CP1 June 2009 Answers

1. Work in a coordinate system rotated such that the slope is along the y axis. The gravitational force is then $\mathbf{g}' = (-g \sin\phi, -g\cos\phi)$, and the trajectory is

$$\begin{aligned} x'(t) &= v\cos\theta t - \frac{1}{2} g \sin\phi t^2 \\ y'(t) &= v\sin\theta t - \frac{1}{2} g \cos\phi t^2 \end{aligned}$$

The projectile meets the slope where $y'(t) = 0$

$$t = \frac{2v\sin\theta}{g\cos\phi}$$

and the range is $x'(t)$ evaluated at the above time.

$$x'(t) = \frac{2v^2\sin\theta}{g\cos\phi}(\cos\theta - \sin\theta\tan\phi)$$

2. (a) The moment of inertia is evaluated with the integral

$$I = \int_S r^2 dm$$

where the integral is over the surface. In this case, r is the distance from the center of the disk. The integral element dm is $\rho r dr d\phi$ in circular coordinates, with mass density $\rho = M/\pi R^2$.

$$I = 2\pi\rho \int_0^R r^3 dr = \frac{1}{2} M R^2$$

(b) Use the parallel axis theorem to evaluate the moment of inertia around a point halfway between the center and the rim of the disk:

$$I' = I_{cm} + Mr^2 = \frac{1}{2} M R^2 + M\left(\frac{R}{2}\right)^2 = \frac{3}{4} M R^2$$

3. We get a differential equation for ω by evaluating the change in angular momentum due to each force. The drag torque is given

$$\frac{dL_D}{dt} = -\gamma\omega$$

For the constant power input, we note that the power can be defined as

$$P = \frac{dE}{dt}$$

where E is the total energy, which is simply $\frac{1}{2} I\omega^2 = L^2/2I$ (the kinetic energy) in this case. Therefore

$$P = \frac{L}{I}\frac{dL}{dt}$$

when acting by itself, so it's contribution to dL/dt is

$$\frac{dL_P}{dt} = \frac{IP}{L}$$

The total dL/dt is

$$\frac{dL}{dt} = \frac{dL_D}{dt} + \frac{dL_P}{dt} = IP/L - \gamma\omega$$

Then substituting in $L = I\omega$, we get the differential equation

$$I\frac{d\omega}{dt} = \frac{P}{\omega} - \gamma\omega$$

Note that steady-state is achieved when $d\omega/dt = 0$, which gives us $\omega_0^2 = P/\gamma$. Then the differential equation becomes

$$\frac{d\omega^2}{dt} + \frac{2\gamma}{I}\omega^2 = \frac{2\gamma}{I}\omega_0^2$$

This is a first-order linear inhomogeneous equation. The initial condition is $\omega(0) = 0$, so the solution is

$$\omega^2(t) = \omega_0^2(1 - e^{-2\gamma t/I})$$

Inverting this to get t gives the desired answer

$$t = -\frac{I}{2\gamma}\ln\left(1 - \frac{\omega^2}{\omega_0^2}\right)$$

4. We choose a cylindrical coordinate system, with the plane of the orbit in $z = 0$. Differentiating the coordinate transformation $\mathbf{r} = (r\cos\phi, r\sin\phi, z)$, we get for the velocity

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

We write down the total energy

$$\begin{aligned} E &= \frac{1}{2}mv^2 - \frac{GMm}{r} \\ &= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{mr^2}{2}\dot{\phi}^2 - \frac{GMm}{r} \\ &= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{2mr^2} - \frac{GMm}{r} \end{aligned}$$

where we have introduced the constant of the motion, $J = mr^2\dot{\phi}$, which is the angular momentum.

If $M = m$, we would replace m in the first two terms by the reduced mass $\mu = m/2$. The last term remains unchanged.

5. The particles have $\gamma = 184$. The half-life in the lab frame is $\tau' = \ell/c$, with $\ell = 1$ km. This half-life has been dilated from the rest frame, so the proper half life is $\tau = \tau'/\gamma$. Plugging in, we get 18 ns.

