A3 Quantum Mechanics Collection Answers
TT 2010

A1.

(a). A complex number whose mod-square is the probability of some event.

(b). A set of amplitudes is *complete* if the amplitude for any event can be calculated from the amplitudes in the set. A quantum state is the state of the system that a complete set of amplitudes specifies.

(c). $|\psi\rangle$ denotes a quantum state. A satisfactory answer would be "a complete set of amplitudes", although strictly $|\psi\rangle$ denotes a quantum state (which can be specified by several different complete sets of amplitudes).

(d). $\psi(x) \equiv \langle x|\psi\rangle$

A2. Since the momentum is certainly p, the ket $|p\rangle \equiv |\psi\rangle$ must be the eigenket of $\hat{p}_x$ with eigenvalue p: $\langle x|\hat{p}_x|p\rangle = p|p\rangle$. Multiplying both sides of this equation by $\langle x|$ we obtain the first-order linear o.d.e

$$-i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = p\langle x|p\rangle \;\Rightarrow\; \frac{\partial}{\partial x}\langle x|p\rangle + \frac{ip}{\hbar}\langle x|p\rangle = 0.$$

The integrating factor is $e^{ipx/\hbar}$, so the solution is

$$\langle x|p\rangle = \text{const} \times e^{ipx/\hbar}.$$

Since there is no uncertainty in the momentum, the position must be entirely undetermined, which is why the amplitude is independent of x. If the wavefunction were normalized to unit probability of being somewhere, the amplitude to be at any particular place would be vanishingly small.

Since $\int dp|p\rangle\langle p|$ is the identity operator, $\int_{-\infty}^{\infty} dp\langle x'|p\rangle\langle p|x\rangle = \langle x'|x\rangle$ is the amplitude for the system to be at x' given that it is certainly at x. Hence it must vanish for $x' \neq x$.

A3. Hermitian: $H^\dagger = H$. A and C are Hermitian.

A4. We can immediately write

$$|\psi(t)\rangle = N\left(e^{-iE_1t/\hbar}|1\rangle + e^{-iE_2t/\hbar}|2\rangle\right)$$

At time $\tau = \pi\hbar/(E_2 - E_1)$,

$$\begin{aligned}
|\psi(\tau)\rangle &= N\left(e^{-i\pi E_1/(E_2-E_1)}|1\rangle + e^{-i\pi E_2/(E_2-E_1)}|2\rangle\right) \\
\langle\psi(\tau)|\psi(0)\rangle &= N^2\left(e^{i\pi E_1/(E_2-E_1)} + e^{i\pi E_2/(E_2-E_1)}\right) \\
&= N^2 e^{i\pi E_1/(E_2-E_1)}(1 + e^{i\pi}) \\
&= 0
\end{aligned}$$

and thus the system at time $t = \tau$ is orthogonal to that at $t = 0$.

A5. Commutator and product:

$$\begin{aligned}
[\hat{A}, \hat{A}^\dagger] &= \frac{1}{2m\hbar\omega}[m\omega\hat{x} + i\hat{p}, m\omega\hat{x} - i\hat{p}] \\
&= \frac{1}{2\hbar}(-[\hat{x}, i\hat{p}] + [i\hat{p}, \hat{x}]) \\
&= 1 \\
\hat{A}\hat{A}^\dagger &= \frac{1}{2m\hbar\omega}(m\omega\hat{x} + i\hat{p})(m\omega\hat{x} - i\hat{p}) \\
&= \frac{1}{2m\hbar\omega}\{(m\omega\hat{x})^2 + \hat{p}^2 - m\omega i[\hat{x}, \hat{p}]\} \\
&= \frac{1}{\hbar\omega}\{\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{1}{2}\hbar\omega\}
\end{aligned}$$

Note that $[\hat{A}, \hat{A}^\dagger] = \hat{A}\hat{A}^\dagger - \hat{A}^\dagger\hat{A} = 1$, so

$$\begin{aligned}
\hat{A}^\dagger\hat{A} &= \hat{A}\hat{A}^\dagger - 1 \\
&= \frac{1}{\hbar\omega}\{\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - \frac{1}{2}\hbar\omega\} \\
\hbar\omega\hat{A}^\dagger\hat{A} &= \hat{H} - \frac{1}{2}\hbar\omega \\
\hat{H} &= \hbar\omega(\hat{A}^\dagger\hat{A} + \frac{1}{2})
\end{aligned}$$

for the Hamiltonian of a harmonic oscillator.

