CP1 Mechanics Collection Answers

TT 2010

CM1. (a) Mass m, initial velocity v at angle α above the horizontal. Define the coordinate system such that the mass starts at the origin. Horizontal motion:

$$x(t) = vt\cos\alpha$$

Vertical motion (from $F = ma$):

$$\begin{aligned} m\frac{d^2y}{dt^2} &= -mg \\ y(t) &= vt\sin\alpha - \frac{1}{2}gt^2 \end{aligned}$$

Note that $y(0) = 0$ is already incorporated into $y(t)$.

The condition for the mass hitting the ground:

$$\begin{aligned} 0 &= y(t_1) \\ &= t_1(v\sin\alpha - \frac{1}{2}gt_1) \\ t_1 &= \frac{2v}{g}\sin\alpha \end{aligned}$$

This is the time taken to return to the horizontal surface. The range is thus

$$x(t_1) = vt_1\cos\alpha = \frac{2v^2}{g}\sin\alpha\cos\alpha = \frac{v^2}{g}\sin 2\alpha$$

The maximum height is found when

$$\begin{aligned} 0 &= \frac{dy}{dt} \\ &= v\sin\alpha - gt \end{aligned}$$

so the time is

$$t_{1/2} = \frac{v}{g}\sin\alpha$$

(This can also be found simply as half the time to fall to the surface.) Plugging into y:

$$\begin{aligned} y_{max} &= y(t_{1/2}) \\ &= \frac{v^2}{2g}\sin^2\alpha \end{aligned}$$

(b) Impact parameter here is the distance between the target (Jupiter) and a straight-line trajectory. Energy and angular momentum far away from Jupiter are

$$\begin{aligned} L &= mvb \\ E &= \frac{1}{2}mv^2 \end{aligned}$$

where m is the mass of the satellite. At the point of closest approach (radius r_0)

$$\begin{aligned} L &= m v_0 r_0 \\ E &= \frac{1}{2} m v_0^2 - \frac{GMm}{r_0} \end{aligned}$$

where v_0 is the velocity at the point of closest approach and M is the mass of Jupiter. We use conservation of L to obtain $v_0 = vb/r_0$, which we then plug into the energy conservation equation:

$$\begin{aligned} 0 &= r_0^2 v^2 + r_0 2GM - v^2 b^2 \\ r_0 &= [\sqrt{(G^2M^2 + v^4 b^2)} - GM]/v^2 \end{aligned}$$

(c) In circular orbit, the centripetal acceleration is

$$\begin{aligned} \frac{GM}{r_0^2} &= \frac{v_0^2}{r_0} \\ v_0^2 &= \frac{GM}{r_0} \end{aligned}$$

After the boost, the angular momentum at perigee is $L = m v_p r_0$, and at apogee $L = m v_a r_a$. Equating the two gives $v_a = v_p r_0 / r_a$. Energy conservation, in the meantime, gives

$$\begin{aligned} E = \frac{1}{2} m v_p^2 - \frac{GMm}{r_0} &= \frac{1}{2} m v_a^2 - \frac{GMm}{r_a} \\ &= \frac{1}{2} m v_p^2 \left(\frac{r_0}{r_a}\right)^2 - \frac{GMm}{r_a} \\ v_p^2 \left(1 - \left(\frac{r_0}{r_a}\right)^2\right) &= \frac{2GM}{r_0} - \frac{2GM}{r_a} \\ &= 2 v_0^2 \left(1 - \frac{r_0}{r_a}\right) \end{aligned}$$

where we've solved for v_p and used the formula for v_0^2 from the earlier circular orbit. This equation simplifies to

$$\left(\frac{v_p}{v_a}\right)^2 = \frac{2 r_a}{r_a + r_0}$$

as required.

CM2. (a) The kinetic energy of the box is

$$T_{box} = \frac{1}{2} M \dot{x}^2$$

The position of the pendulum is $(x + \ell \sin\theta, \ell(1 - \cos\theta))$. The kinetic energy is therefore

$$\begin{aligned} T_m &= \frac{1}{2} m \left((\dot{x} + \ell \cos\theta)^2 + (\ell \sin\theta \dot{\theta})^2 \right) \\ &= \frac{1}{2} m (\dot{x}^2 + \ell^2 \dot{\theta}^2 + 2 \dot{x} \dot{\theta} \ell \cos\theta \end{aligned}$$

while the potential energy is

$$V_m = mg\ell(1 - \cos\theta)$$

Therefore the Lagrangian is

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\dot{x}\dot{\theta}\cos\theta - mg\ell + ml\ell\cos\theta$$

The equation of motion for x is

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \\ &= \frac{d}{dt}\Big((M+m)\dot{x} + m\ell\dot{\theta}\cos\theta\Big) \\ &= (M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta \end{aligned}$$

which is equivalent to saying that the linear momentum of the total center of mass of the system is conserved. This can be checked using the definition of the horizontal center-of-mass position

$$x_{cm} = \frac{Mx + m(x + \ell\sin\theta)}{M+m}$$

and differentiating.

The equation of motion for θ is

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \\ 0 &= \ell\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta \end{aligned}$$

(b) If we assume that both θ and $\dot{\theta}$ are small, these two equations reduce to

$$\begin{aligned} 0 &= (M+m)\ddot{x} + m\ell\ddot{\theta} \\ 0 &= \ell\ddot{\theta} + \ddot{x} + g\theta \end{aligned}$$

where we've taken $\cos\theta \longrightarrow 1$ and $\sin\theta \longrightarrow \theta$. Eliminating $\ddot{x}$, we get the equation

$$0 = \ddot{\theta} + \left(\frac{M+m}{M}\frac{g}{\ell}\right)\theta$$

which gives the frequency

$$\omega = \left(\frac{M+m}{M}\right)^{\frac{1}{2}}\left(\frac{g}{\ell}\right)^{\frac{1}{2}}$$

CM3. Invariant quantity: a scalar value which doesn't change when a Lorentz transformation is applied. The rest mass is the invariant of the 4-momentum vector, for which the components are $(E, \vec{p})$.

We define the 4-momenta $P_1 = (E_1, p_1)$ and $P_2 = (E_2, -E_2)$, recalling that for a photon, the magnitude of the momentum is equal to the energy (within a factor of c). The final momenta are P_e and $P_\gamma = (E, \pm E)$.

$$\begin{aligned}
P_e + P_\gamma &= P_1 + P_2 \\
P_e &= P_1 + P_2 - P_\gamma \\
m^2 &= m^2 + 2P_1P_2 - 2P_1P_\gamma - 2P_2P_\gamma \\
0 &= P_1P_2 - P_1P_\gamma - P_2P_\gamma \\
&= E_1E_2 + p_1E_2 - EE_1 + p_1p_\gamma - EE_2 - E_2p_\gamma
\end{aligned}$$

At this point, we have to choose a sign for p_γ. If we assume $p_\gamma < 0$ (photon still travelling backwards), we find that $E = E_2$, or in other words, there wasn't actually an interaction. If we assume $p_\gamma > 0$, then we find

$$\begin{aligned}
0 &= (E_1 + p_1)E_2 - E(E_1 - p_1 + 2E_2) \\
E &= \frac{(E_1 + p_1)E_2}{(E_1 - p_1) + 2E_2}
\end{aligned}$$