6. Relativistic energy and momentum are $E = \gamma mc^2$ and $p = \gamma mv$, where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We use the Lorentz transformation of energy and momentum to evaluate E' and p' in a frame boosted with velocity βc (note that γ below is different from the one above).

$$\begin{aligned}
p' &= \gamma(p - \beta E/c) \\
E' &= \gamma(E - \gamma pc) \\
E'^2 - p'^2 c^2 &= \gamma^2(E - \beta pc)^2 - \gamma^2(pc - \beta E)^2 \\
&= \gamma^2(E^2(1-\beta^2) - p^2c^2(1-\beta^2)) \\
&= E^2 - p^2c^2
\end{aligned}$$

7. The kinetic energy T is simply the motion of the particle in three dimensions, $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, and the potential can be written $V = mgz + \frac{1}{2}kz^2$, where we've taken $z = 0$ to indicate the equilibrium position (in principle, this is the equilibrium position if there was no gravity, but it makes no difference). The Lagrangian is thus

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz - \frac{1}{2}kz^2$$

To get the equation of motion for vertical displacements,

$$\begin{aligned}
\frac{\partial L}{\partial z} &= -mg - kz \\
\frac{\partial L}{\partial \dot{z}} &= m\dot{z} \\
0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} \\
&= m\ddot{z} + mg + kz
\end{aligned}$$

8. Let v be the speed of the incoming particle, and $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities of the outgoing particles. Linear momentum and kinetic energy are conserved:

$$\begin{aligned}
\frac{1}{2}mv^2 &= \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \\
m\mathbf{v} &= m\mathbf{v}_1 + m\mathbf{v}_2
\end{aligned}$$

Squaring the momentum equation and subtracting the energy equation results in

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

which has the stated meaning.

9. (a) For a swing in a vertical plane, the kinetic energy is $T = \frac{1}{2}I\dot{\theta}^2$, and the potential energy $-mgl\cos\theta/2$ (evaluating from the center of mass of the rod). The Lagrangian is

$$L = \frac{1}{2}I\dot{\theta}^2 + \frac{mgl}{2}\cos\theta$$

which results in the equation of motion

$$0 = I\ddot{\theta} + \frac{mgl}{2}\sin\theta$$

We can also substitute in the moment of inertia for a unifom rod around one end, $ml^2/3$.

In the small-angle approximation, $\sin\theta \approx \theta$, giving us the approximate equation of motion

$$0 = \ddot{\theta} + \frac{3g}{2l}\theta$$

This describes simple harmonic oscillation with frequency $\omega^2 = 3g/2l$. The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2l}{3g}}$$

(b) If the rod swings in any direction from the pivot, then the angular velocity factor in the kinetic energy becomes $v^2 = r^2(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2)$. (This can be checked by calculating the kinetic energy of each mass element along the rod, as a function of θ and ϕ.) The potential energy term remains unchanged. The Lagrangian is

$$L = \frac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{mgl}{2}\cos\theta$$

The cyclic coordinate is ϕ, which leads to the conservation of "rotational" angular momentum, $I\sin^2\theta\dot{\phi}$.

There is no energy input to the system, so the Hamiltonian should also be a constant of the motion.

10. At any time, linear momentum is conserved by the rocket and the ejected fuel:

$$\begin{aligned}(m + dm)v &= m(v + dv) - dm(u - v) \\ dv &= -u\frac{dm}{m} \\ v(m_f) &= -u\int_{m_i}^{m_f}\frac{dm}{m} = u\log\left(\frac{m_i}{m_f}\right)\end{aligned}$$

If at some point the fuel is left stationary, then $v = u$.

$$\begin{aligned}u &= u\log(m_i/m_f) \\ m_f &= m_i e^{-1}\end{aligned}$$

Therefore the fraction of fuel to initial total mass must be at least

$$\frac{m_i - m_f}{m_i} > 1 - e^{-1}$$

If the fuel burn rate is $dm/dt = -\alpha m$, then from $mdv = -udm$ we get

$$\begin{aligned}dv &= -\frac{u}{m}dm \\ \frac{dv}{dt} &= -\frac{u}{m}\frac{dm}{dt} = +\alpha u\end{aligned}$$

which is simply constant acceleration, so the distance grows as $x(t) = \frac{1}{2}\alpha u t^2$. We obtain α by using the data that $m_i/m_f = 2$ at $\tau_{1/2} = 600$ s, and that $m(t) = m_i e^{-\alpha t}$, yielding

$$\alpha = \frac{\log 2}{\tau_{1/2}}$$

At the same time, x is the distance to the stationary fuel, *i.e.*, when $m_i/m_f = e$, which happens at $t = 1/\alpha$. Therefore

$$x = \frac{u}{2\alpha} = \frac{1000 \text{ m/s}}{2\log 2} 600 \text{ s} = 432 \text{ km}$$

