B2
Symmetry and Relativity
Revision 1
TT 2024

"Open book" B2 2018 Q1

1. (a) Two events in the laboratory frame S are characterized by 4-vectors $\mathbf{D} = (ct_d, \mathbf{x}_d)$ and $\mathbf{G} = (ct_g, \mathbf{x}_g)$ where $\mathbf{x}_d$ and $\mathbf{x}_g$ are the corresponding 3-vectors. Write down the conditions for these events to be connected by space-like and time-like intervals. Can two events be connected by a null vector? Explain the answer and draw a schematic (ct, x) diagram indicating all significant features.

Q1 (a)

Consider two arbitrary 4-vectors whose dot product is zero, i.e. $Y^\mu X_\mu = 0$. ~~List the conditions under which the dot product of the 4-vectors is equal to zero.~~ Under what conditions is the inner product zero when the vectors are different combinations of timelike, spacelike, and null?

~~Consider the following 4-vectors: the 4-acceleration **A**, the 4-wavevector **K**, and the 4-current **J**. For which pairs of these vectors are the dot products equal to zero?~~ Justify your answers. [7]

Q1 (b)

(b) For a particle of mass m moving along a world-line X^μ in the inertial reference frame S, define the 4-momentum $\mathbf{P}$ and 4-acceleration $\mathbf{A}$. Using components of 4-momentum $\mathbf{P}$ calculate the dot product $P^\mu P_\mu$. Show that if 3-velocity and 3-acceleration are parallel to each other then the 4 acceleration invariant is $A^\mu A_\mu = \gamma^6 a^2$. [6]

Q1 (c)

(c) Consider the motion of a particle under a pure (rest mass preserving), inverse square law force $\mathbf{f} = \alpha\mathbf{r}/r^3$ where α is a constant. Taking into account that $\mathbf{f}$ is a central force, derive the energy conservation equation showing that $\gamma mc^2 - \alpha/r$ is constant. ~~Give an example of such a force.~~ Give examples of such a force. [5]

Q1 (d)

(d) An electron is accelerated from rest across a gap of $L = 5\,\mathrm{m}$ by a constant electric field of strength $10\,\mathrm{MV\,m^{-1}}$. Find γ at the other end of the gap. Calculate how long it takes for the electron to reach the other end of the gap. Sketch $\gamma(t)$ and $\beta(t)$, indicating the significant values. How much will β change if the accelerating field is reduced by 20%? If, instead of an electron, a proton is accelerated, how will γ and β change? [7]

Q2 (a)

2. (a) A photon can be defined by a 4-wavevector $\mathbf{K}$. Write down the components of this 4-vector and the relationship between them. Considering a photon propagating in a vacuum, define its phase and group velocities in terms of 4-wavevector $\mathbf{K}$ components. Show that the phase ϕ of the wave is Lorentz invariant. Is the phase velocity v_{ph} Lorentz invariant? Explain your answer. [6]

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Q2 (b)

(b) Can a single photon in a vacuum decay into a single massive particle with mass $m \neq 0$ and another photon? Prove your answer. Show that an electron-positron pair can be produced during collisions of photons. Find the minimum number of the photons required and find the minimal energy required for the electron-positron pair to appear, assuming that the photons participating in the collision have the same frequency. [7]

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Q2 (c)

(c) A free electron, with initial velocity in the x direction $\mathbf{v}_0 = (v_0, 0, 0)$ is injected into a vacuum vessel. Show and prove which components of the electron 4-momentum are conserved if inside the vessel there is: (i) a constant electric field (no magnetic field) $\mathbf{E} = (E_x, 0, 0)$ directed along the electron initial velocity; (ii) the magnetic field $\mathbf{B} = (0, B_y, 0)$ with magnetic field lines perpendicular to the electron initial velocity and directed along the y coordinate (no electric field). [5]

Q2 (d)

(d) A plane mirror moves uniformly in the direction of its normal $\mathbf{x}_0$ in a laboratory frame S with velocity $\beta_x = v_x/c = 0.99$. A photon has wavelength $\lambda_1 = 1\,\mu\text{m}$ in the laboratory stationary frame and 3-wavevector $\mathbf{k} = (-k_x, k_y, 0)$ with $|k_x| = |k_y|$ in the frame co-moving with the mirror. Along the x-coordinate it moves in the opposite direction to the mirror. The photon is reflected by the mirror. Find (in the laboratory frame) the angle of reflection and the measured wavelength of the reflected photon, λ_2.

