CP1 Collection Answers
HT 2008

Q1. For time dilation, remember that $\Delta x = 0$ in the object's rest frame. Then a direct application of the Lorentz transformation to the difference in time coordinates should give $\Delta t' = \gamma \Delta t$.

Lorentz contraction (not part of question): remember that lengths are measured by taking position measurements at the same time in the measuring frame. Therefore if S is the rest frame of the object and S' the lab frame,

$$\begin{aligned} L &= x_2 - x_1 \\ t_2 &= t_1 \\ L' &= x'_2 - x'_1 \\ t'_2 &= t'_1 \end{aligned}$$

and this should reduce to $L' = L/\gamma$.

The time experienced by the π mesons in their 100 m at $\beta = 0.98$ is $L/\beta\gamma c$, so the number of half-lifes is $L/\beta\gamma c\tau$, where $\tau = 2.6 \times 10^-8$ s. Therefore the fraction surviving is

$$\frac{N(t)}{N_0} = 2^{-L/\beta\gamma c\tau} = 16.5\%$$

($\gamma = 5.0$, number of half-lives is 2.6)

Q2. At the maximum height, all the kinetic energy has been changed into potential energy, therefore $\frac{1}{2}mv_0^2 = mgh$, from which the answer follows.

(a) The total energy of the particle at any point of its trajectory is $E = \frac{1}{2}mv^2 + mgy$, so $dE = \frac{1}{2}m \cdot d(v^2) + mg \cdot dy$. This is zero without air resistance, but here we are given that the force of air resistance is proportional to v^2. Since work is $F\Delta x$, we can write $dE = -\kappa v^2 \cdot dy$, where κ is some constant which later turns out to be km.

(b) The equation is a first-order linear inhomogeneous ODE. Use the formula; or, if you don't remember it, note that

$$\frac{d}{dy}v^2 e^{2ky} = \frac{dv^2}{dy}e^{2ky} + 2kv^2 e^{2ky},$$

so multiply the original ODE by e^{2ky}, rearrange the terms, and integrate. Note the boundary conditions $v^2 = v_0^2$ at $y = 0$ and $v^2 = 0$ at $y = h'$.

(c) $h = 510$ m, $h' = 350$ m.

(d) The terminal velocity is 99 m/s, which is less than the velocity without air resistance (83 m/s), so we would expect that the speed of arrival is somewhat less than about 80 m/s.

Q3. Using a trial solution $e^{\alpha t}$ to the differential equation, we find an equation

$$\alpha^2 + \alpha\gamma + \omega^2 = 0$$

and thus

$$\alpha = -\frac{\gamma}{2} \pm i\omega_0\sqrt{1 - \left(\frac{\gamma}{2\omega_0}\right)^2}$$

(a) From this, we find that the amplitude decreases exponentially as

$$A(t) = A_0 e^{\gamma t/2}$$

(b) We also get the angular frequency from the imaginary part:

$$\begin{aligned}
\omega_f &= \omega_0\sqrt{1 - \left(\frac{\gamma}{2\omega_0}\right)^2} \\
&= \omega_0\left(1 - \frac{1}{2}\left(\frac{\gamma}{2\omega_0}\right)^2 + \ldots\right) \\
&\approx \omega_0\left(1 - \frac{\gamma^2}{8\omega_0^2}\right)
\end{aligned}$$

(c) At maximum displacement, the total energy is $E = \frac{1}{2}kA^2$, so it falls as $e^{-\gamma t}$.

Q4. Energy and momentum conservation equations lead to $m_1/m_2 = 1/3$. With this mass ratio, we get the center-of-mass velocity to be $\frac{1}{4}u$.

The kinetic energies and the total energy of the particles in the center of mass frame after the collision are

$$\begin{aligned}
T_1' &= \frac{9}{32}m_1u^2 \\
T_2' &= \frac{3}{32}m_1u^2 \\
E_{total}' &= T_1' + T_2' = \frac{3}{8}m_1u^2
\end{aligned}$$

This total energy can be checked against the total energy in the lab frame by adding the kinetic energy of the center of mass system.

After the collision with m_3, the velocity of m_1 would be $(7/4)u$. It makes up the distance with m_2, including the $2vt$ separation at the moment of the collision with m_3, in time $(4/5)t$.