11. In circular orbit,

$$\begin{aligned} \frac{GMm}{r_0^2} &= \frac{mv^2}{r_0} \\ v^2 &= \frac{GM}{r_0} \end{aligned}$$

The comet originally has speed $v_0^2 = GM/\alpha R_E$. After collision, the velocity is $v_0 - \Delta v$. In the new orbit, angular momentum is conserved, and if it barely intersects with the Earth's orbit, the it will be travelling perpendicular to the radial direction at that point at a speed v'. Angular momentum conservation gives

$$\begin{aligned} L = (v_0 - \Delta v_1) m r_0 &= v' m R_E \\ v' &= (v_0 - \Delta v_1)\frac{r_0}{R_E} = \alpha(v_0 - \Delta v_1) \end{aligned}$$

Likewise, energy conservation gives

$$E = \frac{1}{2}m(v_0 - \Delta v_1)^2 - \frac{GMm}{\alpha R_E} = \frac{1}{2}mv'^2 - \frac{GMm}{R_E}$$

which gives, on substitution of v' and rearrangement,

$$(v_0 - \Delta v_1)^2 = \frac{GM}{R_E \alpha(\alpha + 1)}$$

Now substituting the expression for v_0 gives

$$\Delta v_1 = \left(1 - \sqrt{\frac{2}{\alpha + 1}}\right) v_0$$

In the perpendicular boost case, L is unchanged, and the conservation equations give

$$\begin{aligned} L &= v_0 m r_0 = v' m R_E \\ E &= \frac{1}{2}m(v_0^2 + \Delta v_2^2) - \frac{GMm}{\alpha R_E} = \frac{1}{2}mv'^2 - \frac{GMm}{R_E} \end{aligned}$$

which yields, after similar manipulations

$$\Delta v_2 = (\alpha - 1)v_0$$

For a Kuiper belt coment, α is given to be 50. The ratio of velocity changes is

$$\frac{\Delta v_2}{\Delta v_1} = \frac{\alpha - 1}{1 - \sqrt{\frac{2}{\alpha+1}}} \approx 61$$

12. (a) This is the fixed-target case. The energy-momentum vector of the positron is (E, p), and the electron $(m_e, 0)$, where $p = \sqrt{E^2 - m_e^2}$ and we've let $c = 1$. The sum is $(E + m_e, p)$. The maximum mass which can be created in such a collision is the norm of this 4-vector, *i.e.*, the energy available in the center of momentum frame.

$$M^2 = (E + m_e)^2 - p^2 = (E + m_e)^2 - (E^2 - m_e^2) = 2m_e E + 2m_e^2$$

The minimum positron energy to create a $M = 200$ GeV particle is therefore

$$E_{min} = \frac{M^2 - 2m_e^2}{2m_e} = 3.9 \times 10^7 \text{ GeV}$$

(which is a lot of energy).

(b) In the collider case, the energy-momentum vector of the positron is (E, p), and the electron $(E, -p)$. The sum is $(2E, 0)$, so $M = 2E$, and therefore the minimum energy required is $E_{min} = M/2 = 100$ GeV.

The new particle, with mass $M = 200$ GeV, decays to two identical particles with mass (rest energy) $m_Z = 91.2$ GeV. In the rest frame of one daughter, we have the energy-momentum vectors

$$(E, p) + (m_Z, 0) = (E + m_Z, p)$$

where $E^2 - p^2 = m_Z^2$ and the mass of the right-hand side vector is the parent particle with mass M. The magnitude of this vector therefore gives

$$M^2 = (E + m_Z)^2 - p^2 = 2m_Z^2 + 2Em_Z$$

which gives the energy

$$\begin{aligned} E &= \frac{M^2 - 2m_Z^2}{2m_Z} = \gamma m_Z \\ \gamma &= \frac{M^2}{2m_Z^2} - 1 \end{aligned}$$

which gives

$$\beta = \frac{M}{M^2 - 2m_Z^2}\sqrt{M^2 - 4m_Z^2} \approx 0.70$$

or 2.1×10^8 cm/s.

This speed was calculated in the rest frame of one of the daughter particles, not in the lab frame. The method of creating the particle is not relevant, so the answer given above should be the same even if method (a) was used.