[7]

Q3 (a)

3. (a) A particle moves along world-line $X^\mu = X^\mu(\tau)$ in the inertial reference frame S. Does special relativity place any bounds on the possible sizes of forces and accelerations for particles of mass m? Justify your answer. Does special relativity place any bounds on the phase velocity of the electromagnetic wave? Derive an expression for the photon's frequency shift during Compton scattering. [6]

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Q3 (b)

(b) In the laboratory frame S two photons propagate along the x-coordinate, separated by a distance x_0. Calculate the distance between the photons in frame S' moving along the x-coordinate with velocity $\mathbf{v} = (v_x, 0, 0)$. [3]

Q3 (c)

(c) An electron and a photon of wavelength $\lambda_1 = 8\,\mu$m are moving toward each other along the x-coordinate and at some point the photon is scattered in a head on collision. Show that the wavelength of the scattered photon can be estimated as $\lambda_2 \approx \lambda_1/(4\gamma^2)$. Calculate the wavelength of the scattered photon when the initial velocity of the electron in the laboratory frame is $\beta_x = 0.999$. [6]

[*Hint: Use the fact that the energies of the photons are much smaller than the rest energy of the electron.*]

Q3 (d)

(d) In the laboratory frame S, a plane monochromatic electromagnetic wave with angular frequency ω and 3-wavevector $\mathbf{k} = (k_x, 0, 0)$ propagates in vacuum. Write down a possible form of the 4-vector potential $\mathbf{A}$. Use this 4-vector potential $\mathbf{A}$ to find the components of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. [4]

Q3 (e)

(e) An ultra-relativistic electron propagates with constant velocity $\mathbf{v} = (0, 0, v_z)$ along the z-axis through a periodic field. The field is defined in the laboratory frame S by a 4-vector potential with only one non-vanishing temporal component $A^\mu = (\frac{\phi}{c}, 0, 0, 0)$ where $\phi = \phi_0 \cos(k_u z)$, $k_u = 2\pi/d$ and d is the period of the field. Find the fields $\mathbf{E}$ and $\mathbf{B}$ in the laboratory frame and in the rest frame of the electron. ~~Can the field observed in the rest frame of the electron be considered as an electromagnetic wave?~~ Compare the field observed in the rest frame of the electron to that of an electromagnetic wave. [6]

Q4 (a)

4. (a) ~~Define polar and axial 3-vectors. Are the electric and magnetic fields polar or axial vectors? Making reference to the Lorentz force law justify your answer.~~ Compare and contrast the Lorentz transformations of vector fields describing momentum, force, current density, and electric and magnetic fields. Explain why the electric field 3-vector is not the space part of a Lorentz 4-vector. [3]

Q4 (b)

(b) ~~Define the 4-current J^μ and write down the 4-current continuity condition in 4-vector form. Show that the Lorentz force acting on a unit volume of charge density ρ can be written as $f_\mu = J^\nu F_{\nu\mu}$. What is the physical meaning of the f_0 component of this 4-vector?~~ Show that, and explain why, the Lorentz force law is invariant with respect to Lorentz transformations. [4]

Q4 (c)

(c) ~~Electromagnetic waves can be described using a 4-vector potential~~

$$\cancel{A^\mu = \Big(0, A_0 \cos[K^\mu X_\mu], A_0 \sin[K^\mu X_\mu], 0\Big)}$$

~~where~~ $\cancel{\mathbf{X} = (ct, x, y, z)}$ ~~and~~ $\cancel{\mathbf{K} = (\omega/c, 0, 0, k_z)}$~~. Using the 4-vector potential~~ $\cancel{A^\mu}$~~, calculate the components of the field strength tensor~~ $\cancel{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha}$~~.~~ In light of the Maxwell equations and the Lorentz force law, show that the electric and magnetic fields behave differently under the parity transformation. What does this imply about the behaviour of electric and (hypothetical) magnetic charge densities? [4]

Q4 (d)