The differential equation is

$$0 = \langle x|\hat{A}|0\rangle = \frac{1}{2m\hbar\omega}\left(m\omega x\langle x|0\rangle + \hbar\frac{\partial}{\partial x}\langle x|0\rangle\right)$$

so we have the first order linear o.d.e.

$$\frac{\partial}{\partial x}\langle x|0\rangle + \frac{m\omega}{\hbar}x\langle x|0\rangle = 0.$$

The integrating factor is $e^{m\omega x^2/2\hbar}$, so the solution is

$$\langle x|0\rangle = \text{const} \times e^{-m\omega x^2/2\hbar}.$$

A6. The first term, with the coefficient A, represents a wave propagating in the positive x direction. Consider the phase at a position x and time t, and the location of that same phase value at subsequent times: as t increases, x must also increase.

A7. For two operators $\hat{A}$ and $\hat{B}$,

$$\begin{aligned}
\hat{A}[\hat{A}, \hat{B}] + [\hat{B}, \hat{A}]\hat{A} &= \hat{A}(\hat{A}\hat{B} - \hat{B}\hat{A}) + (\hat{B}\hat{A} - \hat{A}\hat{B})\hat{B} \\
&= \hat{A}\hat{A}\hat{B} - \hat{B}\hat{A}\hat{A} \\
&= \hat{A}^2\hat{B} - \hat{B}\hat{A}^2 \\
&= [\hat{A}^2, \hat{B}]
\end{aligned}$$

Hence

$$\begin{aligned}[\hat{x}^2, \hat{p}] &= \hat{x}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{x} \\ &= i\hbar\hat{x} + i\hbar\hat{x} \\ &= 2i\hbar\hat{x}\end{aligned}$$

A8. Commutators: $[J_x, J_y] = iJ_z$ and $[J_x, J^2] = 0$.

J^2 has eigenvalues $j(j+1)$ for $j = 0, \frac{1}{2}, 1, \ldots$.

J_x has eigenvalues m with $-j \leq m \leq j$ with integer steps.

B1. The TDSE is

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle.$$

A stationary state $|E\rangle$ has well-defined energy E, so $H|E\rangle = E|E\rangle$ and therefore

$$i\hbar\frac{\partial}{\partial t}|E\rangle = E|E\rangle.$$

This is a first-order linear d.e. with integrating factor $e^{iEt/\hbar}$, so the solution is

$$|E, t\rangle = |E, 0\rangle e^{-iEt/\hbar}.$$

Since Q is independent of time,

$$i\hbar\frac{\partial}{\partial t}\langle\psi|Q|\psi\rangle = i\hbar\frac{\partial\langle\psi|}{\partial t}Q|\psi\rangle + \langle\psi|Qi\hbar\frac{\partial|\psi\rangle}{\partial t}$$

Taking the Hermitian adjoint of the TDSE we have

$$-i\hbar\frac{\partial\langle\psi|}{\partial t} = \langle\psi|H.$$

Using this and the TDSE to replace the derivatives above

$$i\hbar\frac{\partial}{\partial t}\langle\psi|Q|\psi\rangle = -\langle\psi|HQ|\psi\rangle + \langle\psi|QH|\psi\rangle = \langle\psi|[Q, H]|\psi\rangle.$$

Classical mechanics applies in the regime in which it suffices to know the expectation value of variables because their uncertainties are negligible. So Ehrenfest's theorem gives the classical equations of motion.

Since the electric dipole is an observable, its operator will be Hermitian, so $\langle -|Q|+\rangle = q$.

