz transformation → tensor equation unchanged → 3-vector (derived) equation unchanged

You can of course also show this using the Lorentz transformations of **E** and **B** directly

Q4 (c)

(c) ~~Electromagnetic waves can be described using a 4-vector potential~~

$$\cancel{A^\mu = \left(0, A_0 \cos[K^\mu X_\mu], A_0 \sin[K^\mu X_\mu], 0\right)}$$

~~where $\mathbf{X} = (ct, x, y, z)$ and $\mathbf{K} = (\omega/c, 0, 0, k_z)$. Using the 4-vector potential A^μ, calculate the components of the field strength tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$.~~ In light of the Maxwell equations and the Lorentz force law, show that the electric and magnetic fields behave differently under the parity transformation. What does this imply about the behaviour of electric and (hypothetical) magnetic charge densities? [4]

Under space inversion:

$$\mathbf{x} \to -\mathbf{x}$$

$$\mathbf{v} \to -\mathbf{v}$$

$$\mathbf{f} \to -\mathbf{f}$$

$$\nabla \to -\nabla$$

$$\mathbf{f} = q(\boxed{\mathbf{E}} + \boxed{\mathbf{v} \wedge \mathbf{B}})$$

$$\boxed{\mathbf{E} \to -\mathbf{E}}$$

$$\boxed{\mathbf{B} \to \mathbf{B}}$$

Can also see difference in $\nabla \wedge \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0$

Q4 (c)

(c) ~~Electromagnetic waves can be described using a 4 vector potential~~

$$\cancel{A^\mu = \Big(0, A_0 \cos[K^\mu X_\mu], A_0 \sin[K^\mu X_\mu], 0\Big)}$$

~~where $\mathbf{X} = (ct, x, y, z)$ and $\mathbf{K} = (\omega/c, 0, 0, k_z)$. Using the 4-vector potential A^μ, calculate the components of the field strength tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$.~~ In light of the Maxwell equations and the Lorentz force law, show that the electric and magnetic fields behave differently under the parity transformation. What does this imply about the behaviour of electric and (hypothetical) magnetic charge densities? [4]

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad \Longrightarrow \quad \rho \to \rho$$

Electric charge density is a normal scalar

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{B} \propto \rho_m$$

$$\mathbf{B} \to \mathbf{B}$$

$$\nabla \to -\nabla \quad \Longrightarrow \quad \rho_m \to -\rho_m$$

Magnetic charge density is a pseudo-scalar

Q4 (d)

(d) Frame S' moves with a constant 3-velocity $\mathbf{v}$ relative to the laboratory frame S. In S, the components of the electric field and the magnetic field are $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$. Write down the form of the transformed electric and magnetic field in the frame S'. [4]

Components parallel to **v**

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$$

$$\beta = \frac{|\mathbf{v}|}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Components perpendicular to **v**

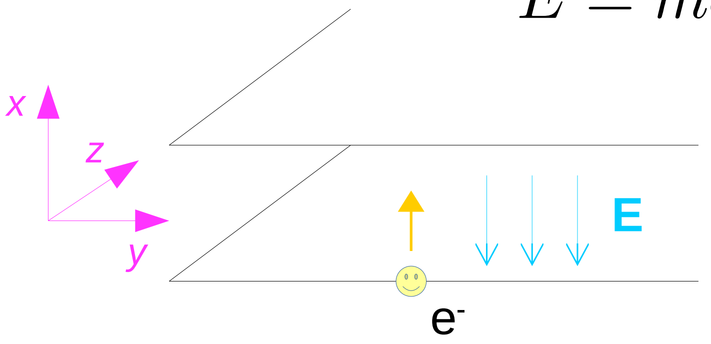
$$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{\mathbf{v}\wedge\mathbf{E}}{c^2})$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v}\wedge\mathbf{B})$$

Remember to define all the terms!

Q4 (e)

(e) An isolated parallel plate capacitor is at rest in the laboratory frame. The plates of the capacitors are parallel to the yz-plane in the laboratory frame S. The distance between the plates in this frame is $x_0 = d$. The capacitor's proper dimensions are fixed. An electron is launched into the gap between the plates from the surface of the plate with negative charge. The electron has zero initial velocity. The electric field in the capacitor between the plates is equal $100\,\mathrm{MV\,m^{-1}}$ and the distance between the plates is $0.1\,\mathrm{m}$. (i) Find the electron energy and velocity at the second plate of the capacitor; (ii) Is it possible to prevent the electron from hitting the positive plate of the capacitor by applying magnetic field. Explain the answer; (iii) Suggest the direction of the magnetic field required to prevent the electron beam from reaching the second plate; (iv) Find the strength of the magnetic field needed to prevent the electron reaching the second plate; (v) Describe and sketch the electron trajectory in the combined fields. [10]

