Classical Mechanics

TT 2010

Attempt 2 of the 3 mechanics (CM) problems.

CM1. Do all three parts:

(a). A mass m is projected at an initial velocity v at angle α above a horizontal surface. Neglecting air resistance, obtain expressions for:

(i) the horizontal range,

(ii) the time taken to return to the horizontal surface,

(iii) the maximum height attained. [9]

(b). An interplanetary spacecraft passes close to Jupiter. When still at a large distance from the planet it has speed v and impact parameter b relative to Jupiter. Show that the craft's distance of closest approach is $[\sqrt{(G^2M^2 + v^4b^2)} - GM]/v^2$, where G is the gravitational constant and M is the mass of Jupiter. [8]

(c). An efficient way to reach the Moon is first to put the spacecraft in a low circular Earth orbit of radius r_0 and speed v_0. The speed is then boosted to v_p, giving an elliptical orbit with apogee at the Moon's orbit r_a and perigee at r_0. Show that [8]

$$\left(\frac{v_p}{v_0}\right)^2 = \frac{2r_a}{r_0 + r_a}.$$

CM2. A box of mass M can slide horizontally (in the $\hat{x}$ direction) on a frictionless surface. A simple pendulum of string length ℓ and mass m is suspended from the center of the top of the box. Let $\hat{y}$ point upwards; the pendulum swings only in the $\hat{x} - \hat{y}$ plane. Denote the horizontal coordinate of the center of mass of the box itself by x, and the angle that the pendulum makes with the vertical by θ. At $t = 0$, the pendulum displacement is $\theta = \theta_0 \neq 0$.

(a). Find the Lagrangian and the equation of motion for the generalized coordinates x and θ. Which conservation law is obtained as a result of the cyclic coordinate? [10]

(b). Find the solution for θ in the small angle approximation (assume both θ and $\dot{\theta}$ are small). Hence show that the pendulum executes simple harmonic oscillations at a frequency [15]

$$\omega = \left(\frac{M+m}{M}\right)^{\frac{1}{2}} \left(\frac{g}{\ell}\right)^{\frac{1}{2}}$$

CM3. What is meant by an *invariant* quantity in the context of special relativity? Illustrate your answer by discussing the relationship between total energy, momentum and rest mass for a particle in special relativity. [8]

In the jet of a quasar, an ultra-relativistic electron collides head-on with a very low energy photon of the cosmic microwave background. No new particles are created. Show that the energy of the photon after the collision is given by

$$E = \frac{(E_1 + p_1 c)E_2}{(E_1 - p_1 c) + 2E_2}$$

where E_1 and p_1 are respectively the energy and the magnitude of the momentum of the incident electron, and E_2 is the energy of the initial photon. [17]