In the energy representation H is the diagonal matrix of its eigenvalues. The matrix of Q has $\langle +|Q|+\rangle$ in the top left corner, etc. So

$$H = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \; Q = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}$$

Hence

$$[Q, H] = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} - \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} = 2qE\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

From the above, in the energy representation,

$$|\psi, t\rangle = \begin{pmatrix} e^{-iEt/\hbar}/\sqrt{2} \\ e^{iEt/\hbar}/\sqrt{2} \end{pmatrix}$$

so by Ehrenfest's theorem,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\langle\psi|Q|\psi\rangle &= 2qe(e^{iEt/\hbar}/\sqrt{2}, e^{-iEt/\hbar}/\sqrt{2}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-iEt/\hbar}/\sqrt{2} \\ e^{iEt/\hbar}/\sqrt{2} \end{pmatrix} \\ &= qE\left(-e^{i2Et/\hbar} + e^{-i2Et/\hbar}\right) \\ &= -i2qE\sin(2Et/\hbar) \end{aligned}$$

We obtain the desired result on dividing through by $i\hbar$.

Integrating the equation wrt t, we find $\langle Q\rangle = q\cos(2Et/\hbar)$, so we could have a charge Q that oscillates simply-harmonically around the origin over a distance s such that $q = sQ$.

B2. The given $\phi(x)$ has, for $x < 0$, an incident and reflected wave; for $0 \leq x \leq a$, positively and negatively travelling waves, but at a different kinetic energy as befitting a different potential; and for $x > a$, a transmitted wave, with the same energy as the incident, as the potential is the same.

$$\begin{aligned} k &= \sqrt{2mE}/\hbar \\ \beta &= \sqrt{2m(E+V_0)}/\hbar \end{aligned}$$

It is assumed that there is no negatively travelling wave from $x = +\infty$.

Matching the values and first derivatives at $x = 0$ and $x = a$ yields the following relationships between the coefficients:

$$\begin{aligned} 1 + r &= c + d \\ k - kr &= \beta c - \beta d \\ ce^{i\beta a} + de^{-i\beta a} &= te^{ika} \\ \beta ce^{i\beta a} - \beta de^{-i\beta a} &= kte^{ika} \end{aligned}$$

For the condition $\beta a = p\pi$, with p a positive integer, we have $e^{\pm i\beta a} = (-1)^p$. The last two conditions give $(c+d) = (c-d)\beta/k$, which then implies $1+r = 1-r$, which can only be true if $r = 0$.

The kinetic energy of the particles inside the potential well is $E + V_0$. The condition $\beta a = p\pi$ results in the energies

$$\begin{aligned} p\pi = \beta a &= a\frac{\sqrt{2m(E+V_0)}}{\hbar} \\ E &= \frac{\hbar^2\pi^2p^2}{2ma^2} - V_0 \end{aligned}$$

which is the same, aside from the V_0, as the infinite square well's energy levels.

When there is no reflection, the boundary conditions give the coefficients

$$\begin{aligned} c &= \frac{1}{2}\left(1+\frac{k}{\beta}\right) \\ d &= \frac{1}{2}\left(1-\frac{k}{\beta}\right) \end{aligned}$$

and thus the wavefunction

$$\phi(x) = \cos\beta x + i\frac{k}{\beta}\sin\beta x$$

B3. Bookwork definitions and description of how they are obtained.

The characteristic orbital radius of muonic hydrogen is like that of electronic hydrogen, but with the reduced mass μ replaced with that for a muon. One might use the ratio of muon to electron mass (207), but more accurate would be the ratio of reduced masses (186), since the hydrogen nucleus is less than 5 times more massive than the muon. Using $a_0 = 5.292 \times 10^{-11}$ m, we obtain $a_\mu = 2.85 \times 10^{-13}$ m, or 285 fm. This means that muon will spend much more time at lower radii than an electron. In particular, the muon will be more sensitive to the nuclear charge distribution than will the electron.