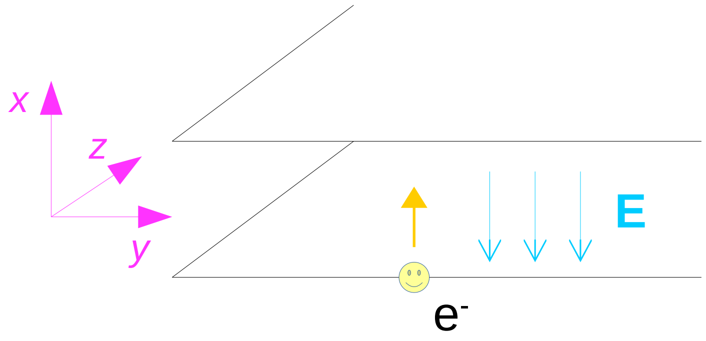


$$E = mc^2 + \Delta E = 0.511\text{ MeV} + (100\text{ MV/m})(0.1\text{ m})e$$
$$= 10.511\text{ MeV}$$
$$\gamma = E/mc^2 \approx 20.569$$
$$\beta = 0.99882$$

Pretty much like Q1(d)

Q4 (e)

(ii) Is it possible to prevent the electron from hitting the positive plate of the capacitor by applying magnetic field. Explain the answer.



$$\mathbf{f} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

Early **v** in +x direction

B field along y or z directions would deflect and bend trajectory back

(iii) Suggest the direction of the magnetic field required to prevent the electron beam from reaching the second plate.

See above. Any direction in yz plane would work (symmetric around x axis).

Q4 (e) (iv)

Find the strength of the magnetic field needed to prevent the electron reaching the second plate.

- Uniform **B** field → circular motion
- In that frame S', **B**≠0 and **E**=0
- Field invariant
- ➔ Therefore |**B**| > |**E**|/c in any frame

$$F^{\mu\nu}F_{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right)$$

S' velocity in S (not electron **v**!)

Choose

$$\mathbf{B} = B\hat{\mathbf{y}}$$

$$0 = \mathbf{E}'_\perp = \gamma_u(\mathbf{E}_\perp + \mathbf{u} \wedge \mathbf{B})$$

$$0 = -E\hat{\mathbf{x}} + u_z B(\hat{\mathbf{z}} \wedge \hat{\mathbf{y}})$$

$$u_z = -\frac{E}{B}$$

S' speed determined by **B** strength

Q4 (e) (iv)

Find the strength of the magnetic field needed to prevent the electron reaching the second plate.

Find **B**' strength in frame with no **E**

$$\mathbf{E} = -E\hat{\mathbf{x}}$$

$$\mathbf{B} = B\hat{\mathbf{y}}$$

$$\mathbf{u} = -\frac{E}{B}\hat{\mathbf{z}}$$

$$\mathbf{E}'_{||} = \mathbf{E}_{||} = 0$$

$$\mathbf{B}'_{||} = \mathbf{B}_{||} = 0$$

$$\begin{aligned}\mathbf{B}'_{\perp} &= \gamma_u \left(\mathbf{B}_{\perp} - \frac{\mathbf{u} \wedge \mathbf{E}}{c^2} \right) \\ &= \gamma_u B \left(1 - \frac{E^2}{c^2 B^2} \right) \hat{\mathbf{y}} \\ &= \frac{B}{\gamma_u} \hat{\mathbf{y}}\end{aligned}$$

$$\gamma_u = \left(1 - \frac{E^2}{c^2 B^2} \right)^{-1/2}$$

Q4 (e) (iv)

Find the strength of the magnetic field needed to prevent the electron reaching the second plate.

Note that this is for the electron, not the frame

$$\gamma' m \frac{d\mathbf{v}'}{dt} = -e(\dot{x}, \dot{y}, \dot{z}) \wedge (0, B', 0) = eB'(\dot{z}, 0, -\dot{x})$$

$$\ddot{x} = \frac{eB'}{\gamma' m}\dot{z}$$

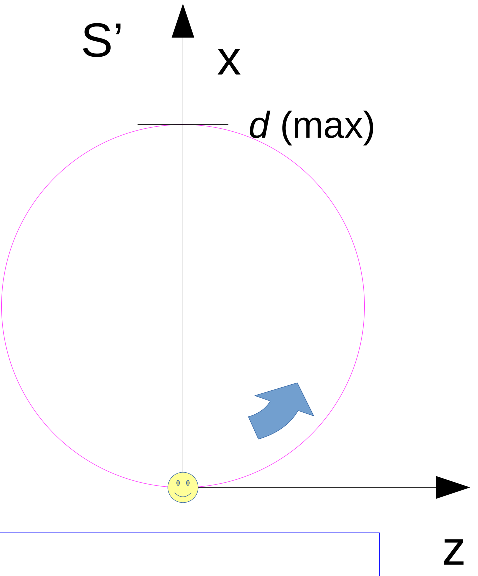
$$\ddot{z} = -\frac{eB'}{\gamma' m}\dot{x}$$

