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Energy and Momentum in Special Relativity

Energy (E) and momentum ($\vec{p}$) form a 4-vector, much as "space-time" also refers to a 4-vector of time and space.

4-vectors are much like the usual 3-vectors which describe space, and we can compare the energy-momentum vector (P) with the simple momentum 3-vector ($\vec{p}$):

	$\vec{p}$	P
components	(p_x, p_y, p_z)	$(E, p_x, p_y, p_z) = (E, \vec{p})$
inner product	$\vec{p}_1 \cdot \vec{p}_2$ $= p_{x1}p_{x2} + p_{y1}p_{y2} + p_{z1}p_{z2}$	$P_1 \cdot P_2$ $= E_1E_2 - (p_{x1}p_{x2} + p_{y1}p_{y2} + p_{z1}p_{z2})c^2$ $= E_1E_2 - \vec{p}_1 \cdot \vec{p}_2 c^2$
"norm"	$\lvert\vec{p}\rvert^2 = \vec{p} \cdot \vec{p} = p_x^2 + p_y^2 + p_z^2$ (length)^2	$\lvert P\rvert^2 = P \cdot P$ $= E^2 - (p_x^2 + p_y^2 + p_z^2)c^2 = m^2c^4$ (rest mass)^2
invariant wrt	spatial rotations	Lorentz transformations (space-time rotations)
conservation	$\sum E_{before} = \sum E_{after}$ $\sum \vec{p}_{before} = \sum \vec{p}_{after}$	$\sum P_{before} = \sum P_{after}$

In either case, the vectors form linear spaces (or vector spaces, if you prefer to think of them that way), *i.e.*, sums of vectors produce other vectors, vectors multiplied by scalars are also vectors, and addition, scalar multiplication, and inner products obey associative and distributive laws.

Expressions which contain only geometric objects (*e.g.*, vectors, scalars, tensors) hold true in any reference frame—in fact, this is one way to summarize *general* relativity as well. (Note that Maxwell's equations, which are often written as four equations, can also be written as a single equation with a tensor and a vector—so we know it has the form of a physical law!)

An expression which contains a *component* of a geometric object, such as the energy or the 3-momentum, only holds true in one reference frame.

A common technique in first-year special relativity problems is as follows:

1. write the energy-momentum conservation equation with 4-vectors (true in all frames)
2. square both sides—this leaves the invariants (still true in all frames)
3. evaluate each side (*i.e.*, the 4-momenta *before* and the 4-momenta *after*) in the most convenient frame.

A classic example is Compton scattering, which is only slightly more complicated to derive than the above because you manipulate the four-momentum conservation equation before you square it. The question is, what is the change in energy and direction of a photon scattering off an electron at rest? From 4-momentum conservation of the process $e + \gamma \rightarrow e + \gamma$,

$$P_e + P_\gamma = P'_e + P'_\gamma$$

where P_e and P_γ are the initial 4-momenta and P'_e and P'_γ the final 4-momenta. Physically, we expect that the electron will acquire some momentum, but we can ignore it in the problem (of course, we can always figure it out later), so we isolate P'_e on one side of the equation:

$$P'_e = P_e + P_\gamma - P'_\gamma$$

which we then square (*i.e.*, take an inner product with itself):

$$\begin{aligned}
P'^2_e &= (P_e + P_\gamma - P'_\gamma)^2 \\
&= P^2_e + P^2_\gamma + P'^2_\gamma + 2P_e \cdot P_\gamma - 2P_e \cdot P'_\gamma - 2P_\gamma \cdot P'_\gamma \\
m^2_e c^4 &= m^2_e c^4 + 2P_e \cdot P_\gamma - 2P_e \cdot P'_\gamma - 2P_\gamma \cdot P'_\gamma \\
0 &= P_e \cdot P_\gamma - P_e \cdot P'_\gamma - P_\gamma \cdot P'_\gamma
\end{aligned}$$

Note that $P^2_\gamma = P'^2_\gamma = 0$ in any frame (photons are massless). Also notice that the final momentum of the electron has dropped entirely out of the problem. All that remains is to evaluate this equation in the most convenient frame, which is the one in which the electron was initially at rest:

$$\begin{aligned}
P_e &= (m_e c^2, 0) \\
P_\gamma &= (E_\gamma, \vec{p}_\gamma) \\
P'_\gamma &= (E'_\gamma, \vec{p}'_\gamma)
\end{aligned}$$

where $E_\gamma = |\vec{p}_\gamma| c$ and $E'_\gamma = |\vec{p}'_\gamma| c$. This results in

$$\begin{aligned}
0 &= m_e c^2 (E_\gamma - E'_\gamma) - E_\gamma E'_\gamma + \vec{p}_\gamma \cdot \vec{p}'_\gamma \\
&= m_e c^2 (E_\gamma - E'_\gamma) - E_\gamma E'_\gamma + E_\gamma E'_\gamma \cos\theta \\
E'_\gamma &= \frac{E_\gamma m_e c^2}{E_\gamma (1 - \cos\theta) + m_e c^2}
\end{aligned}$$

where θ is the angle between the initial and final photon directions.