The potential inside a uniformly charged sphere of radius $r_0 = 1$ fm is

$$U(r) = \frac{e^2}{4\pi\epsilon_0 r_0}\left(\frac{r^2}{2r_0^2} - \frac{3}{2}\right)$$

which can be found from Gauss's law and matching the potential at $r = r_0$ to the normal Coulomb potential. We have taken $Z = 1$ already. The perturbing potential is therefore

$$H_1 = \frac{e^2}{4\pi\epsilon_0 r_0}\left(\frac{r^2}{2r_0^2} - \frac{3}{2} + \frac{r_0}{r}\right)$$

and the first-order perturbation is

$$\begin{aligned} \Delta E &= \langle 100|H_1|100\rangle \\ &= \int d^3x \varphi_{100}^* H_1 \varphi_{100} \\ &= \frac{e^2}{\pi a_\mu^3 \epsilon_0 r_0}\int_0^{r_0}\left(\frac{r^4}{2r_0^2} - \frac{3}{2}r^2 + r_0 r\right)e^{2r/a_\mu}dr \\ &\approx \frac{e^2 r_0^2}{10\pi a_\mu^3 \epsilon_0} \end{aligned}$$

where a_μ is the muonic radius, and the integral is approximated by the value at $r = 0$, as $2r_0/a_\mu \approx 0.007$; the correction is less than 1%.

To calculate the actual value, it is easiest to extract $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ and use $\hbar c = 197$ MeVfm, resulting in

$$\begin{aligned} \Delta E &\approx \frac{2r_0^2}{5a_\mu^3}\alpha\hbar c \\ &= 0.025 \text{ eV} \end{aligned}$$

B4. The Hamiltonian is $H = p^2/2m$. It is a scalar so it commutes with L^2. Consequently there is a complete set of mutual eigenkets of H and L^2. [2]

We can express H in terms of p_r and L^2:

$$H = \frac{p^2}{2m} = \frac{1}{2m}\left(p_r^2 + \frac{\hbar^2 L^2}{r^2}\right)$$

For kets $|E, l\rangle$ that are simultaneous eigenkets of l, this expression immediately reduces to the required expression. [3]

The second term in the bracket is the kinetic energy of tangential motion (classically $\frac{1}{2}mv_t^2$) and the first term is the radial kinetic energy (classically $\frac{1}{2}mv_r^2$). [1]

For the given form of A_l

$$\begin{aligned} A_l^\dagger A_l &= \frac{1}{2m}\left(-ip_r - \frac{(l+1)\hbar}{r}\right)\left(ip_r - \frac{(l+1)\hbar}{r}\right) \\ &= \frac{1}{2m}\left(p_r^2 + \frac{(l+1)^2\hbar^2}{r^2} + i(l+1)\hbar[p_r, r^{-1}]\right) \end{aligned}$$

But $[p_r, r^{-1}] = -r^{-2}[p_r, r] = i\hbar r^{-2}$ so

$$A_l^\dagger A_l = \frac{1}{2m}\left(p_r^2 + \frac{(l+1)^2\hbar^2}{r^2} - \frac{(l+1)\hbar^2}{r^2}\right) = H_l$$

as required. [6]

Clearly,

$$[A_l, A_l^\dagger] = \frac{1}{2m}\left[\left(ip_r - \frac{(l+1)\hbar}{r}\right), \left(-ip_r - \frac{(l+1)\hbar}{r}\right)\right] = -\frac{(l+1)i\hbar}{m}[p_r, r^{-1}] = \frac{(l+1)\hbar^2}{mr^2}$$

But $H_{l+1} - H_l = \{(l+1)(l+2) - l(l+1)\}\hbar^2/(2mr^2) = (l+1)\hbar^2/mr^2$. [3]

Now

$$A_l H_l = A_l A_l^\dagger A_l = (A_l^\dagger A_l + [A_l, A_l^\dagger])A_l = (H_l + H_{l+1})A_l = H_{l+1}A_l$$

So if we multiply both sides of $E|E, l\rangle = H_l|E, l\rangle$ by A_l, we get

$$EA_l|E, l\rangle = A_l H_l|E, l\rangle = H_{l+1}A_l|E, l\rangle,$$

which establishes that $A_l|E, l\rangle \propto |E, l+1\rangle$ is a state with the same energy but more angular momentum. To see whether there is an upper limit on the angular momentum, we evaluate

$$|A_l|E, l\rangle|^2 = \langle E, l|A_l^\dagger A_l|E, l\rangle = E > 0$$

so there is no limit to the angular momentum. Physically, when there is no confining potential, with a given energy the particle can move at a given speed along a path that is as far from the origin as it pleases. Hence its angular momentum is not constrained by its energy. [5]