(d) Frame S' moves with a constant 3-velocity $\mathbf{v}$ relative to the laboratory frame S. In S, the components of the electric field and the magnetic field are $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$. Write down the form of the transformed electric and magnetic field in the frame S'. [4]

Q4 (e)

(e) An isolated parallel plate capacitor is at rest in the laboratory frame. The plates of the capacitors are parallel to the yz-plane in the laboratory frame S. The distance between the plates in this frame is $x_0 = d$. The capacitor's proper dimensions are fixed. An electron is launched into the gap between the plates from the surface of the plate with negative charge. The electron has zero initial velocity. The electric field in the capacitor between the plates is equal $100\,\mathrm{MV\,m^{-1}}$ and the distance between the plates is $0.1\,\mathrm{m}$. (i) Find the electron energy and velocity at the second plate of the capacitor; (ii) Is it possible to prevent the electron from hitting the positive plate of the capacitor by applying magnetic field. Explain the answer; (iii) Suggest the direction of the magnetic field required to prevent the electron beam from reaching the second plate; (iv) Find the strength of the magnetic field needed to prevent the electron reaching the second plate; (v) Describe and sketch the electron trajectory in the combined fields. [10]

Mock exam question

- No previous exam questions, so we'll pretend
  - Hopefully this will provide further illustration of the concepts
- Let $X \equiv x^{\mu}\sigma_{\mu}$
  - σ_0=I identity matrix, σ_i 2x2 Pauli matrices
  - x^{μ} are components of a 4-vector
  - Derive an expression relating the determinant of X to the length of x

Mock exam question

- Derive an expression relating the determinant of X to the length of x

$$X \equiv x^\mu \sigma_\mu = \begin{pmatrix} ct+z & x-iy \\ x+iy & ct-z \end{pmatrix}$$

$$\det(x^\mu \sigma_\mu) = (ct+z)(ct-z) - (x-iy)(x+iy)$$
$$= c^2t^2 - z^2 - x^2 - y^2 = -x^\mu x_\mu$$

X is another representation of a 4-vector,
as a 2x2 matrix rather than a column vector

Mock exam question

- Let M be an arbitrary 2x2 complex matrix with det(M) = 1
- Define the transformed matrix $X' = MXM^\dagger$
- Show that the length of the corresponding 4-vector is unchanged

$$\det X' = (\det M)(\det X)(\det M^\dagger) = \det X$$

$$\Rightarrow (x')^\mu (x')_\mu = x^\mu x_\mu$$

Mock exam question

- Show that the set of M matrices forms a group under matrix multiplication
  - Closure: $\det(M_1 M_2) = (\det M_1)(\det M_2) = 1$
  - Associativity: same as for matrix multiplication
  - Identity element: M=I (same as for matrices)
  - Inverse: $$\det(M^{-1}) = \frac{1}{\det M} = 1$$
- In fact the group is the “Special Linear” group SL(2,c)

Mock exam question

- How many degrees of freedom does M have?
  - 8 components (2 per element)
  - 2 constraints → 6 real parameters
- Interpretation?
  - Transformation leaves length of a 4-vector unchanged, but changes its components → Lorentz transformation
  - Full Lorentz group also has 6 real parameters: 3 for rotations, 3 for boosts

Mock exam question

- Now let's look at the subset of matrices which are unitary. You can assume it's a subgroup.
- Show that elements of the unitary subgroup leaves the time component of the 4-vector unchanged

$$
\begin{aligned}
UXU^\dagger &= Ux^\mu \sigma_\mu U^\dagger \\
&= Ux^0 U^\dagger + Ux^j \sigma_j U^\dagger \\
&= x^0 I + Ux^j \sigma_j U^\dagger
\end{aligned}
$$

Mock exam question

- How many degrees of freedom does U have?
  - Unitarity condition → 4 constraints
  - But remember also that we required detU=1 → 3 real parameters
- Interpretation?
  - U only affects spatial components, but leaves length unchanged → 3D rotations (also only 3 real angle parameters)

Mock exam question

- A further comment on SU(2), the group of U: one also finds that the Pauli matrices are related to the generators of SU(2)
- Exponentiation, e.g.:

$$R_1(\theta) = e^{-i\theta J_1} = e^{-i\theta\sigma_1/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

- This is just a spin-1/2 rep: need 4π rotation!