$$\omega = \frac{eB'}{\gamma' m}$$

$$x(t) = \frac{d}{2}(1 - \cos\omega t)$$

$$z(t) = \frac{d}{2}\sin\omega t$$

$$\dot{z}(t=0) = \beta c = \frac{deB'}{2\gamma' m}$$



How do we calculate this?

$$p = \beta'\gamma' mc = \frac{d}{2}eB'$$

Larmor radius for radius $d/2$

Q4 (e) (iv)

Find the strength of the magnetic field needed to prevent the electron reaching the second plate.

Electron initially at rest in S
→ initial momentum **p** in S' is from boost

$$\beta' = \beta_u, \gamma' = \gamma_u$$

$$p = \frac{d}{2}eB' = \frac{deB}{2\gamma_u}$$

$$\beta'\gamma' mc = \frac{E}{cB}\left(1 - \frac{E^2}{c^2B^2}\right)^{-1/2} mc = \frac{deB}{2}\left(1 - \frac{E^2}{c^2B^2}\right)^{1/2}$$

$$\Rightarrow \quad B^2 = \frac{E^2}{c^2} + \frac{2mc}{de}\frac{E}{c}$$

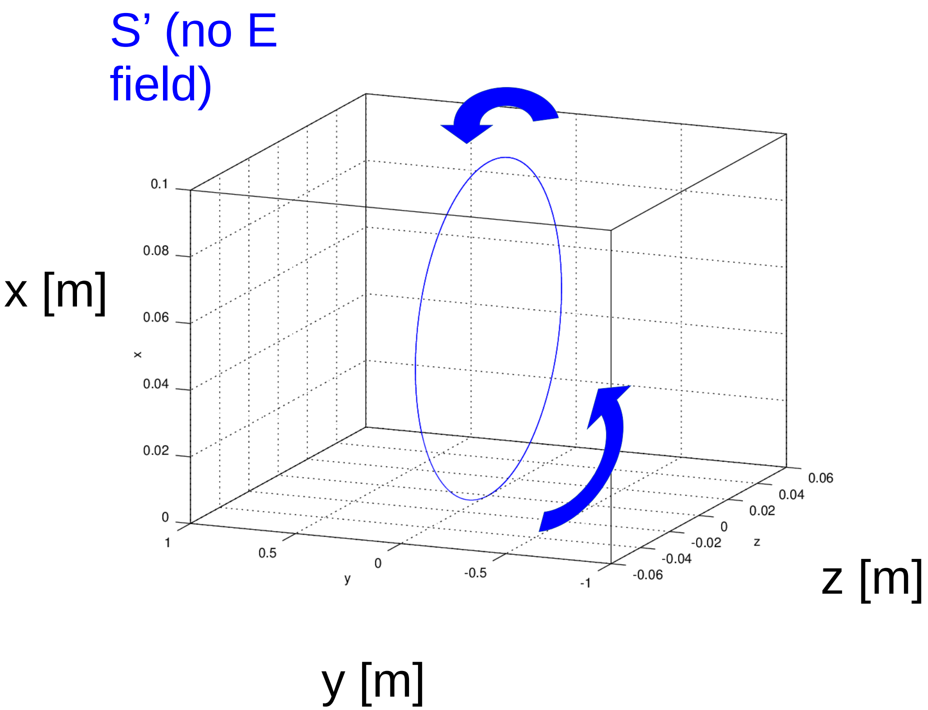
$$B \approx 0.35 \text{ T}$$ (minimum)

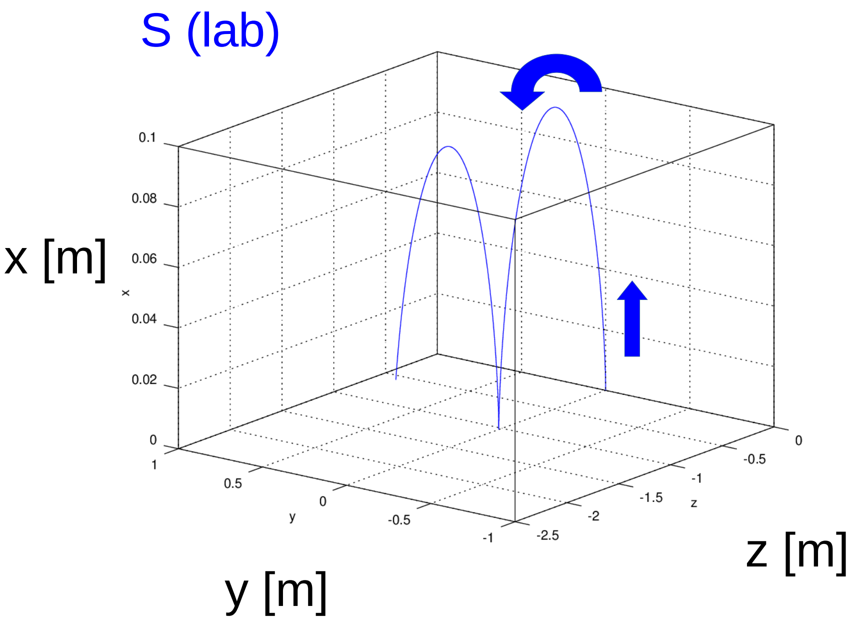
Note $\frac{E}{c} = \frac{100 \times 10^6 \text{ V/m}}{3 \times 10^8 \text{ m/s}} = \frac{1}{3} \text{ T}$

$$\frac{2mc}{de} = \frac{2(0.511 \times 10^6 \text{ V})}{(0.1 \text{ m})(3 \times 10^8 \text{ m/s})} \approx 0.034 \text{ T}$$

Q4 (e) (v)

Describe and sketch the electron trajectory in the combined fields.





Mock exam question

- No previous exam questions, so we'll pretend
  - Hopefully this will provide further illustration of the concepts
- Let $X \equiv x^\mu \sigma_\mu$
  - $\sigma_0=I$ identity matrix, σ_i 2x2 Pauli matrices
  - x^μ are components of a 4-vector
  - Derive an expression relating the determinant of X to the length of x

Mock exam question

- Derive an expression relating the determinant of X to the length of x

$$X \equiv x^\mu \sigma_\mu = \begin{pmatrix} ct + z & x - iy \\ x + iy & ct - z \end{pmatrix}$$

$$\det(x^\mu \sigma_\mu) = (ct + z)(ct - z) - (x - iy)(x + iy)$$
$$= c^2 t^2 - z^2 - x^2 - y^2 = -x^\mu x_\mu$$

X is another representation of a 4-vector,
as a 2x2 matrix rather than a column vector

Mock exam question

- Let M be an arbitrary 2x2 complex matrix with det(M) = 1
- Define the transformed matrix $X' = MXM^\dagger$
- Show that the length of the corresponding 4-vector is unchanged

$$\det X' = (\det M)(\det X)(\det M^\dagger) = \det X$$

$$\Rightarrow (x')^\mu (x')_\mu = x^\mu x_\mu$$

Mock exam question

- Show that the set of M matrices forms a group under matrix multiplication
  - Closure: $\det(M_1 M_2) = (\det M_1)(\det M_2) = 1$
  - Associativity: same as for matrix multiplication
  - Identity element: M=I (same as for matrices)
  - Inverse: $$\det(M^{-1}) = \frac{1}{\det M} = 1$$
- In fact the group is the "Special Linear" group SL(2,c)

Mock exam question

- How many degrees of freedom does M have?
  - 8 components (2 per element)
  - 2 constraints → 6 real parameters
- Interpretation?
  - Transformation leaves length of a 4-vector unchanged, but changes its components → Lorentz transformation
  - Full Lorentz group also has 6 real parameters: 3 for rotations, 3 for boosts

Mock exam question

- Now let’s look at the subset of matrices which are unitary. You can assume it’s a subgroup.
- Show that elements of the unitary subgroup leaves the time component of the 4-vector unchanged

$$\begin{aligned} UXU^\dagger &= Ux^\mu \sigma_\mu U^\dagger \\ &= Ux^0 U^\dagger + Ux^j \sigma_j U^\dagger \\ &= x^0 I + Ux^j \sigma_j U^\dagger \end{aligned}$$

Mock exam question

- How many degrees of freedom does U have?
  - Unitarity condition → 4 constraints
  - But remember also that we required detU=1 → 3 real parameters
- Interpretation?
  - U only affects spatial components, but leaves length unchanged → 3D rotations (also only 3 real angle parameters)

Mock exam question

- A further comment on SU(2), the group of U: one also finds that the Pauli matrices are related to the generators of SU(2)
- Exponentiation, e.g.:

$$R_1(\theta) = e^{-i\theta J_1} = e^{-i\theta\sigma_1/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

- This is just a spin-1/2 rep: need 4π rotation!